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SEMI-SIMPLE RADICAL CLASSES

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The purpose of this paper is to characterize all semi-simple radical classes (those classes of rings which are semi-simple classes and at the same time radical classes).

Andrunakievic has shown that the class of Boolean rings is a semi-simple radical class. More recently, Armendariz has considered such classes.

For "I is an ideal of the ring R" we shall write "$I \lhd R$".

Following Divinsky [6], but substituting classes of rings for ring properties, we define:

(i) A nonempty class of rings $\mathscr{C}$ is a *radical class* if and only if $\mathscr{C}$ satisfies the following conditions:

(A) Homomorphic images of rings in $\mathscr{C}$ are in $\mathscr{C}$.

(B) Every ring R has an ideal $\mathscr{C}(R) \in \mathscr{C}$ such that if $I \lhd R$ and $I \in \mathscr{C}$ then $I \subseteqq \mathscr{C}(R)$.

(C) The only ideal of the factor ring $R/\mathscr{C}(R)$ which is in $\mathscr{C}$ is the zero ideal.

(ii) If $\mathscr{C}$ is a radical class, a ring R is $\mathscr{C}$ *semi-simple* if and only if $\mathscr{C}(R) = (0)$.

(iii) A nonempty class of rings $\mathscr{C}$ is a *semi-simple class* if and only if $\mathscr{C}$ satisfies the following conditions:

(E) Every nonzero ideal of a ring in $\mathscr{C}$ can be homomorphically mapped onto a nonzero ring in $\mathscr{C}$.

(F) If every nonzero ideal of a ring R can be homomorphically mapped onto a nonzero ring in $\mathscr{C}$ then $R \in \mathscr{C}$.

2. Rings without nilpotent elements.

Our purpose in this section is to establish:

THEOREM 2.1.[1] *A ring R without nilpotent elements is isomorphic to a subdirect sum of rings without proper divisors of zero.*

It will be convenient to first prove:

LEMMA 2.2. *If R has no nilpotent elements and $0 \neq x \in R$ then*
(i) $x_r = \{y \in R \colon xy = 0\} \lhd R$ *and* $x_r = x_l = \{y \in R \colon yx = 0\}$,
(ii) $x \notin x_l$,

[1] The author wishes to thank the referee for pointing out that this result has also been obtained by V. Andrunakievic and Ju. M. Rjabuhin, *Rings without nilpotent elements, and completely simple ideals*, Dokl. Akad. Nauk. SSR. **180**, 9-11 (Translation, Soviet Mathematics **9** (1968), 565-568).

(iii) *if* $r \in R$ *and* $rx \in x_l$ *then* $r \in x_l$,

(iv) *the factor ring* R/x_l *has no nilpotent elements.*

Proof. Let R be a ring with no nilpotent elements and $0 \neq x \in R$. If $a \in R$ and $ax = 0$ then $(xa)^2 = 0$ so $xa = 0$. Similarily if $xa = 0$ then $ax = 0$. This establishes (i). Since $x^2 \neq 0$, (ii) is clear. If a, $b \in R$ and $ab^2 = 0$ then $(bab)^2 = 0$ so $bab = 0$, but then $(ab)^2 = 0$ so $ab = 0$. From this (iii) and (iv) follow immediately.

To prove the theorem it is sufficient to find, for each $0 \neq x \in R$, an ideal $I(x)$ of R for which $R/I(x)$ has no proper divisors of zero and $x \notin I(x)$. Let $Z(x) = \{I \lhd R: x \notin I$, if $rx \in I$ then $r \in I$, and R/I has no nilpotent elements$\}$. By 2.2 $x_l \in Z(x)$ so $Z(x) \neq \emptyset$ and it is clear that the union of an ascending chain in $Z(x)$ is also in $Z(x)$. Thus we may choose, by Zorn's Lemma, $I(x)$ maximal in $Z(x)$.

