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COEFFICIENT MULTIPLIERS OF H^p AND B^p SPACES

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This paper describes the coefficient multipliers of $H^p (0 < p < 1)$ into $\ell^q (p \leqq q \leqq \infty)$ and into $H^q (1 \leqq q \leqq \infty)$. These multipliers are found to coincide with those of the larger space B^p into $\ell^q (1 \leqq q \leqq \infty)$ and into $H^q (1 \leqq q \leqq \infty)$. The multipliers of H^p and B^p into $B^q (0 < p < 1, 0 < q < 1)$ are also characterized.

A function f analytic in the unit disk is said to be of class $H^p (0 < p < \infty)$ if

$$M_p(r, f) = \left\{\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta\right\}^{1/p}$$

remains bounded as $r \to 1$. H^∞ is the space of all bounded analytic functions. It was recently found ([2], [4]) that if $p < 1$, various properties of H^p extend to the larger space B^p consisting of all analytic functions f such that

$$\int_0^1 (1 - r)^{1/p-2} M_1(r, f) dr < \infty .$$

Hardy and Littlewood [8] showed that $H^p \subset B^p$.

A complex sequence $\{\lambda_n\}$ is called a *multiplier* of a sequence space A into a sequence space B if $\{\lambda_n a_n\} \in B$ whenever $\{a_n\} \in A$. A space of analytic functions can be regarded as a sequence space by identifying each function with its sequence of Taylor coefficients. In [4] we identified the multipliers of H^p and $B^p (0 < p < 1)$ into ℓ^1. We have also shown ([2], Th. 5) that the sequence $\{n^{1/q-1/p}\}$ multiplies B^p into B^q. We now extend these results by describing the multipliers of $H^p (0 < p < 1)$ into $\ell^q (p \leqq q \leqq \infty)$, of B^p into $\ell^q (1 \leqq q \leqq \infty)$, and of both H^p and B^p into $B^q (0 < q < 1)$. We also extend a theorem of Hardy and Littlewood (whose proof was never published) by characterizing the multipliers of H^p and B^p into $H^q (0 < p < 1 \leqq q \leqq \infty)$. In almost every case considered, the multipliers of B^p into a given space are the same as those of H^p.

2. Multipliers into ℓ^q.

We begin by describing the multipliers of H^p and B^p into ℓ^∞, the space of bounded complex sequences.

THEOREM 1. *For $0 < p \leqq 1$, a sequence $\{\lambda_n\}$ is a multiplier of H^p into ℓ^∞ if and only if*

$$\lambda_n = O(n^{1-1/p}) . \tag{1}$$

For $p < 1$, *the condition* (1) *also characterizes the multipliers of* B^p *into* ℓ^∞.

Proof. If $f(z) = \sum a_n z^n$ is in B^p, then by Theorem 4 of [2],

$$a_n = o(n^{1/p-1}) . \tag{2}$$

If $f \in H^1$, then $a_n \to 0$ by the Riemann-Lebesgue lemma. This proves the sufficiency of (1). Conversely, suppose $\{\lambda_n\}$ is a multiplier of H^p into ℓ^∞. Then the closed linear operator

$$\Lambda : f \longrightarrow \{\lambda_n a_n\}$$

maps H^p into ℓ^∞. Thus Λ is bounded, by the closed graph theorem (which applies since H^p is a complete metric space with translation invariant metric; see [1], Chapter 2). In other words,

$$\sup_n |\lambda_n a_n| = ||\Lambda(f)|| \leqq K ||f|| . \tag{3}$$

Now let

$$g(z) = (1 - z)^{-1-1/p} = \sum b_n z^n ,$$

where $b^n \sim Bn^{1/p}$; and choose $f(z) = g(rz)$ for fixed $r < 1$. Then by (3)

$$|\lambda_n| n^{1/p} r^n \leqq C(1 - r)^{-1} .$$

The choice $r = 1 - 1/n$ now gives (1). Note that $\{\lambda_n\}$ multiplies H^p or B^p into ℓ^∞ if and only if it multiplies into c_0 (the sequences tending to zero).

