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COMPLETELY INJECTIVE SEMIGROUPS

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A semigroup S with identity is termed completely right injective if every right unitary S-system is injective. The semigroup S is called completely injective if every right and left unitary S-system is injective. We prove that S is completely injective if and only if S is a semigroup with zero, where every right ideal and every left ideal of S is generated by an idempotent. This condition is equivalent to the statement that S is an inverse semigroup with zero, whose idempotents are dually well-ordered.

If S is completely injective and if e is an idempotent in S, then eSe, and every two-sided ideal of S, is completely injective.

A completely injective semigroup S is termed *central* if S is the union of groups. If S is completely injective and S has a finite number of right ideals, or if the two-sided ideals of S are local, then S is central.

2. **Main theorems.** *Throughout this paper S will always denote a semigroup with* 1, *and all S-systems will be unitary.* The set of idempotents of any semigroup T will be denoted by $E(T)$.

Using 2.2, and the proof of 2.7 of [3], we have the first part of the following theorem.

THEOREM 2.1. *If S is completely right injective, then every right ideal is generated by an idempotent. Thus the right ideals form a chain under set inclusion, which is dually well-ordered. In addition, S contains a zero element.*

Proof. From the proof of 2.6 of [3], we have S contains a left zero 0. Now $0S = \{0\}$ is contained in every right ideal. Hence for $a \in S$, then $0S \subseteqq a0S$. Thus $0 = a0x = a0$.

LEMMA 2.2. *Let $e, f \in E(S)$.*

(i) *If every right ideal of S is generated by an idempotent, then $Se \subseteqq Sf$ implies $eS \subseteqq fS$.*

(ii) *If every left ideal of S is generated by an idempotent, then $eS \subseteqq fS$ implies $Se \subseteqq Sf$.*

(iii) *If every right and left ideal of S is generated by an idempotent, then $Se \subseteqq Sf$ if and only if $eS \subseteqq fS$. In particular, $Se = Sf$ if and only if $eS = fS$.*

Proof. Clearly, $eS \subseteqq fS$ if and only if $fe = e$, and $Se \subseteqq Sf$ if and only if $ef = e$. To prove (i), suppose $Se \subseteqq Sf$. Then $ef = e$. Either $eS \subseteqq fS$ or $fS \subset eS$. The latter is impossible for then $f \neq e$ and $ef = f$.

Part (ii) is proved in a similar way, while (iii) follows from (i) and (ii).

LEMMA 2.3. *If every right and left ideal of S is generated by an idempotent, then S is an inverse semigroup. Moreover, $E(S)$ is a chain under the natural partial ordering, which is dually well-ordered.*

Proof. The fact that S is regular follows from Lemma 1.13 of [1, p. 27]. We shall now prove (i) of Lemma 1.17 of [1, p. 28] to show that S is inverse.

For $e, f \in E(S)$, then either $eS \subseteqq fS$ or $fS \subseteqq eS$. If $eS \subseteqq fS$, then by 2.2 we have $Se \subseteqq Sf$. Hence $fe = ef = e$, and $e \leqq f$ under the natural partial ordering. Thus the idempotents of S commute, and form a chain.

Since $E(S)$ is commutative, then $e \leqq f$ if and only if $eS \subseteqq fS$. By 2.1, the dual well-ordering of the right ideals implies that any nonempty subset $\{e_\alpha \mid \alpha \in I\}$ of $E(S)$ contains a greatest element; namely the idempotent which generates $\bigcup_{\alpha \in I} e_\alpha S$.

LEMMA 2.4. *If T is an inverse semigroup and $e \in E(T)$, then aea^{-1} and $a^{-1}ea$ are in $E(T)$.*

Proof. Since $a^{-1}a \in E(T)$ and $E(T)$ is commutative, then

$$(aea^{-1})(aea^{-1}) = a(a^{-1}a)e^2a^{-1} = aea^{-1} .$$

Similarly, $(a^{-1}ea)^2 = a^{-1}ea$.

FIRST MAIN THEOREM 2.5. *A semigroup S is completely injective if and only if S is a semigroup with zero, and every left and right ideal of S is generated by an idempotent.*

Proof. From 2.1, we have the "only if" part of this theorem.

Suppose now S is a semigroup with zero, and every right and left ideal is generated by an idempotent. From 2.3, S is an inverse semigroup and $E(S)$ is dually well-ordered. Using the same technique employed in the proof of Theorem 2.6 of [3], we show every right S-system is injective. A similar argument shows left S-systems are injective.

