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CONCERNING SEMI-STRATIFIABLE SPACES

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In this paper, a class of spaces, called semi-stratifiable spaces is introduced. This class of spaces lies between the class of semi-metric spaces and the class of spaces in which closed sets are G_δ. This class of spaces is invariant with respect to taking countable products, closed maps, and closed unions. In a semi-stratifiable space, bicompactness and countable compactness are equivalent properties. A semi-stratifiable space is F_σ-screenable.

A T_1-space is semi-metric if and only if it is semi-stratifiable and first countable. A completely regular space is a Moore space if and only if it is a semi-stratifiable p-space.

The concept of semi-stratifiable spaces as a generalization of semi-metric spaces (see Corollary 1.4) is due to E. A. Michael. It appears that all properties of semi-metric spaces which do not depend on first countability also hold in semi-stratifiable spaces. The class of semi-stratifiable spaces contains all stratifiable spaces [3], all cosmic spaces [13], and all spaces with a σ-locally finite [15] or σ-discrete [2] network.

Some of the results of this paper were announced in [5].

Most terms which are not defined in this paper are used as in Kelley [10].

1. Preliminaries.

DEFINITION 1.1. A topological space X is a *semi-stratifiable* space if, to each open set $U \subset X$, one can assign a sequence $\{U_n\}_{n=1}^\infty$ of closed subsets of X such that

(a) $\bigcup_{n=1}^\infty U_n = U$,

(b) $U_n \subset V_n$ whenever $U \subset V$, where $\{V_n\}_{n=1}^\infty$ is the sequence assigned to V.

A correspondence $U \to \{U_n\}_{n=1}^\infty$ is a *semi-stratification* for the space X whenever it satisfies conditions (a) and (b) of Definition 1.1.

By comparing the above definition with Definition 1.1 of [3], one can see that, if the correspondence $U \to \{U_n\}_{n=1}^\infty$ is a stratification for X, then $U \to \{U'_n\}_{n=1}^\infty$, where $U'_n = \mathrm{Cl}\, U_n$, is a semi-stratification for X. In [8], Heath gives an example of a (paracompact) semi-stratifiable space which is not stratifiable.

THEOREM 1.2. *A necessary and sufficient condition for a topological space X to be semi-stratifiable is that there be a sequence*

$\{g_i\}_{i=1}^{\infty}$ *of functions from* X *into the collection of open sets of* X *such that* (i) $\bigcap_{i=1}^{\infty} g_i(x) = Cl\{x\}$ *for each* x, *and* (ii) *if* y *is a point of* X *and* $\{x_i\}_{i=1}^{\infty}$ *is a sequence of points in* X, *with* $y \in g_i(x_i)$ *for all* i, *then* $\{x_i\}_{i=1}^{\infty}$ *converges to* y.

Proof. Let $U \rightarrow \{U_n\}_{n=1}^{\infty}$ be a semi-stratification for X. For each i, define the function g_i by $g_i(x) = X - (X - \mathrm{Cl}\,\{x\})_i$. The sequence $\{g_i\}_{i=1}^{\infty}$ satisfies conditions (i) and (ii) of the theorem.

Conversely, let $\{g_i\}_{i=1}^{\infty}$ satisfy conditions (i) and (ii) of the theorem. For each n and each open set U, let $U_n = X - \bigcup\{g_n(x) \colon x \in X - U\}$. Then correspondence $U \rightarrow \{U_n\}_{n=1}^{\infty}$ is a semi-stratification for X.

DEFINITION 1.3. A topological space X is *semi-metric* if there is a distance function d defined on X such that

(1) $d(x, y) = d(y, x) \geqq 0$,

(2) $d(x, y) = 0$ if and only if $x = y$,

(3) x is a limit point of a set M if and only if $\inf\{d(x, y) \colon y \in M\} = 0$.

See [7, 11].

With the aid of Theorem 3.2 of [7] we have the following relationship between semi-stratifiable spaces and semi-metric spaces.

COROLLARY 1.4. *A* T_1*-space is a semi-metric space if and only if it is a first countable semi-stratifiable space.*

2. Properties of semi-stratifiable spaces.

THEOREM 2.1. *The countable product of semi-stratifiable spaces is semi-stratifiable.*

Proof. For each i, let X_i be a semi-stratifiable space and $\{g_{ij}\}_{j=1}^{\infty}$ be a sequence of functions on X_i satisfying the conditions of Theorem 1.2. Let $X = \prod_{i=1}^{\infty} X_i$ and let π_i be the projection of X onto X_i. For each i, j and each x in X, let $h_{ij}(x) = g_{ij}(\pi_i(x))$ if $j \leqq i$ and $h_{ij}(x) = X_i$ if $j > i$. Now let $g_j(x) = \prod_{i=1}^{\infty} h_{ij}(x)$ for each j and x. The sequence $\{g_i\}_{i=1}^{\infty}$ satisfies the conditions of Theorem 1.2 and, hence, X is semi-stratifiable.