As a corollary we may show that the estimate (2) is best possible in a rather strong sense. For functions of class H^p, this estimate is due to Hardy and Littlewood [8]. Evgrafov [6] later showed that if $\{\delta_n\}$ tends monotonically to zero, then there is an $f \in H^p$ for which $a_n \neq O(\delta_n n^{1/p-1})$. A simpler proof was given in [5]. The result may be reformulated: if $a_n = O(d_n)$ for all $f \in H^p$, then $d_n n^{1-1/p}$ cannot tend monotonically to zero. We can now sharpen this statement as follows.

COROLLARY. *If* $\{d_n\}$ *is any sequence of positive numbers such that* $a_n = O(d_n)$ *for every function* $\sum a_n z^n$ *in* H^p, *then there is an* $\varepsilon > 0$ *such that*

$$d_n n^{1-1/p} \geqq \varepsilon > 0 , \qquad n = 1, 2, \cdots .$$

Proof. If $a_n = O(d_n)$ for every $f \in H^p$, then $\{1/d_n\}$ multiplies H^p into ℓ^∞. Thus $1/d_n = O(n^{1-1/p})$, as claimed.

We now turn to the multipliers of H^p and B^p into $\ell^q (q < \infty)$, the space of sequences $\{c_n\}$ with $\sum |c_n|^q < \infty$. The following theorem generalizes a previously known result [4] for ℓ^1.

THEOREM 2. *Suppose* $0 < p < 1$.

(i) *A complex sequence* $\{\lambda_n\}$ *is a multiplier of* H^p *into* $\ell^q (p \leqq q < \infty)$ *if and only if*

$$\sum_{n=1}^{N} n^{q/p} |\lambda_n|^q = O(N^q) . \tag{4}$$

(ii) *If* $1 \leqq q < \infty$, $\{\lambda_n\}$ *is a multiplier of* B^p *into* ℓ^q *if and only if* (4) *holds.*

(iii) *If* $q < p$, *the condition* (4) *does not imply that* $\{\lambda_n\}$ *multiplies* H^p *into* ℓ^q; *nor does it imply that* $\{\lambda_n\}$ *multiplies* B^p *into* ℓ^q *if* $q < 1$.

Proof. (i) A summation by parts (see [4]) shows that (4) is equivalent to the condition

$$\sum_{n=N}^{\infty} |\lambda_n|^q = O(N^{q(1-1/p)}) . \tag{5}$$

Assume without loss of generality that $\lambda_n \geqq 0$ and $\sum_{n=1}^{\infty} \lambda_n^q = 1$. Let $s_1 = 0$ and

$$s_n = 1 - \left\{\sum_{k=n}^{\infty} \lambda_k^q\right\}^{1/\beta}, \qquad n = 2, 3, \cdots ,$$

where $\beta = q(1/p - 1)$. Note that s_n increases to 1 as $n \to \infty$. By a theorem of Hardy and Littlewood ([8], p. 412), $f \in H^p (0 < p < 1)$ implies

$$\int_0^1 (1 - r)^{\beta-1} M_1^q(r, f)dr < \infty , \qquad p \leqq q < \infty . \tag{6}$$

Thus if $f(z) = \sum a_n z^n$ is in H^p and $\{\lambda_n\}$ satisfies (4) with $p \leqq q < \infty$, it follows that

$$\begin{aligned}
\infty &> \sum_{n=1}^{\infty} \int_{s_n}^{s_{n+1}} (1 - r)^{\beta-1} M_1^q(r, f)dr \\
&\geqq \sum_{n=1}^{\infty} |a_n|^q \int_{s_n}^{s_{n+1}} (1 - r)^{\beta-1} r^{nq} dr \\
&\geqq \sum_{n=1}^{\infty} |a_n|^q (s_n)^{nq} \int_{s_n}^{s_{n+1}} (1 - r)^{\beta-1} dr \\
&= \frac{1}{\beta} \sum_{n=1}^{\infty} |a_n|^q (s_n)^{nq} \{(1 - s_n)^\beta - (1 - s_{n+1})^\beta\} \\
&= \frac{1}{\beta} \sum_{n=1}^{\infty} |a_n|^q (s_n)^{nq} \lambda_n^q ,
\end{aligned}$$