If $a \in R$ and $a \notin I(x)$ let $J = \{y \in R: ay \in I(x)\} \supseteqq I(x)$. Then $J/I(x) = (a + I(x))_r$ in $R/I(x)$ and by 2.2 (i) $(a + I(x))_l = (a + I(x))_r \lhd R/I(x)$. Since $a \notin I(x)$, $ax \notin I(x)$ so $x \notin J$. If $rx \in J$ then $arx \in I(x)$ so $ar \in I(x)$, hence $r \in J$. Finally by 2.2 (iv)$R/J \cong R/I(x)/J/I(x)$ has no nilpotent elements, so $J \in Z(x)$. Hence $J = I(x)$ so $R/I(x)$ has no proper divisors of zero.

Note 2.3. The generalized nil radical Ng of Andrunakievic [4] and Thierrin [10] (see also [6]) is the upper radical with respect to the class of rings without proper divisors of zero. A ring R is Ng semi-simple if and only if R is isomorphic to a subdirect sum of rings without proper divisors of zero. In this context, 2.1 can be restated as: A ring R is Ng semi-simple if and only if R has no nilpotent elements.

3. $\mathscr{B}_1$-rings.

If $x \in R$, let $[x] =$ the subring of R generated by x.

DEFINITION 3.1. R is a $\mathscr{B}_1$-ring $.\equiv.$ for all $x \in R$, $[x] = [x]^2$.

Let R be a ring and $x \in R$. Clearly $[x] = [x]^2$ if and only if $x \in [x]^2$ if and only if there are integers $a_2, \cdots, a_k$ such that $x = \sum_{i=2}^{k} a_i x^i$. Using this it is clear that homomorphic images of $\mathscr{B}_1$-rings are $\mathscr{B}_1$-rings and that if A/B and B are $\mathscr{B}_1$-rings then A is a $\mathscr{B}_1$-ring. It then easily follows that the class of $\mathscr{B}_1$-rings (which we shall denote by $\mathscr{B}_1$) is a radical class.

LEMMA 3.2. *A nonzero $\mathscr{B}_1$-ring without proper divisors of zero is a field of prime characteristic which is algebraic over its prime subfield.*

Proof. Let R be a nonzero $\mathscr{B}_1$-ring without proper divisors of zero. If x is a nonzero element of R there are integers $a_2, \cdots, a_k$ such that $x = \sum_{i=2}^{k} a_i x^i$, hence $e_x = \sum_{i=2}^{k} a_i x^{i-1}$ is an identity for $[x]$. Since x is not a zero divisor e_x is an identity for R. If $w \in R$, $w \neq 0$, $e_w \in [w] = [w]^2$ so $e_w \in [w] \cdot w \subseteqq Rw$ thus $R = Rw$. Since R is nonzero, R is a division ring.

Let e be the identity of R. Then $[2e] = [2e]^2 = [4e]$ so $Ne = 0$ for some positive integer N. Consequently the characteristic of R is a prime and since $e = e_w \in [w]$ for all nonzero $w \in R$, R is algebraic over its prime subfield. Therefore, by Theorem 2, page 183 of Jacobson [7] R is a field.

COROLLARY 3.3. *If R is a $\mathscr{B}_1$-ring then R is isomorphic to a subdirect sum of algebraic fields of prime characteristic. So, in particular, R is commutative.*

Proof. If $x \in R$, $x^N = 0$ and $R \in \mathscr{B}_1$, then $[x] = [x]^2 = \cdots = [x]^N = (0)$ so $x = 0$. Hence $\mathscr{B}_1$-rings do not have nilpotent elements so the corollary follows from 2.1 and 3.2.

THEOREM 3.4. *A ring R is a $\mathscr{B}_1$-ring if and only if every finitely generated subring of R is isomorphic to a finite direct sum of finite fields.*

Proof. Let $R \in \mathscr{B}_1$ and R' be a finitely generated subring of R. Then $R' \in \mathscr{B}_1$ and hence is commutative, so by the Hilbert Basis Theorem R' has maximum condition on ideals. If $P' \neq R'$ and P' is a prime ideal of R' then P' is a maximal ideal of R' since by 3.2 R'/P' is a field. Since R' is finitely generated, commutative, and $[g]$ has an identity for each generator g of R', R' has an identity. Then by Theorem 2, page 203 of [11] R' has minimum condition on ideals. But then R' is a commutative Wedderburn ring so R' is isomorphic to a finite direct sum of fields each of which must be finite since they are finitely generated, algebraic and of prime characteristic.