Let M, P, and R be S-systems where $P \subseteqq R$. If $f: P \to M$ is a

S-homomorphism of P_S into M_S, let (P_0, f_0) be the maximal pair defined in the proof of 2.6 of [3]. To show M is injective, it suffices to show $P_0 = R$. Suppose $r \in R$, $r \notin P$, and let $A = \{a \in S \mid ra \in P_0\}$. As in 2.6 of [3], we will reach a contradiction for $P_0 \neq R$, if we can show the existence of an S-homomorphism $h: rS \to M$ which agrees with f_0 on $P_0 \cap rS$. If A is empty, the argument is the same as in 2.6 of [3].

Suppose A is nonempty. Then $A = eS$, for $e \in E(S)$. Let h be the same mapping, $h(rs) = zes$ for all $s \in S$, defined in 2.6 of [3]. We need only show that h is single-valued. The argument of 2.6 of [3] will then complete the proof.

As shown in 2.6 of [3], h will be single-valued if and only if $rs_1 = rs_2$ implies $res_1 = res_2$, for all $s_1, s_2 \in S$. Since S is inverse, then $res_1 = r(es_1s_1^{-1}s_1) = r(s_1s_1^{-1})es_1 = (rs_1)s_1^{-1}es_1 = rs_2s_1^{-1}es_1$. Likewise $res_2 = rs_1s_2^{-1}es_2$. Since es_1, and es_2 belong to A, then res_1 and res_2 belong to P_0. Therefore $s_2s_1^{-1}es_1$ and $s_1s_2^{-1}es_2$ belong to A. Since $A = eS$, then $s_2s_1^{-1}es_1 = es_2s_1^{-1}es_1$; consequently $res_1 = res_2s_1^{-1}es$. Likewise $res_2 = res_1s_2^{-1}es_2$. Using the fact that idempotents commute and Lemma 2.4, we have

$$\begin{aligned} res_1 &= (res_2)s_1^{-1}es_1 = (res_1s_2^{-1}es_1)s_1^{-1}es_1 \\ &= res_1(s_2^{-1}es_2)(s_1^{-1}es_1) = res_1(s_1^{-1}es_1)(s_2^{-1}es_2) \\ &= r(es_1s_1^{-1}es_1)s_2^{-1}es_2 = res_1s_2^{-1}es_2 = res_2 \,. \end{aligned}$$

SECOND MAIN THEOREM 2.6. *A semigroup T is completely injective if and only if T is an inverse semigroup with zero, and $E(T)$ is dually well-ordered.*

Proof. The definition of completely injective implies such semigroups contain an identity 1. Using 2.5 and 2.3, we have the necessity.

Conversely, suppose T is inverse with zero and $E(T)$ is dually well-ordered. Using the argument in the proof of Lemma 2.1 of [4], the greatest element of $E(T)$ is the identity element of T.

Let R be any right ideal of T. By Theorem 1.13 of [1, p. 27], the principal right ideals of T are generated by idempotents. Therefore $E(T) \cap R$ is not empty. Since $E(T)$ is dually well-ordered, then $E(T) \cap R$ contains a greatest element f. It follows $R = fT$. In this way every right and left ideal is generated by an idempotent. Applying 2.5 we have T is completely injective.

If S is completely injective, it is of interest to note that the $\mathscr{R}$-classes of S, defined in [1, p. 47], are of the form $eS \backslash fS$, where fS is maximal in eS.

EXAMPLE 2.7. N. R. Reilly [4] called a semigroup T an ω-semigroup if and only if there exists a one-to-one may φ of $E(T)$, which is commutative, onto the set of nonnegative integers such that

$$\varphi(e) \leqq \varphi(f)$$

if and only if $f \leqq e$. Thus $E(T)$ is dually well-ordered. Applying 2.6, for any inverse ω-semigroup T, we have $T^0 = T \cup 0$ is completely injective. The bisimple ω-semigroups are concrete examples of inverse ω-semigroups. In particular, the bicyclic semigroup of [1, p. 43] with zero adjoined is completely injective. These provide examples of completely injective semigroups, which are not the union of groups, as discussed in [3].

A trivial example of a completely right injective semigroup which is not completely left injective is a right zero semigroup containing two or more elements with 0 and 1 adjoined. In fact, applying the technique of 2.5, the authors have shown that if S is a right 0-simple semigroup containing an idempotent $e \neq 0$, then $S^1 = S \cup 1$ is completely right injective.

