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ON ULAM'S CONJECTURE FOR SEPARABLE GRAPHS

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Ulam's conjecture, that every graph of order greater than two is determined up to isomorphism by its collection of maximal subgraphs, is verified for the case of separable graphs which have no pendant vertices. Partial results are then obtained for the case of graphs with pendant vertices.

Unless otherwise stated, the graphs dealt with in this paper will be finite and undirected, and may have loops and multiple edges. Any definitions and notation not given below can be found in Berge [1].

A *part* G^1 of a graph G is a subset of the vertices and edges of G. The end-vertices of edges in G^1 need not themselves be in G^1. If G^1 is a part of G, $G - G^1$ denotes that partial subgraph of G which is obtained by deleting G^1 and all edges of G which are joined to vertices of G^1. Now let S be some distinguished set of parts of a graph, and let $S(X) = \{X^i\}$ be the labelled set of these parts in the graph X. We call two graphs G, H *S-equivalent* if $|S(G)| = |S(H)| = M(< \infty)$ and, possibly after relabelling, $G - G^i \cong H - H^i (1 \leqq i \leqq M)$. G^i, H^i will be referred to as *corresponding* parts.

In [8] Ulam proposed the following conjecture.

CONJECTURE A. Vertex-equivalent graphs of order greater than two are isomorphic.

Kelly [7] verified this conjecture for trees and, by exhaustion, for all graphs up to order seven. A related conjecture, intuitively simpler but also as yet unsolved, was suggested by Harary [4].

CONJECTURE B. Edge-equivalent graphs with more than three edges are isomorphic.

Harary and Palmer [5] strengthened Kelly's theorem by showing that pendant vertex-equivalent trees are isomorphic, and hence proved as corollaries both Conjectures A and B for trees. In [2] the author took this one stage further and showed that peripheral vertex-equivalent trees are isomorphic. For directed graphs, Harary and Palmer [5, 6] have proved that vertex-equivalent weak tournaments of order greater than four are isomorphic, that edge-equivalent tournaments are isomorphic, and that pendant vertex-equivalent directed trees with at least three pendant vertices are isomorphic. In [7] Kelly also stated that Conjecture A holds for disconnected graphs. A proof is given by Harary in [4]. We here examine this conjecture for graphs of connectivity one.

2. Separable graphs with no pendant vertices. Throughout this section G, H will be taken to be vertex-equivalent graphs of order N.

A *cut-vertex* of a graph is a vertex whose removal disconnects the graph. A connected graph is *separable* if it contains a cut-vertex. A *block* is a maximal connected subgraph that is not separable. A *pendant vertex* is a vertex joined to just one other vertex.

THEOREM 1. *Vertex-equivalent separable graphs with no pendant vertices are isomorphic.*

We first need two lemmas.

LEMMA 1.1. (*Kelly* [7].) *Let Y be any graph of order less than N. Suppose there are α distinct subgraphs of G isomorphic to Y and that vertex u_i of G is in α_i of these subgraphs; that there are β distinct subgraphs of H isomorphic to Y and that vertex v_i (where v_i is the vertex corresponding to u_i) is in β_i of these subgraphs. Then*

$$\alpha = \beta, \text{ and } \alpha_i = \beta_i (1 \leqq i \leqq N) .$$

LEMMA 1.1 remains true if 'subgraph' is replaced by 'partial subgraph' throughout. We shall refer to this version as Lemma 1.1 (a).

Note. It is an easy consequence of Lemma 1.1 that corresponding vertices have the same number of loops and are of the same degree; it is also clear that G and H have the same number of pendant vertices.

LEMMA 1.2. *Suppose G has blocks $B_1, B_2, \cdots, B_m$* (where $m > 1$) *and H has blocks $C_1, C_2, \cdots, C_n$. Then $m = n$, and the blocks can be relabelled so that $B_i \cong C_i (1 \leqq i \leqq n)$.*

Proof. Let B_i have order b_i and C_i have order c_i. We may assume that $b_1 \geqq b_2 \geqq \cdots \geqq b_m$ and $c_1 \geqq c_2 \geqq \cdots \geqq c_n$. The proof will be by induction. Suppose we have already shown that $B_1 \cong C_1, B_2 \cong C_2, \cdots, B_k \cong C_k$. Since G and H have the same order, $k = m$ if and only if $k = n$, and in that case $G \cong H$. Otherwise $k < \min(m, n)$ and we may suppose that $b_{k+1} \geqq c_{k+1}$. In Lemma 1.1 take Y to be isomorphic to B_{k+1}. If Y occurs γ times in $\bigcup_1^k B_i$, then Y also occurs γ times in $\bigcup_1^k C_i$. Now Y occurs at least $\gamma + 1$ times in G, since $Y \cong B_{k+1}$.