by the definition of s_n. But by (5),

$$\left\{\sum_{k=n}^{\infty} \lambda_k^q\right\}^{1/\beta} \leqq \frac{C}{n},$$

which shows, by the definition of s_n, that

$$(s_n)^{nq} \geqq (1 - C/n)^{nq} \longrightarrow e^{-Cq} > 0 .$$

Since these factors $(s_n)^{nq}$ are eventually bounded away from zero, the preceding estimates show that $\sum |a_n|^q \lambda_n^q < \infty$. In other words, $\{\lambda_n\}$ is a multiplier of H^p into ℓ^q if it satisfies the condition (4).

(ii) The above proof shows that $\{\lambda_n\}$ multiplies B^p into ℓ^1 under the condition (4) with $q = 1$. (This was also shown in [4].) The more general statement (ii) now follows by showing that if $\{\lambda_n\}$ satisfies (4), then the sequence $\{\mu_n\}$ defined by

$$\mu_n = |\lambda_n|^q \, n^{(1/p-1)(q-1)}$$

satisfies (4) with $q = 1$. Hence $\{\mu_n\}$ is a multiplier of B^p into ℓ^1, and in view of (2), $\{\lambda_n\}$ is a multiplier of B^p into ℓ^q. Alternatively, it can be observed that $f \in B^p$ implies (6) for $1 \leqq q < \infty$, so that the foregoing proof applies directly. Indeed, if $f \in B^p$, then (as shown in [2], proof of Theorem 3)

$$M_1(r, f) = O((1 - r)^{1-1/p}) ;$$

hence, if $1 \leqq q < \infty$,

$$\int_0^1 (1 - r)^{q(1/p-1)-1} M_1^q(r, f) dr \leqq C \int_0^1 (1 - r)^{1/p-2} M_1(r, f) dr < \infty .$$

(iii) That (4) does not imply $\{\lambda_n\}$ multiplies H^p into $\ell^q (q < p)$ or B^p into $\ell^q (q < 1)$, follows from the fact [4] that the series

$$\sum_{n=1}^{\infty} n^{q(1-1/p)-1} |a_n|^q$$

may diverge if $f \in H^p$ and $q < p$, or if $f \in B^p$ and $q < 1$.

To show the necessity of (4), we again appeal to the closed graph theorem. If $\{\lambda_n\}$ multiplies H^p into $\ell^q (0 < p < \infty, 0 < q < \infty)$, then

$$\Lambda : f \longrightarrow \{\lambda_n a_n\}$$

is a bounded operator:

$$\left\{\sum_{n=0}^{\infty} |\lambda_n a_n|^q\right\}^{1/q} \leqq C \, || f || , \qquad f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^p .$$

Choosing $f(z) = g(rz)$ as in the proof of Theorem 1, we now find

$$\left\{\sum_{n=1}^{\infty} n^{q/p} |\lambda_n|^q r^{nq}\right\}^{1/q} \leqq C(1-r)^{-1} ;$$

and (4) follows after terminating this series at $n = N$ and setting $r = 1 - 1/N$. Note that the argument shows (4) is necessary even if $p \geqq 1$ or $q < p$.