The converse is obvious; in fact, if $x \in R'$ and R' is isomorphic to a finite direct sum of finite fields then there is an integer $n(x) \geqq 2$ such that $x^{n(x)} = x$. Thus we have:

COROLLARY 3.5. *R is a $\mathscr{B}_1$-ring if and only if for each $x \in R$ there exists an integer $n(x) \geqq 2$ such that $x^{n(x)} = x$.*

A class of rings $\mathscr{C}$ is said to be *hereditary* if $I \lhd R \in \mathscr{C}$ implies that $I \in \mathscr{C}$. Analogously we say:

DEFINITION 3.6. A class of rings $\mathscr{C}$ is *strongly hereditary* $. \equiv$ if S is a subring of $R \in \mathscr{C}$ then $S \in \mathscr{C}$.

PROPOSITION 3.7. *If $\mathscr{F}$ is a strongly hereditary finite set of finite fields then a ring R is isomorphic to a subdirect sum of fields in $\mathscr{F}$ if and only if every finitely generated subring of R is isomorphic to a finite direct sum of fields in $\mathscr{F}$.*

Proof. Since $\mathscr{F}$ is a finite set of finite fields there exists an integer $N \geqq 2$ such that $x^N = x$ for all $x \in F \in \mathscr{F}$.

Let R have ideals $I_\alpha: \alpha \in A$ such that $R/I_\alpha \cong F_\alpha \in \mathscr{F}$ and $\cap \{I_\alpha: \alpha \in A\} = (0)$. Let R' be a finitely generated subring of R. Then $R' \in \mathscr{B}_1$ since $x^N = x$ for all $x \in R \supseteqq R'$, so by 3.4 $R' \cong A_1 \oplus \cdots \oplus A_k$ and the A_i are finite fields. Choose $a_i \in R'$ such that $[a_i] \cong A_i$. Then $a_i \neq 0$ so $a_i \notin I_{\beta_i}$ for some $\beta_i \in A$ but $I_{\beta_i} \cap [a_i] \lhd [a_i]$ so $I_{\beta_i} \cap [a_i] = (0)$. Therefore $A_i \cong [a_i] \cong [a_i] + I_{\beta_i}/I_{\beta_i}$ is isomorphic to a subring of F_{β_i}. Since $\mathscr{F}$ is strongly hereditary R' is isomorphic to a finite direct sum of fields in $\mathscr{F}$.

Conversely, if every finitely generated subring of R is isomorphic to a finite direct sum of fields in $\mathscr{F}$, R must be a $\mathscr{B}_1$-ring since again $x^N = x$ for all $x \in R$. Thus by 3.3 there are ideals $I_\alpha: \alpha \in A$ of R such that $\cap \{I_\alpha: \alpha \in A\} = (0)$ and R/I_α is a field of prime characteristic; moreover, R/I_α must be a finite field since $x^N - x = 0 \in I_\alpha$ for all $x \in R$. Therefore, for each $\alpha \in A$, there exists $x_\alpha \in R$ such that $[x_\alpha] + I_\alpha/I_\alpha = R/I_\alpha$. But then R/I_α is a homomorphic image of $[x_\alpha]$ so R/I_α is isomorphic to a field in $\mathscr{F}$.

4. Semi-simple radical classes.

LEMMA 4.1. *If $\mathscr{C}$ is a class of rings such that subdirect sums of rings in $\mathscr{C}$ are in $\mathscr{C}$ and $\mathscr{C}$ satisfies (A) then $\mathscr{C}$ is strongly hereditary.*

Proof. Let $R \in \mathscr{C}$ and S be a subring of R.