Hence Y occurs at least once in $\bigcup_{k+1}^{n} C_i$. Therefore Y is isomorphic to a subgraph of C_j for some $j > k$. But

$$\text{order of } Y = \text{order of } B_{k+1} = b_{k+1} \geqq c_{k+1} \geqq c_j = \text{order of } C_j$$

and hence $Y \cong C_j$. Without loss of generality we may take $j = k + 1$. Hence, $B_{k+1} \cong C_{k+1}$. Induction is started by the same argument, with $k = 0$. Therefore, $m = n$ and $B_i \cong C_i$ for all i.

Proof of Theorem 1. Let B_1 be a 'pendant' block of G (that is, a block containing just one cut-vertex of G) such that no pendant block of G or H has order less than b_1, the order of B_1. (The assumption that B_1 is in G results in no loss of generality.) Let u be the cut-vertex joining B_1 to the rest of G. Write $G_1 = G - (B_1 - u)$ and denote by G_1^s the graph obtained from G_1 by adding s isolated vertices and joining each to u by one edge. Then G_1^1 is a proper partial subgraph of G and hence, by Lemma 1.1 (a), there is a partial subgraph H_1^1 of H isomorphic to G_1^1, i.e., $\psi(G_1^1) = H_1^1$, $\psi(u) = v$, say. (Note: u and v are not necessarily corresponding vertices.) Let H_1 be the graph obtained from H_1^1 by deleting its pendant vertex. Then it is clear that $H_1 = \psi(G_1)$ and, by Lemma 1.2, H_1 has one block fewer than H, that block, C_1 say, being isomorphic to B_1. Now H is obtained from H_1^1 by adding $b_1 - 2$ vertices and some edges. Since no pendant block of H has order less than that of B_1, and since H has no pendant vertices, it is easy to see that those edges can only be incident with v, p (the pendant vertex of H_1^1) and the $b_1 - 2$ new vertices. Thus v is a cut-vertex of H and it follows that the subgraph of H on v, p, and these $b_1 - 2$ new vertices is isomorphic to C_1. It now remains to show that there is an isomorphism of B_1 and C_1 mapping u onto v. Denote by B_1^1 the graph obtained from B_1 by adding an isolated vertex and joining it to u by one edge. Define C_1^1 analogously. It will suffice to prove that $B_1^1 \cong C_1^1$.

By Lemma 1.1, G and H have the same number of cut-vertices of the same degree, and hence u and v have the same degree, say $r + s + 2t$, where t is the common number of loops at u and v. Suppose that, apart from loops, r edges of G_1 and s edges of B_1 are joined to u. Then it is clear that r edges of H_1, and hence s edges of C_1, are joined to v. If B_1^1 occurs α times in G_1^s, it occurs $\alpha + r$ times in G. But then B_1^1 occurs α times in H_1^s (since $G_1^s \cong H_1^s$) and $\alpha + r$ times in H (by Lemma 1.1). This implies that $B_1^1 \cong C_1^1$ and hence $G \cong H$.

3. Graphs with pendant vertices. The *trunk* $T(G)$ of a graph G is that subgraph of G which remains after repeated removal of pendant vertices. A *limb* L of G is a nontrivial maximal connected subgraph of G having just one vertex in common with $T(G)$. This vertex is called the *root* of L. G is the union of its trunk and limbs. The situation is illustrated in Figure 1.

Note. It is clear that these definitions only have meaning for graphs containing cycles of length greater than two.