THEOREM 2.2. *A semi-stratifiable space is hereditarily semi-stratifiable.*

Theorem 2.2 can be proved by taking the natural restriction to the subspace of a semi-stratification of the larger space. In the case of closed subspaces, all semi-stratifications on the subspace can be constructed in this manner.

THEOREM 2.3. *If Y is a closed subspace of a semi-stratifiable space X and $U \to \{U_n\}_{n=1}^\infty$ is a semi-stratification for Y, then there is a semi-stratification $V \to \{V_n\}_{n=1}^\infty$ for X such that $(V \cap Y)_n = (V_n \cap Y)$.*

Proof. If $W \to \{W_n\}_{n=1}^\infty$ is any semi-stratification for X, then let $V_n = (V \cap Y)_n \cup (V - Y)_n$. The correspondence $V \to \{V_n\}_{n=1}^\infty$ is a semi-stratification for X satisfying $(V \cap Y)_n = V_n \cap Y$.

By applying Theorem 2.3 with respect to the common subspace, we obtain the following theorem:

THEOREM 2.4. *The union of two closed (in the union) semi-stratifiable spaces is semi-stratifiable.*

DEFINITION 2.5. A topological space is *F_σ-screenable* if every open cover has a σ-discrete closed refinement which covers the space.

Theorem 2.6 generalizes McAulely's Lemma 1 of [12].

Theorem 2.6. *A semi-stratifiable space is F_σ-screenable.*

Proof. Let X be a semi-stratifiable space with a semi-stratification $U \to \{U_n\}_{n=1}^\infty$. Let $\{O_\alpha : \alpha \in I\}$ be an open cover of X and let I be well-ordered. For each natural number n, define: $H_{1n} = (O_1)_n$ and, for each $\alpha > 1$, $H_{\alpha n} = (O_\alpha)_n - \cup\{O_\beta : \beta \in I, \beta < \alpha\}$. For each natural number n, let $\mathscr{H}_n = \{H_{\alpha n} : \alpha \in I\}$. Then $\mathscr{H}_n$ is a discrete collection of closed sets. By the well-ordering on I, $\mathscr{H} = \cup_{n=1}^\infty \mathscr{H}_n$ covers X.

DEFINITION 2.7. A topological space is *$\aleph_1$-compact* if every uncountable subset has a limit point.

THEOREM 2.8. *In a semi-stratifiable T_1-space X, the following are equivalent* (1) *X is Lindelöf,* (2) *X is hereditarily separable, and* (3) *X is $\aleph_1$-compact.*

Proof. (1) $\Rightarrow$ (2) Let X be a Lindelöf semi-stratifiable space. Since a Lindelöf space in which open sets are F_σ is hereditarily Lindelöf, it is sufficient to show that X is separable. Let $\{g_i\}_{i=1}^\infty$ be a sequence of functions satisfying the conditions of Theorem 1.2. For each i, $\{g_i(x) : x \in X\}$ is an open cover of X and, since X is Lindelöf, there is a countable subset D_i of X such that $\{g_i(x) : x \in D_i\}$ is an open cover of X. The set $D = \cup_{i=1}^\infty D_i$ is a countable dense subset of X.

(2) $\Rightarrow$ (3) The proof of this part is well-known.

(3) $\Rightarrow$ (1) Let X be an $\aleph_1$-compact semi-stratifiable T_1-space. Let $\mathscr{G}$ be an open cover of X and suppose that $\mathscr{G}$ has no countable subcover. By Theorem 2.6, $\mathscr{G}$ has a closed refinement $\mathscr{H} = \bigcup_{n=1}^{\infty} \mathscr{H}_n$ where each $\mathscr{H}_n$ is discrete. Since $\mathscr{G}$ has no countable subcover, there is an n such that $\mathscr{H}_n$ is uncountable. Let X' be a subset of X consisting of exactly one point of each nonempty element of $\mathscr{H}_n$. The set X' is uncountable and has no limit point.

Theorem 2.8 cannot be strengthened by replacing hereditarily separable by separable. The example of a Moore space which is not metrizable due to R. L. Moore (see [9]) is an example of a separable semi-stratifiable space which is not Lindelöf.

Since, in a Lindelöf space, bicompact is equivalent to countably compact, Theorem 2.8 has the following corollary:

COROLLARY 2.9. *In a semi-stratifiable T_1-space, bicompact is equivalent to countably compact.*

3. Mappings.

It is a well-known theorem [17] that the closed compact image of a separable metric space is a separable metric space. However, there are closed images of separable metric spaces which are not first countable. The following theorem gives a property of metric spaces which is preserved by closed maps.