Set $R_i = R$ for all $i \in Z^+ =$ the set of positive integers. Now the (discrete) direct sum $\sum \{R_i: i \in Z^+\}$ is an ideal of the direct product (complete direct sum) $\prod \{R_i: i \in Z^+\}$. If $s \in S$ let $\hat{s}(i) = s$ for all $i \in Z^+$. Then $S \rightarrow \varDelta(S) = \{\hat{s}: s \in S\}$ is an embedding of S into $\prod \{R_i: i \in Z^+\}$. $\varDelta(S) + \sum \{R_i: i \in Z^+\}$ is a subdirect sum of copies of R and hence is in $\mathscr{C}$, so

$$S \cong \varDelta(S) \cong \frac{\varDelta(S) + \sum \{R_i: i \in Z^+\}}{\sum \{R_i: i \in Z^+\}} \in \mathscr{C} .$$

Using a theorem of Amitsur [1] which states that every ring is a homomorphic image of a subdirect sum of total matrix rings of finite order over the ring of all integers, Armendariz in [5] proves that if a hypernilpotent radical class $\mathscr{C}$ is a semi-simple class, then $\mathscr{C}$ contains all rings. A hypernilpotent radical class is a hereditary radical class which contains all nilpotent rings.

THEOREM 4.2. *If $\mathscr{C}$ is a semi-simple radical class and $\mathscr{C} \nsubseteq \mathscr{B}_1$ then $\mathscr{C}$ consists of all rings.*

Proof. Let $\mathscr{C}$ be a semi-simple radical class. If $\mathscr{C} \nsubseteq \mathscr{B}_1$ then there is a $R \in \mathscr{C}$ and $x \in R$ such that $[x] \neq [x]^2$. In [8] Kurosh shows that for any semi-simple class $\mathscr{S}$, subdirect sums of rings in $\mathscr{S}$ are in $\mathscr{S}$. Thus, by 4.1, $[x] \in \mathscr{C}$ and since $[x]^2 \lhd [x]$, $[x]/[x]^2 \in \mathscr{C}$. Now $[x]/[x]^2$ is a zero ring on a cyclic group and since $\mathscr{C}$ satisfies (F), $C^\infty =$ the zero ring on the infinite cyclic group is in $\mathscr{C}$. This implies (see [3] and [6]) that $\mathscr{C}$ contains all nilpotent rings. Since $\mathscr{C}$ is a semi-simple class (see [2] and [6]) $\mathscr{C}$ is hereditary, hence $\mathscr{C}$ is hypernilpotent. Therefore, by [5], $\mathscr{C}$ is the class of all rings.

THEOREM 4.3. *If $\mathscr{C}$ is not the class of all rings then the following are equivalent:*

(1) *$\mathscr{C}$ is a semi-simple radical class,*

(2) *there is a strongly hereditary finite set $\mathscr{C}(F)$ of finite fields such that: $R \in \mathscr{C}$ if and only if R is isomorphic to a subdirect sum of fields in $\mathscr{C}(F)$,*

(3) *there is a strongly hereditary finite set $\mathscr{C}(F)$ of finite fields such that: $R \in \mathscr{C}$ if and only if every finitely generated subring of R is isomorphic to a finite direct sum of fields in $\mathscr{C}(F)$.*

Proof. By 3.7 we have that (2) and (3) are equivalent.

Assume that $\mathscr{C}$ satisfies condition (3). Clearly $\mathscr{C}$ satisfies (A) and (E).

If $B \lhd A$ and both A/B and B are in $\mathscr{C}$ and A' is a finitely generated subring of A then $A' + B/B \cong A'/A' \cap B$ is isomorphic to a finite direct sum of fields in $\mathscr{C}(F)$. A slight modification of the proof given for Proposition 1 on page 241 of Jacobson [7] shows that $A' \cap B$ is finitely generated as a ring. Thus $A' \cap B$ is also isomorphic to a finite direct sum of fields in $\mathscr{C}(F)$ and so $A' \cong A'/A' \cap B \oplus A' \cap B$. Therefore $A \in \mathscr{C}$. From this it is easy to show that if $\mathscr{C}(R) =$ the sum of all ideals of R which are in $\mathscr{C}$ then $\mathscr{C}(R) \in \mathscr{C}$ and $\mathscr{C}(R/\mathscr{C}(R)) = (0)$. Thus, $\mathscr{C}$ satisfies (B) and (C).