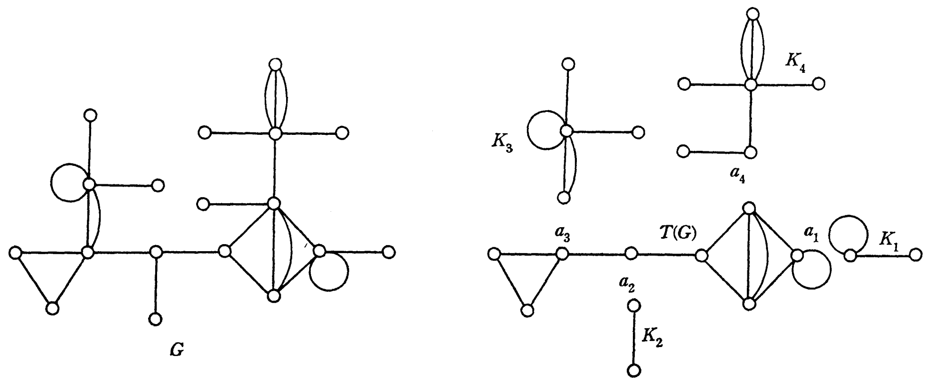

Figure 1.

Suppose G and H are vertex-equivalent graphs. We omit the case when G and H have two blocks, one of which is of order two. This seems to be very difficult to deal with, and is not amenable to the methods used here. If G has only one pendant vertex, on a limb of order greater than two, then H also has these properties and $G \cong H$. For let u be the pendant vertex of G, and suppose that there are l loops and m other edges incident with u. Then G is obtained from $G - u$ by adding an isolated vertex with l loops, and joining it by m multiple edges to the unique pendant vertex of $G - u$. If vertex v of H corresponds to u, then H can be reconstructed from $H - v$ in the same way. Now suppose that G and H have exactly two pendant vertices such that (i) each maximal subgraph of G obtained by the deletion of a pendant vertex has just one limb, of order two; (ii) there is a maximal subgraph of G containing two isolated vertices. Then, by an argument similar to that used above, we again have $G \cong H$. These cases will be excluded from the following discussion.

For the rest of this section we make the weaker assumption that G, H are pendant vertex-equivalent connected graphs. Since Harary and Palmer's theorem for trees extends quite easily to graphs with cycles of length at most two, we consider here only graphs having cycles of length greater than two. It is immediate that G and H have isomorphic trunks. We now look at their limbs. θ, ϕ, ϕ', respectively will denote isomorphisms of $G - u_1$ onto $H - v_1$, $G - u_2$ onto $H - v_2$ and $G - u_2'$ onto $H - v_2'$, where (u_1, v_1), (u_2, v_2), and (u_2', v_2') are pairs of corresponding pendant vertices, and $v_2' = \theta(u_2)$.

LEMMA 2.1. *Let G and H have limbs $K_1, K_2, \cdots, K_m$, and $L_1, L_2, \cdots, L_n$, respectively, where K_i has order k_i and L_i has order l_i. Assume that $k_1 \leqq k_2 \leqq \cdots \leqq k_m$, and that $l_1 \leqq l_2 \leqq \cdots \leqq l_n$. Then $m = n$ and $k_i = l_i (1 \leqq i \leqq n)$.*

Proof. Suppose that $k_i = l_i (1 \leqq i < r)$ but that $k_r \neq l_r$, say $k_r < l_r$. Let $u_1 \in K_r$. If $k_r > 2$, the combined order of the r smallest limbs in $G - u_1$ is $\sum_{i=1}^r k_i - 1$, and the combined order of the r smallest limbs in $H - v_1$ is at least $\sum_{i=1}^r l_i - 1 > \sum_{i=1}^r k_i - 1$. This contradicts the fact that $\theta(G - u_1) = H - v_1$. Therefore $k_r = 2$ and $l_r > 2$. Now there is a pendant vertex u_2 in some $K_i (1 \leqq i \leqq r)$ such that $v_2 \notin L_j$ for all $j < r$. Then $G - u_2$ has $m - 1$ limbs and $H - v_2$ has n limbs. Therefore $m = n + 1$ and this implies that $l_i > 2 (1 \leqq i \leqq n)$ and that $k_i = 2 (1 \leqq i \leqq n + 1)$. Since G and H clearly have the same order

$$n + 1 \sum_{1}^{n+1} (k_i - 1) = \sum_{1}^{n} (l_i - 1) \geqq 2n$$

and therefore $n = 1$ and $l_1 = 3$. But this is precisely the case excluded at the beginning of this section.