THEOREM 3.1. *The closed image of a semi-stratifiable space is semi-stratifiable.*

Proof. Let f be a closed continuous function from a semi-stratifiable space X onto a topological space Y. Let $U \to \{U_n\}_{n=1}^{\infty}$ be a semi-stratification for X. For each open set V of Y and each natural number n, let $V_n = f((f^{-1}[V])_n)$. The correspondence $V \to \{V_n\}_{n=1}^{\infty}$ is a semi-stratification for Y.

Theorem 3.1 does not remain true if closed is replaced by open.

Theorem 3.1 and Corollary 1.4 imply that the closed image of a semi-metric space is semi-stratifiable. However, it can be shown that the subspace of βN (the Stone-Čech compactification of the natural numbers) consisting of N together with one point of $\beta N - N$ is a semi-stratifiable space which cannot be the closed image of a semi-metric space. It is an open question whether the spaces which are closed images of semi-metric spaces are precisely the semi-stratifiable Fréchet-Urysohn spaces [2].

4. Moore spaces.

In this section, we wish to give necessary and sufficient conditions for a semi-stratifiable space to be a Moore space.

DEFINITION 4.1. A sequence $\{\mathscr{G}_n\}_{n=1}^{\infty}$ of open covers of a topological space X is a *development* for X if (1) $\mathscr{G}_{i+1}$ is a refinement for $\mathscr{G}_i$ and (2), if x is a point of X and U is an open set in X containing x, then there is a natural number k such that $\mathrm{St}(x, \mathscr{G}_k) \subset U$. A *Moore space* is a regular T_1-space which has a development. See [7, 14].

A Moore space is semi-metric and, hence, is semi-stratifiable. The following example, due to McAuley [11], shows that this implication cannot be reversed.

EXAMPLE 4.2. Let X be the x-axis of the Cartesian plane E^2. Let d denote the usual distance function in E^2 and, if $p \neq q$, let $\alpha(p, q)$ denote the nonobtuse angle (in radians) formed by X and the line through p and q. Define a distance function D on E^2 as follows: $D(p, p) = 0$ and, if $p \neq q$, $D(p, q) = d(p, q) + \alpha(p, q)$. A basis for the topology on E^2 is $\{U_\varepsilon(p): p \in E^2, \varepsilon > 0\}$ where $U_\varepsilon(p) = \{q: D(p, q) < \varepsilon\}$. Let S denote E^2 with this topology. If S were a Moore space, it would be second countable, since it is Lindelöf. But S is not second countable since any basis contains uncountably many elements.

DEFINITION 4.3. A T_1-space X is said to be *quasi-complete* provided that there is a sequence $\{\mathscr{B}_n\}_{n=1}^{\infty}$ of open covers of X with the following property: if $\{A_n\}_{n=1}^{\infty}$ is a decreasing sequence of nonempty closed subsets of X and if there exists an element $x_0 \in X$ such that, for each n, there is a $\mathscr{B}_n \in \beta_n$ with $A_n \cup \{x_0\} \subset B_n$, then $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$.

DEFINITION 4.4 (Borges [4]). A T_1-space X is a *$w\Delta$-space* if there exists a sequence $\{\mathscr{B}_n\}_{n=1}^{\infty}$ of open covers of X such that, if $\{A_n\}_{n=1}^{\infty}$ is a decreasing sequence of nonempty closed subsets of X and there exists $x_0 \in X$ for which $A_n \subset \mathrm{St}(x_0, \mathscr{B}_n)$ for all n, then $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$.

Definition 4.3 is at least formally weaker then Definition 4.4. It is an open question whether all quasi-complete spaces are $w\Delta$-spaces.

Theorem 4.5, due to Heath [7], gives a sufficient condition for a space to be a Moore space.

THEOREM 4.5. *A regular T_1-space X is a Moore space provided that there is a sequence $\{g_i\}_{i=1}^{\infty}$ of functions from X into the topology on X with the following properties*: (A) *For each x in X, $\{g_i(x)\}_{i=1}^{\infty}$ is a decreasing local base at x.* (B) *If y is a point of X and $\{x_i\}_{i=1}^{\infty}$ is a sequence in X with $y \in g_i(x_i)$ for each i, then $\{x_i\}_{i=1}^{\infty}$ converges to y.* (C) *If y is a point of X, U is an open subset of X containing y, and $\{x_i\}_{i=1}^{\infty}$ is a sequence in X such that, for each n, $y \in g_n(x_n)$ and there* *is a natural number* k *with* $\mathrm{Cl}\,[g_{n+k}(x_{n+k})] \subset g_n(x_n)$, *then there is a natural number* m *with* $g_m(x_m) \subset U$.