COROLLARY 1. *If $\{n_k\}$ is a lacunary sequence of positive integers $(n_{k+1}/n_k \geqq Q > 1)$, and if $f(z) = \sum a_n z^n$ is in $H^p(0 < p < 1)$, then*

$$\sum_{k=1}^{\infty} n_k^{q(1-1/p)} |a_{n_k}|^q < \infty , \qquad p \leqq q < \infty .$$

COROLLARY 2. *If $f(z) = \sum a_n z^n$ is in $H^p(0 < p < 1)$, then $\sum n^{p-2} |a_n|^p < \infty$.*

The first corollary extends a theorem of Paley [13] that $f \in H^1$ implies $\{a_{n_k}\} \in \ell^2$. The second is a theorem of Hardy and Littlewood [7]. It is interesting to ask whether the converse to Corollary 1 (with $q = p$) is valid. That is, if $\{c_k\}$ is a given sequence for which

$$\sum_{k=1}^{\infty} n_k^{p-1} |c_k|^p < \infty ,$$

then is there a function $f(z) = \sum a_n z^n$ in H^p with $a_{n_k} = c_k$? We do not know the answer.

Hardy and Littlewood [9] also proved that $\{\lambda_n\}$ multiplies H^1 into H^2 (alias ℓ^2) if (and only if)

$$\sum_{n=1}^{N} n^2 |\lambda_n|^2 = O(N^2) .$$

From this it is easy to conclude that (4) characterizes the multipliers of H^1 into ℓ^q, $2 \leqq q < \infty$. Indeed, let $\{\lambda_n\}$ satisfy (4) and let $\mu_n = |\lambda_n|^{q/2}$. Then, by the Hardy-Littlewood theorem, $\{\mu_n\}$ multiplies H^1 into ℓ^2 (see [3], p. 253). Hence $\{\lambda_n\}$ multiplies H^1 into ℓ^q. (See also Hedlund [12].)

On the other hand, the condition (4) is *not* sufficient if $p = 1$ and $q < 2$. This may be seen by choosing a lacunary series

$$f(z) = \sum_{k=1}^{\infty} c_k z^{n_k} , \qquad n_{k+1}/n_k \geqq Q > 1 ,$$

with $\sum |c_k|^2 < \infty$ but $\sum |c_k|^q = \infty$ for all $q < 2$. The sequence $\{\lambda_n\}$ with $\lambda_n = 1$ if $n = n_k$ and $\lambda_n = 0$ otherwise then satisfies (4) but does not multiply H^1 into ℓ^q, $q < 2$.

3. Multipliers into B^q. The following theorem may be regarded as a generalization of our previous result ([2], Th. 5) that if $f \in B^p$, then its fractional integral of order $(1/p - 1/q)$ is in B^q. (A fractional integral of negative order is understood to be a fractional derivative.)

THEOREM 3. *Suppose $0 < p < 1$ and $0 < q < 1$. Let ν be the positive integer such that $(\nu + 1)^{-1} \leqq p < \nu^{-1}$. Then $\{\lambda_n\}$ is a multiplier of H^p or B^p into B^q if and only if $g(z) = \sum_{n=0}^{\infty} \lambda_n z^n$ has the property*

$$M_1(r, g^{(\nu)}) = O((1 - r)^{1/p - 1/q - \nu}) . \tag{7}$$

Proof. Let $\{\lambda_n\}$ satisfy (7), let $f(z) = \sum a_n z^n$ be in B^p, and let $h(z) = \sum \lambda_n a_n z^n$. Then

$$h(\rho z) = \frac{1}{2\pi}\int_0^{2\pi} f(\rho e^{it}) g(ze^{-it}) dt , \qquad 0 < \rho < 1 .$$

Differentiation with respect to z gives

$$\rho^\nu h^{(\nu)}(\rho z) = \frac{1}{2\pi}\int_0^{2\pi} f(\rho e^{it}) g^{(\nu)}(ze^{-it}) e^{-i\nu t} dt . \tag{8}$$

Hence

$$\begin{aligned} \rho^\nu M_1(r\rho, h^{(\nu)}) &\leqq M_1(r, g^{(\nu)}) M_1(\rho, f) \\ &\leqq C(1 - r)^{1/p - 1/q - \nu} M_1(\rho, f) , \end{aligned}$$

where $r = |z|$. Taking $r = \rho$, we now see that $f \in B^p$ implies $h^{(\nu)} \in B^s$, $1/s = 1/q + \nu$. Thus $h \in B^q$, by Theorem 5 of [2].

