PACIFIC JOURNAL OF MATHEMATICS
Vol. 31, No. 2, 1969

QUASI-INJECTIVE MODULES AND STABLE TORSION CLASSES

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In this note we examine the $\mathscr{T}$-torsion submodule of quasi-injective R-modules, R a ring with unit, where $\mathscr{T}$ is a torsion class in the sense of S. E. Dickson. We show that for a stable torsion class $\mathscr{T}$, the $\mathscr{T}$-torsion submodule of any quasi-injective module is a direct summand, while if $\mathscr{T}$ contains all Goldie-torsion modules, then every epimorphic image of a quasi-injective module has its $\mathscr{T}$-torsion submodule as a direct summand. In addition, we show that for a stable torsion class $\mathscr{T}$, all $\mathscr{T}$-torsion-free modules are injective if and only if $R = T(R) \oplus K$ (ring direct sum), with K Artinian semisimple.

All R-modules will be unitary left R-modules. Originally our results were obtained for torsion classes closed under submodules. However, the referee has kindly pointed out how this assumption can be omitted throughout, supplying a proof in the case of Theorem 2.3. We take this opportunity to express our gratitude.

1. Following S. E. Dickson [2], a class $\mathscr{T}$ $(\neq \varnothing)$ of R-modules is a *torsion class* if $\mathscr{T}$ is closed under factors, extensions, and arbitrary direct sums. The torsion class $\mathscr{T}$ is *stable* if $\mathscr{T}$ is closed under essential extensions. Every torsion class $\mathscr{T}$ determines in every R-module A a unique maximal $\mathscr{T}$-submodule $T(A)$, the *$\mathscr{T}$-torsion submodule* of A, and $T(A/T(A)) = 0$, i.e., $A/T(A)$ is *$\mathscr{T}$-torsion-free*. The R-module A *splits* if $T(A)$ is a direct summand of A. For further properties of torsion classes the reader is referred to [6].

The class $\mathscr{G}$ of Goldie-torsion modules is the smallest torsion class containing all factor modules A/B where B is essential in A, and their isomorphic copies. As shown in [1], $\mathscr{G} = \{A \mid Z_2(A) = A\}$ where $Z_1(A) =$ singular submodule of A and $Z_2(A)/Z_1(A) =$ singular submodule of $A/Z_1(A)$ (see also [4]).

An R-module A is *quasi-injective* provided every homomorphism from any submodule of A into A can be extended to an endomorphism of A. For any R-module A, $E(A)$ will denote the injective envelope of A.

2. The proof of the following is straightforward and so will be omitted.

PROPOSITION 2.1. *If the torsion class $\mathscr{T}$ is stable then every* *injective R-module splits. If $\mathscr{T}$ is closed under submodules, the converse holds and either condition is equivalent to $T(E(A)) = E(T(A))$ for all R-modules A.*

The next lemma can be found in [5, Proposition 2.3].

LEMMA 2.2. *If A is a quasi-injective R-module and $E(A) = M \oplus N$ then $A = (M \cap A) \oplus (N \cap A)$.*

We now have

THEOREM 2.3. *Let $\mathscr{T}$ be a stable torsion class. Then every quasi-injective R-module A splits, $A = T(A) \oplus N$ where N is quasi-injective and $\mathscr{T}$-torsion-free.*

Proof. Choose a submodule N of A maximal with respect to $T(A) \cap N = 0$. Then $E(A) = E(T(A)) \oplus E(N)$, hence by Lemma 2.2, $A = A \cap E(T(A)) \oplus A \cap E(N)$. Since $\mathscr{T}$ is stable $A \cap E(T(A)) = T(A)$ and hence $A = T(A) \oplus N$ with $N = A \cap E(N)$ quasi-injective and $\mathscr{T}$-torsion-free.

Since the class $\mathscr{G}$ of Goldie-torsion modules is stable, it follows that $G(A)$ is a direct summand of A whenever A is quasi-injective; this was obtained by M. Harada in [5, Th. 1.7].

Let $\mathscr{T}$ be a torsion class; a submodule B of an R-module A is *$\mathscr{T}$-closed* if $T(A/B) = 0$.

LEMMA 2.4. *Let $\mathscr{T}$ be a torsion class and let B be a $\mathscr{T}$-closed submodule of the R-module A. If M is any R-module and $f \in \mathrm{Hom}_R(M, A)$ then $N = f^{-1}(B)$ is $\mathscr{T}$-closed in M.*

Proof. If C/N is a $\mathscr{T}$-submodule of M/N then $f(C)/B$ is a $\mathscr{T}$-submodule of A/B. Hence $f(C) \subseteqq B$ since B is $\mathscr{T}$-closed and so N is $\mathscr{T}$-closed.