3. Properties of completely injective semigroups.

In §'s 3 *and* 4, *S will always denote a completely injective semigroup.* We begin this section with a discussion of a one-to-one correspondence between the lattices of right ideals and of left kernel congruences belonging to S-endomorphisms of ${}_SS$. The left kernel congruence belonging to a S-endomorphism g of ${}_SS$ is defined to be that left congruence ρ on S given by $a\rho b$ if and only if $g(a) = g(b)$.

DEFINITION 3.1. If K is a subset of S, let $\rho(K)[\lambda(K)]$ denote the right [left] congruence of S defined by: $(a, b) \in \rho(K)[(a, b) \in \lambda(K)]$ if and only if $ka = kb[ak = bk]$ for all $k \in K$. If σ is a right [left] congruence on S, let $\ell(\sigma)[r(\sigma)]$ denote the set of all $s \in S$ such that if $a\sigma b$, then $sa = sb[as = bs]$. Clearly, $\ell(\sigma)[r(\sigma)]$ is a left [right] ideal of S.

PROPOSITION 3.2. *If* $e \in E(S)$, *then* $r(\lambda(eS)) = eS$ *and* $\ell(\rho(Se)) = Se$.

Proof. If $b \in r(\lambda(eS))$, then $\lambda(e) \subseteqq \lambda(b)$. Thus the mapping $g: xe \rightarrow xb$ is an S-homomorphism of Se onto Sb. Now $b = g(e) = eg(e)$. Therefore $b \in eS$ and $r(\lambda(eS)) \subseteqq eS$. Since the opposite inclusion is immediate, we have equality. Similarly, $\ell(\rho(Se)) = Se$.

The left congruence $\lambda(eS)$, wehere $e \in E(S)$, is the left kernel congruence belonging to the S-endomorphism $h: {}_SS \rightarrow {}_SS$, where $h(x) = xe$ for all $x \in S$. Conversely, every left kernel congruence belonging to a S-endomorphism h of ${}_SS$ is of this form. Indeed, the left kernel congruence belonging to h is $\lambda(h(1)S)$, which equals $\lambda(eS)$ for some $e \in E(S)$.

Since $e_1S \subseteqq e_2S$ implies $\lambda(e_1S) \supseteqq \lambda(e_2S)$, then 3.2 implies that the mapping $eS \rightarrow \lambda(eS)$ is a one-to-one inclusion reversing correspondence between the lattice of right ideals of S and the set $\mathscr{K}$ of all left kernel congruences belonging to S-endomorphisms of ${}_SS$. Thus we have the following theorem.

THEOREM 3.3. *The lattice of right ideals of S and the lattice of left kernel congruences belonging to S-endomorphisms of ${}_SS$ are dual isomorphic.*

Thus S satisfies the minimum condition (D.C.C.) on right ideals if and only if $\mathscr{K}$ satisfies the maximum condition (A.C.C.). These results are similar to results for quasi-Frobenius rings.

Note that if $\sigma \in \mathscr{K}$, then $\lambda(\mathfrak{r}(\sigma)) = \sigma$. It is not difficult to show this relation is not true for an arbitrary left congruence on S.

Next we show certain subsystems of S are completely injective.

THEOREM 3.4. *For every $e \in E(S)$, eSe is completely injective.*

Proof. We show every left and right ideal of eSe is generated by an idempotent. Let L be a left ideal of eSe. It follows directly that $L = SL \cap eSe$. Now $SL = Sf$, for some $f \in E(S)$. Using Lemma 1.19 of [1, p. 30], we have $L = Sf \cap eSe = Sf \cap Se \cap eS = Sfe \cap eS = (eSe)f$. If $ef = e$, then $L = eSe$. If $ef = f$, then $f = efe \in eSe$, and $L = (eSe)f$. A similar argument holds for right ideals.

If H is a two-sided ideal of S, we have by 2.2 and Theorem 1.17(ii) of [1, p. 28], that $H = eS = Se$. Hence $H = eS \cap Se = eSe$ and we can write

COROLLARY 3.5. *Every two-sided ideal of S is completely injective.*

4. Central completely injective semigroups. Throughout this section, S denotes a completely injective semigroup and T an arbitrary semigroup. If $E(T)$ is contained in the center of T, then T is termed *central*. In [3], the authors determined a structure for central completely injective semigroups. We use the fact that an inverse semigroup T is central if and only if T is the union of groups (see the proof of 2.8 of [3]). Applying this together with 2.6 we have 4.1 and 4.2.