Conversely, let $\{\lambda_n\}$ multiply H^p into B^q. Then by the closed graph theorem,

$$\Lambda : \sum a_n z^n \longrightarrow \sum \lambda_n a_n z^n$$

is a bounded operator from H^p to B^q. If $(\nu + 1)^{-1} \leqq p < \nu^{-1}$, let

$$f(z) = \nu!\, z^\nu (1 - z)^{-\nu - 1} = \sum_{n=\nu}^{\infty} a_n z^n ,$$

where $a_n = n!/(n - \nu)!$, and observe that

$$h(z) = \sum_{n=\nu}^{\infty} \lambda_n a_n z^n = z^\nu g^{(\nu)}(z) . \tag{9}$$

Let $f_r(z) = f(rz)$ and $h_r(z) = h(rz)$. Since Λ is bounded, there is a constant C independent of r such that

$$\| h_r \|_{B^q} = \| \Lambda(f_r) \| \leqq C \| f_r \|_{H^p} .$$

In other words,

$$\int_0^1 (1-t)^{1/q-2} M_1(tr, h)dt \leqq CM_p(r, f) \\ = O((1-r)^{1/p-\nu-1}) .$$

It follows that

$$M_1(r^2, h)\int_r^1 (1-t)^{1/q-2}dt = O((1-r)^{1/p-\nu-1}) ,$$

or

$$M_1(r^2, h) = O((1-r)^{1/p-1/q-\nu}) .$$

But in view of (9), this proves (7).

COROLLARY. *The sequence* $\{\lambda_n\}$ *multiplies* B^p *into* B^p *if and only if*

$$M_1(r, g') = O\left(\frac{1}{1-r}\right) . \tag{10}$$

Proof. If $p = q$, the condition (10) is equivalent to (7). (see [8], p. 435.) This corollary is essentially the same as a result of Zygmund ([14], Th. 1), who found the multipliers of the Lipschitz space Λ_α or λ_α into itself. Because of the duality between these spaces and B^p (see [2], §§3, 4), the multipliers from Λ_α to Λ_α and from λ_α to λ_α $(0 < \alpha < 1)$ are the same as those from B^p to B^p. Similar remarks apply to the spaces Λ_* and λ_*, also considered in [14].

4. **Multipliers into H^q.** By combining Theorem 3 with the simple fact that $f' \in B^{1/2}$ implies $f \in H^1$, it is possible to obtain a sufficient condition for $\{\lambda_n\}$ to multiply H^p into H^q, $0 < p < 1 \leqq q \leqq \infty$. However, this method leads to a sharp result only in the case $q = 1$. The following theorem provides the complete answer.

THEOREM 4. *Suppose* $0 < p < 1 \leqq q \leqq \infty$, *and let* $(\nu+1)^{-1} \leqq p < \nu^{-1}$, $\nu = 1, 2, \cdots$. *Then* $\{\lambda_n\}$ *is a multiplier of* H^p *or* B^p *into* H^q *if and only if* $g(z) = \sum_{n=0}^{\infty} \lambda_n z^n$ *has the property*

$$M_q(r, g^{(\nu+1)}) = O((1-r)^{1/p-\nu-2}) . \tag{11}$$

Hardy and Littlewood ([9], [10]) stated in different terminology that (11) implies $\{\lambda_n\}$ is a multiplier of H^p into $H^q (0 < p < 1 \leqq q < \infty)$, but they never published the proof. Our proof will make use of the following lemma.

LEMMA. *Let* f *be analytic in the unit disk, and suppose*

$$\int_0^1 (1-r)^\alpha M_q(r, f')dr < \infty ,$$

where $\alpha > 0$ *and* $1 \leqq q \leqq \infty$. *Then*

$$\int_0^1 (1-r)^{\alpha-1} M_q(r, f)dr < \infty .$$

Proof of Lemma. Without loss of generality, assume $f(0) = 0$, so that

$$f(re^{i\theta}) = \int_0^r f'(se^{i\theta})e^{i\theta}ds .$$

The continuous form of Minkowski's inequality now gives

$$M_q(r, f) \leqq \int_0^r M_q(s, f')ds . \tag{12}$$

Hence an interchange of the order of integration shows that

$$\int_0^1 (1-r)^{\alpha-1} M_q(r, f)dr \leqq \frac{1}{\alpha}\int_0^1 (1-s)^\alpha M_q(s, f')ds ,$$

which proves the lemma.