If $\mathscr{T}$ is a torsion class containing the class $\mathscr{G}$, then a $\mathscr{T}$-closed submodule B of the R-module A has no essential extension in A; hence if A is quasi-injective then B is a direct summand of A by [3, Corollary 3, p. 24]. Another way of showing this has been suggested by the referee: Choose K maximal in A with respect to $K \cap B = 0$. Then $E(A) = E(K) \oplus E(B)$ so $A = (A \cap E(K)) \oplus (A \cap E(B)) = K \oplus (A \cap E(B))$. Now $A/B \cong K \oplus (A \cap E(B))/B$ and since $\mathscr{G} \subseteqq \mathscr{T}$, $A \cap E(B) = B$.

THEOREM 2.5. *If $\mathscr{T}$ is a torsion class containing $\mathscr{G}$, and the R-module A is an epimorphic image of a quasi-injective R-module then A splits.*

Proof. Let M be quasi-injective, $f: M \to A$ an epimorphism. Then $T(A)$ is a $\mathscr{T}$-closed submodule of A, hence by Lemma 2.4, $N = f^{-1}(T(A))$ is $\mathscr{T}$-closed in M. By the previous remark, N is a direct summand of M, say $M = N \oplus P$. Then $f(P) \cap T(A) = 0$ and so $A = T(A) \oplus f(P)$.

We note that the previous theorem is a generalization of [7, Th. 1.1] and the method employed is that of [8, Th. 2.10].

3. In [1, Th. 3.1] it was shown that a ring $R = G(R) \oplus K$ (ring direct sum), where K is semisimple with minimum condition, if and only if all $\mathscr{G}$-torsion-free modules are injective. In this section we prove this result for any stable torsion class $\mathscr{T}$.

LEMMA 3.1. *Let $R = S \oplus K$, where S is semisimple with minimum condition. Then any R-module A satisfying $KA = 0$ is an injective R-module.*

Proof. If $KA = 0$ then A is an injective S-module. Let I be a left ideal of R; then $I = S_1 \oplus K_1$ where $S_1 \subseteqq S$, $K_1 \subseteqq K$ are left ideals of R. Also $1 = u + v, u \in S, v \in K$. If $f: I \to A$ is an R-homomorphism, then for any $b \in K_1$, $0 = vf(b) = f(b)$. There is an S-homomorphism $g: S \to A$ coinciding with f on I, and this yields an R-homomorphism $g^*: R \to A$ coinciding with f on I if we define $g^*(s + k) = g^*(s)$.

THEOREM 3.2. *Let $\mathscr{T}$ be a stable torsion class. Then all $\mathscr{T}$-torsion-free R-modules are injective if and only if $R = T(R) \oplus K$, where K is a semisimple ring with minimum condition.*

Proof. Assume A is injective whenever $T(A) = 0$. Since the class $\mathscr{F}$ of $\mathscr{T}$-torsion-free modules is closed under submodules [2], every submodule of any $A \in \mathscr{F}$ is injective, hence is a direct summand, and so every $A \in \mathscr{F}$ is completely reducible. Let M be any R-module and assume $T(M) \neq 0$. If $T(M)$ is essential in M then $T(M) = M$, since $\mathscr{T}$ is stable. Otherwise select B maximal relative to $B \cap T(M) = 0$. Then $T(B) = 0$ and so B is injective. Thus $M = B \oplus U$. Now $M/U \cong B \in \mathscr{F}$ so that $T(M) \subseteqq U$ by [2, Proposition 2.4]. The maximal property of B ensures that $T(M)$ is essential in U and so $U = T(M)$. In particular $R = T(R) \oplus K$, where K is a completely reducible R-module since $K \in \mathscr{F}$. The decomposition is two-sided since right multiplications are R-homomorphisms and both classes $\mathscr{T}$ and $\mathscr{F}$ are closed under factors.

Conversely, assume $R = T(R) \oplus K$, where K is a semi-simple ring with minimum condition. We note that for any R-module A, if $T(A) = 0$ then $T(R)A = 0$. For if $A \in \mathscr{F}$ and $0 \neq a \in A$ then $T(R)a$ is an epimorphic image of $T(R)$ and so $T(R)a \subseteqq T(A) = 0$. That every $\mathscr{T}$-torsion-free module is injective now follows from Lemma 3.1.

We conclude with the following example. Let R be the ring of lower triangular 2×2 matrices over a finite field and let $\mathscr{T}$ be the smallest torsion class containing all projective simple R-modules. Note that $\mathscr{T}$ contains nonzero R-modules since every simple in the socle of R is in $\mathscr{T}$. Moreover R is a hereditary Artinian ring with (rad $R)^2 = 0$ so by [9, Theorem B] every nonprojective simple is injective. Since R is not semisimple, it has nonzero $\mathscr{T}$-torsion-free R-modules. If $T(A) = 0$ for an R-module $A \neq 0$ then socle (A) contains no projective simples. Since R is Noetherian and socle (A) is essential in A, A is injective. It is readily verified that socle $(R) = T(R)$. Thus the condition that T be stable is needed in Theorem 3.2, even when T is closed under submodules.

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Received October 7, 1968.

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