THEOREM 4.6. *A regular* T_1*-space is a Moore space if it is a quasi-complete semi-stratifiable space.*

Proof. Let X be a regular quasi-complete semi-stratifiable T_1-space. Let $\{\mathcal{B}_n\}_{n=1}^{\infty}$ be a sequence satisfying the conditions of Definition 4.3 and let $\{h_n\}_{n=1}^{\infty}$ be a sequence satisfying the conditions of Theorem 1.2. For each x in X, let $B_n(x)$ be a member of $\mathcal{B}_n$ containing x. For each x, let $g_1(x)$ be an open subset of X containing x such that $\mathrm{Cl}\,g_1(x) \subset B_1(x) \cap h_1(x)$ and let $g_{n+1}(x)$ be an open subset of X containing x such that $\mathrm{Cl}\,g_{n+1}(x) \subset B_{n+1}(x) \cap h_{n+1}(x) \cap g_n(x)$. The sequence $\{g_n\}_{n=1}^{\infty}$ satisfies the conditions of Theorem 4.5 and, hence, X is a Moore space.

By Proposition 2.8 of [4], we have the following corollary:

COROLLARY 4.7. *If* X *is a regular* T_1*-space, then the following are equivalent*:

(1) X *is a Moore space.*

(2) X *is a semi-stratifiable* $w\Delta$*-space.*

(3) X *is a semi-stratifiable quasi-complete space.*

If X is a completely regular T_1-space, let βX denote its Stone-Čech compactfication. The following definition is due to Arhangel'skii [1, 2].

DEFINITION 4.8. A completely regular T_1-space X is a *p-space* provided that there is a sequence $\{\mathcal{B}_n\}_{n=1}^{\infty}$ of collections of open subsets of βX such that each $\mathcal{B}_n$ covers X and $\bigcap_{n=1}^{\infty} \mathrm{St}\,(x, \mathcal{B}_n) \subset X$ for each point x in X.

LEMMA 4.9. *A p-space is quasi-complete.*

Proof. Let X be a p-space and let $\{\mathcal{B}_n\}_{n=1}^{\infty}$ satisfy the conditions of Definition 4.8. For each n, let $\mathcal{B}'_n$ be an open cover of X such that, if $B \in \mathcal{B}'_n$, then $\mathrm{Cl}_{\beta X}\,B$ is contained in some member of $\mathcal{B}_n$. Let $\{A_n\}_{n=1}^{\infty}$ be a decreasing sequence of closed subsets of X and x be a point of X such that there is a $B_n \in \mathcal{B}'_n$ with $A_n \cup \{x\} \subset B_n$ for each n. Since $\mathrm{Cl}_{\beta X}\,A_n$ is compact, $\bigcap_{n=1}^{\infty} \mathrm{Cl}_{\beta X}\,A_n \neq \varnothing$. But $\mathrm{Cl}_{\beta X}\,A_n \subset \mathrm{St}\,(x, \mathcal{B}_n)$ and $\bigcap_{n=1}^{\infty} \mathrm{Cl}_{\beta X}\,A_n \subset X$. Thus, $\bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} \mathrm{Cl}_{\beta X}\,A_n$. Hence, X is quasi-complete.

It can be seen that, in completely regular spaces, the concepts of $w\Delta$-spaces, p-spaces, and quasi-complete spaces are related. The exact relationship between these three concepts is an open problem.

THEOREM 4.10. *A completely regular T_1-space is a Moore space if and only if it is a semi-stratifiable p-space.*

Proof. Lemma 4.9 and Theorem 4.6 show that a semi-stratifiable p-space is a Moore space.

Conversely, let X be a completely regular Moore space and let $\{\mathscr{G}_n\}_{n=1}^{\infty}$ be a development for X. By the remark following Definition 4.1, X is semi-stratifiable. For each n, let

$$\mathscr{B}_n = \{\beta X - \mathrm{Cl}_{\beta X}(X - G): G \in \mathscr{G}_n\}\ .$$

The sequence $\{\mathscr{B}_n\}_{n=1}^{\infty}$ satisfies the conditions of Definition 4.8. Since $G \subset \beta X - \mathrm{Cl}_{\beta X}(X - G)$, $\mathscr{B}_n$ covers X. If $x \in X$ and $y \in \beta X - X$, let U and V be disjoint open subsets of βX containing x and y, respectively. There is a k where $\mathrm{St}(x, \mathscr{G}_k) \subset U \cap X$. Then $y \notin \mathrm{St}(x, \mathscr{B}_k)$. Hence, X is a p-space.

Since a locally compact Hausdorff space is a p-space, we have the following corollary:

COROLLARY 4.11. *A locally compact semi-stratifiable Hausdorff space is a Moore space.*

In Theorem 4.10, the condition of complete regularity can be replaced with regularity by using the Wallman compactification [6, 16] instead of the Stone-Čech compactification. Appropriate changes will also have to be made in Definition 4.8.

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