Proof of Theorem 4. Suppose first that $\{\lambda_n\}$ satisfies (11). Given $f(z) = \sum a_n z^n$ in B^p, we are to show that $h(z) = \sum \lambda_n a_n z^n$ belongs to H^q. By (8), with ν replaced by $(\nu+1)$, we have

$$\rho^{\nu+1} \,|\, h^{(\nu+1)}(\rho z) \,| \leqq \frac{1}{2\pi}\int_0^{2\pi} |f(\rho e^{it})| \, |g^{(\nu+1)}(ze^{-it})| \, dt .$$

Since $q \geqq 1$, it follows from Jensen's inequality ([11], § 6.14) that

$$\begin{aligned} \rho^{\nu+1} M_q(r\rho, h^{(\nu+1)}) &\leqq M_1(\rho, f) M_q(r, g^{(\nu+1)}) \\ &\leqq C(1-r)^{1/p-\nu-2} M_1(\rho, f) , \end{aligned}$$

where $r = |z|$ and (11) has been used. Now set $r = \rho$ and use the hypothesis $f \in B^p$ to conclude that

$$\int_0^1 (1-r)^\nu M_q(r, h^{(\nu+1)})dr < \infty .$$

But by successive applications of the lemma, this implies

$$\int_0^1 M_q(r, h')dr < \infty .$$

Thus, in view of the inequality (12), it follows that $h \in H^q$, which was to be shown.

Conversely, suppose $\{\lambda_n\}$ is a multiplier of H^p into H^q for arbitrary $q(0 < q \leqq \infty)$. Then by the closed graph theorem,

$$\Lambda: \sum a_n z^n \longrightarrow \sum \lambda_n a_n z^n$$

is a bounded operator from H^p to H^q. An argument similar to that used in the proof of Theorem 3 now leads to the estimate (11).

COROLLARY. *If $0 < p < 1 \leqq q \leqq \infty$ and $f \in B^p$, then its fractional integral $f_\alpha \in H^q$, where $\alpha = 1/p - 1/q$. This is false if $q < 1$.*

This corollary can also be proved directly. Indeed, since ([2], Th. 5) the fractional integral of order $(1/p - 1/s)$ of a B^p function is in B^s $(0 < s < 1)$, and since ([8], p. 415) the fractional integral of order $(1 - 1/q)$ of an H^1 function is in $H^q(1 \leqq q \leqq \infty)$, it suffices to show that $f' \in B^{1/2}$ implies $f \in H^1$. But this is easy; it follows from (12) with $q = 1$. That the corollary is false for $q < 1$ is a consequence of the fact ([2], Th. 5) that the fractional derivative of order $(1/p - 1/q)$ of every B^q function is in B^p.

The converse is also false. That is, if $f \in H^q$, its fractional derivative of order $(1/p - 1/q)$ need not be in $B^p(0 < p < 1 \leqq q \leqq \infty)$. As before, this reduces to showing that $f \in H^1$ does not imply $f' \in B^{1/2}$. To see this, let $f(z) = \sum c_k z^{n_k}$, where $\{n_k\}$ is lacunary, $\{c_k\} \in \ell^2$, and $\{c_k\} \notin \ell^1$. Then $f \in H^2 \subset H^1$, but $f' \notin B^{1/2}$, since it was shown in [4] (Th. 3, Corollary 2) that

$$\sum_{k=1}^{\infty} n_k^{1-1/p} \, |a_{n_k}| < \infty$$

whenever $\sum a_n z^n \in B^p$ and $\{n_k\}$ is a lacunary sequence.

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