In the light of Lemma 2.1 we now assume that G and H have limbs $\{K_i\}_1^n$ and $\{L_i\}_1^n$ respectively, arranged so that $k_1 \leqq k_2 \leqq \cdots \leqq k_n$, where k_i is the (common) order of K_i and L_i; also that K_i has root a_i in $T(G)$ and that L_i has root b_i in $T(H)$. The notation $(U, u) \cong (V, v)$ (or $\alpha(U, u) = (V, v)$) will be used to denote that graphs U and V are isomorphic under an isomorphism (α) mapping vertex $u \in U$ onto vertex $v \in V$.

THEOREM 2. *The limbs of G and H can be arranged so that $(K_i, a_i) \cong (L_i, b_i) (1 \leqq i \leqq n)$.*

Proof. (a) $n \geqq 2$. Let $u_1 \in K_1, u_2 \in K_2$. Then $G - u_1$ has limbs of order $k_1 - 1, k_2, \cdots, k_n$, and hence so has $H - v_1$. We may therefore assume that $v_1 \in L_1$. Then $\theta(G - u_1) = H - v_1 \Rightarrow$

$$(1) \qquad \theta(K_i, a_i) = (L_i, b_i) \qquad (2 \leqq i \leqq n)\ .$$

There are now three subcases:

(i) $k_2 > k_1 + 1$. Then $\phi(G - u_2) = H - v_2 \Rightarrow \phi(K_1, a_1) = (L_1, b_1)$.

(ii) $k_2 = k_1 + 1$. From (1), $\theta(u_2) = v_2' \in L_2$ and hence $\theta(K_2 - u_2, a_2) = (L_2 - v_2', b_2)$. $\phi(G - u_2) = H - v_2 \Rightarrow$ either $\phi(K_1, a_1) = (L_1, b_1)$ or $\phi(K_2 - u_2, a_2) = (L_1, b_1)$. $\phi'(G - u_2') = H - v_2' \Rightarrow$ either $\phi'(K_1, a_1) = (L_1, b_1)$ or $\phi'(K_1, a_1) = (L_2 - v_2', b_2)$. Therefore at least one of $\phi, \phi', \phi\theta^{-1}\phi'$ (product of isomorphisms) maps (K_1, a_1) onto (L_1, b_1).

(iii) $k_2 = k_1$. Suppose $v_2 \in L_t$. Then $\phi(G - u_2) = H - v_2 \Rightarrow \{\phi(K_i, a_i)\}_{i\neq 2} = \{(L_i, b_i)\}_{i\neq t}$. Put $\phi_r = \phi(\theta^{-1}\phi)^r$. We shall show that, for some r, either

$$\phi_r(K_1, a_1) = (L_1, b_1) \text{ or } \phi_r(K_1, a_1) = (L_2, b_2) . \tag{2}$$

For assume otherwise. Then $\phi_r(K_1, a_1)$ is well-defined for all r and we may put $\phi_r(K_1, a_1) = (L_{j_r}, b_{j_r})$, where $j_r > 2$. Thus

$$\theta^{-1}\phi_r(K_1, a_1) = \theta^{-1}(L_{j_r}, b_{j_r}) = (K_{j_r}, a_{j_r}) . \tag{3}$$

Since n is finite there are integers $p, q, (p < q)$ such that $(L_{j_p}, b_{j_p}) = (L_{j_q}, b_{j_q})$. Then $\phi_p(K_1, a_1) = \phi_q(K_1, a_1)$, i.e., $\theta^{-1}\phi_{q-p-1}(K_1, a_1) = (K_1, a_1)$, contradicting (3). Hence (2) follows. From (1)$v_2' \in L_2$ and, by an analogous argument to the above, $\phi'(G - u_2') = H - v_2' \Rightarrow$ either $\phi_s'(K_1, a_1) = (L_1, b_1)$ or $\phi_s'(K_2, a_2) = (L_1, b_1)$ for some s, where $\phi_s' = \phi'(\theta^{-1}\phi')^s$. Therefore for some r, s, at least one of $\phi_r, \phi_s', \phi_s'\theta^{-1}\phi_r$ maps (K_1, a_1) onto (L_1, b_1). (i), (ii) and (iii) together imply that an isomorphism θ_1 exists such that

$$\theta_1(K_1, a_1) = (L_1, b_1) .$$

This completes the proof of part (a).