THEOREM 4.1. *A semigroup T with 1 is central completely injective if and only if T is an inverse semigroup with 0, $E(T)$ is* *dually well-ordered, and T is a union of groups.*

THEOREM 4.2. *S is central if and only if S is a union of groups.*

Certainly, there are many conditions on an inverse semigroup which imply that it is the union of groups. For example, 7.4 of [2, p. 41] would imply that S is central if and only if the left and right units of each element are equal.

Next we shall give a condition in terms of local semigroups. Using the terminology of [1, p. 21], if T is a semigroup with 1, then an element a in T is called a *right* [*left*] *unit* provided there exist $x \in T$ such that $ax = 1[xa = 1]$. A left and right unit is called a *unit*.

PROPOSITION 4.3. *A semigroup T with 1 is termed local provided one of the following equivalent conditions are satisfied.*

(i) *Every right unit is a left unit.*

(ii) *The set of nonunits form a proper ideal of T.*

(iii) *T contains an ideal, which is a unique maximal right ideal.*

THEOREM 4.4. *S is central if and only if the two-sided ideals of S are local.*

Proof. If S is central, then each two-sided ideal of S has the form eS, where the fS of 2.11 of [3] satisfies (iii) of 4.3. Thus eS is local.

To prove the converse we shall establish the statement, "all the idempotents of S are contained in its center". For S, $E(S)$ is dually well-ordered. Thus we can list the elements of $E(S)$ as

$$1 = e_0 > e_1 > e_2 \cdots > e_\alpha > \cdots > 0$$

where the subscripts are ordinal numbers less than the ordinal number of $E(S)$. It follows

$$S = e_0S \supset e_1S \supset e_2S \supset \cdots \supset e_\alpha S \supset \cdots \supset 0 .$$

We use transfinite induction to prove the above statement. To show that e_1 is in the center of S, let K be the set of nonunits of S. Since S is local, then K is a two-sided ideal which is a unique maximal right ideal of S. Thus $K = e_1S$ and, as in the discussion preceding 3.5, $e_1S = Se_1$. Hence e_1 is in the center of S.

Assume inductively that all e_α, for $\alpha < \beta$, are in the center. If β is not a limit ordinal, then $\beta = \alpha + 1$, where e_α is in the center. Hence $e_\alpha S$ is local. Using the fact that a right ideal of an ideal of S is itself a right ideal of S, then the argument in the preceding paragraph can be applied to show that e_β is in the center.

If e_β is a limit ordinal, then $\bigcap_{\alpha<\beta} e_\alpha S = e_\beta S$. Since the $e_\alpha S$, for $\alpha < \beta$, are two-sided ideals, then $e_\beta S$ is a two-sided ideal and e_β is in the center.

One could use 4.2 to prove the following result. However, 4.5 follows directly from 7.5 of [2, p. 41]. The second part is a consequence of 3.3.

PROPOSITION 4.5. *If S satisfies the minimum condition for right ideals, or the maximum condition for left kernel congruences belonging to endomorphisms of ${}_S S$, then S is central.*

A right T-system is projective if the usual diagram of right T-systems can be completed. We call a semigroup T with identity 1 *completely right projective* if every right T-system is projective. In ring theory, completely projective is equivalent to completely injective. This is not the case for semigroups, which can be deduced from the following theorem.

THEOREM 4.6. *If T is a completely injective and completely right projective semigroup, then T is a group with zero.*

Proof. Let us denote the right annihilator of x by x^*. For any idempotent e of T, we have $eT \cap e^* = 0$. Since the right ideals of T form a chain, then $e^* = 0$ for any nonzero e in $E(T)$.

Let N be a nonzero right ideal of T. Let T/N denote the Rees factor T-system of T by N defined in [2, p. 252]. Let g denote the natural homomorphism of T onto T/N. Since T/N is projective, there exists a monomorphism h of T/N into T such that $gh = 1$ and hg is idempotent. Consequently $hg(1) = e, e \in E(T)$, and $hg(x) = ex$ for all $x \in T$.

By the definition of g, we have $hg(N) = h(\bar{0}) = 0$. On the other hand, $hg(N) = hg(1)N = eN$. Thus $eN = 0$. The discussion in the first paragraph together with the fact that $N \neq 0$ implies $e = 0$. Hence $T/N = \bar{0}$, and $N = T$. Therefore each element in the semigroup of nonzero elements has a right inverse in T and T is a group with zero.

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