If every nonzero ideal of a ring R can be homomorphically mapped onto a nonzero ring in $\mathscr{C}$ then by 3.7, every nonzero ideal of R can be homomorphically mapped onto a ring in $\mathscr{C}(F)$. Sulinski [9] (see also [6], Theorem 46) shows that this implies that R is isomorphic to a subdirect sum of rings in $\mathscr{C}(F)$ and hence by 3.7 again, $R \in \mathscr{C}$. So

$\mathscr{C}$ satisfies (F) and hence $\mathscr{C}$ is a semi-simple radical class.

Conversely, suppose $\mathscr{C}$ satisfies condition (1). Let $\mathscr{C}(F) =$ the class of all fields which are in $\mathscr{C}$ and define $A = \prod \{R: R \in \mathscr{C}(F)\}$. Since $\mathscr{C}$ is a semi-simple class subdirect sums of rings in $\mathscr{C}$ are in $\mathscr{C}$; thus $A \in \mathscr{C}$. By hypothesis, $\mathscr{C} \subseteqq \mathscr{B}_1$ so by 3.4 all elements of A must be torsion. From this it follows that there is a finite number of primes $p_1, \cdots, p_N$ such that every field in $\mathscr{C}(F)$ is of characteristic p_i for some $1 \leqq i \leqq N$. For each finite field $R \in \mathscr{C}(F)$ choose $a(R)$ such that $[a(R)] = R$ and for each infinite field $R \in \mathscr{C}(F)$ set $a(R) = 0$. Then $a = \{a(R)\}_{R \in \mathscr{C}(R)}$ is in A and by 3.5 $a^K = a$ for some integer $K \geqq 2$. Thus, for all finite fields R in $\mathscr{C}(F)$, the dimension of R over its prime subfield is $\leqq K - 1$. Hence there is only a finite number of finite fields in $\mathscr{C}(F)$. Suppose there is an infinite field $R \in \mathscr{C}(F)$. By 3.2 R is of prime characteristic and is algebraic over its prime subfield so R has an infinite number of non-isomorphic finite subfields. All these subfields are in $\mathscr{C}(F)$ since $\mathscr{C}$ is strongly hereditary by 4.1. This is impossible since there is only a finite number of finite fields in $\mathscr{C}(F)$. Therefore $\mathscr{C}(F)$ is a strongly hereditary finite set of finite fields. If $R \in \mathscr{C}$ then $R \in \mathscr{B}_1$ so by 3.3 R is isomorphic to a subdirect sum of fields all of which are in $\mathscr{C}(F)$ since $\mathscr{C}$ satisfies (A). Conversely, any ring isomorphic to a subdirect sum of rings in $\mathscr{C}(F)$ is in $\mathscr{C}$ since $\mathscr{C}$ is semi-simple class. Thus $\mathscr{C}$ satisfies (2).

REFERENCES

1. S. A. Amitsur, *The identities of* P. I.-*rings*, Proc. Amer. Math. Soc. **4** (1953), 27-34.

2. T. A. Anderson, N. Divinsky, and A. Sulinski, *Hereditary radicals in associative and alternative rings*, Canad. J. Math. **17** (1965), 594-603.

3. ———, *Lower radical properties for associative and alternative rings*, J. London Math. Soc. **41** (1966), 417-24.

4. V. Andrunakievic, *Radicals in associative rings* II, Mat. Sb. **55** (1961), 329-46.

5. E. P. Armendariz, *Closure properties in radical theory*, Pacific J. Math. **26** (1968), 1-8.

6. N. J. Divinsky, *Rings and radicals*, Univ. of Toronto Press, Toronto, 1965.

7. N. Jacobson, *Structure of rings*, Amer. Math. Soc. Coll. Publ. **37** (1964).

8. A. G. Kurosh, *Radicals of rings and algebras*, Mat. Sb. **33** (1953), 13-26.

9. A. Sulinski, *Certain questions in the general theory of radicals*, Mat. Sb. **44** (1958), 273-86.

10. G. Thierrin, *Sur les ideaux complement premiers d'un annaux quelconque*, Bull. Acad. Roy. Belg. **43** (1957), 124-32.

11. O. Zariski and P. Samuel, *Commutative algebra*, Vol. I, Van Nostrand ,Princeton N. J., 1958.

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