(b) $n = 1$. If a_1 is not pendant in K_1, apply the methods of Lemma 2.1 and part (a) of this proof to the sublimbs of K_1 and L_1 branching from a_1 and b_1 respectively. Otherwise let a_1' be the first vertex along K_1 from a_1 that is joined to more than two other vertices of K_1 (such a vertex exists since G has at least two pendant vertices). Similarly let b_1' be the first vertex along L_1 from b_1 that is joined to more than two other vertices of L_1. It is not difficult to see that the subgraph of K_1 lying between a_1 and a_1' is isomorphic to the subgraph of L_1 lying between b_1 and b_1'. Now apply the methods of Lemma 2.1 and part (a) of this proof to the sublimbs of K_1 and L_1 which branch outwards from a_1' and b_1' respectively.

COROLLARY 2.1. *If $k_1 > 2$, then $G \cong H$.*

Proof. Let $u_1 \in K_1$. We may assume that $v_1 \in L_1$. Then $\theta: G - u_1 \to H - v_1$ induces an isomorphism of $T(G)$ and $T(H)$ mapping a_i onto $b_i (1 \leqq i \leqq n)$, possibly after some relabelling. In addition it is clear that $\theta(K_i, a_i) = (L_i, b_i) (2 \leqq i \leqq n)$, and hence also, by Theorem 2, that $(K_1, a_1) \cong (L_1, b_1)$. Therefore $G \cong H$.

In the same way one can prove:

COROLLARY 2.2. *If, for some i, $k_{i+1} - k_i > 1$, then $G \cong H$.*

When there are limbs of order two, one has in general (that is, except for the situation in Corollary 2.2) to deal with a problem concerning the automorphism group of the trunk. Some progress in this direction has been made by Greenwell and Hemminger [3]. However, we have one result which by-passes this difficulty.

COROLLARY 2.3. *If the trunk of G is a complete subgraph then $G \cong H$.*

Proof. By Theorem 2 $(K_i, a_i) \cong (L_i, b_i)(1 \leqq i \leqq n)$. Since $T(G)$ and $T(H)$ are complete, this isomorphism can be extended to an isomorphism of G and H by mapping the vertices of $T(G)$ which are not in the set $\{a_i\}_1^n$ onto distinct vertices of $T(H)$ not in the set $\{b_i\}_1^n$.

We conclude by proposing a conjecture analogous to Conjectures A and B.

CONJECTURE C. Pendant vertex-equivalent graphs with at least k pendant vertices are isomorphic (k to be determined).

For this conjecture to be true we must obviously have $k > 2$. I am indebted to Dr. Peter M. Neumann for pointing out a counter-example when $k = 3$. Figure 2 pictures the two non-isomorphic pendant vertex equivalent graphs.

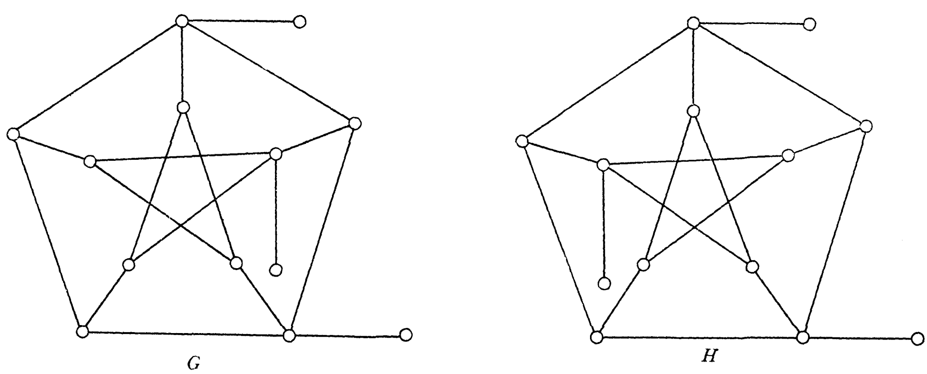

Figure 2.

Dr. Neumann informs me that he has also found a counter-example when $k = 4$.

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