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MÖBIUS FUNCTIONS OF ORDER k

TOM M. APOSTOL

Let k denote a fixed positive integer. We define an arithmetical function μ_k, the Möbius function of order k, as follows:

$$\mu_k(1) = 1\,,$$
$$\mu_k(n) = 0 \text{ if } p^{k+1} \mid n \text{ for some prime } p\,,$$
$$\mu_k(n) = (-1)^r \text{ if } n = p_1^k \cdots p_r^k \prod_{i>r} p_i^{a_i}\,, \qquad 0 \leqq a_i < k\,,$$
$$\mu_k(n) = 1 \text{ otherwise}\,.$$

In other words, $\mu_k(n)$ vanishes if n is divisible by the $(k+1)$st power of some prime; otherwise, $\mu_k(n)$ is 1 unless the prime factorization of n contains the kth powers of exactly r distinct primes, in which case $\mu_k(n) = (-1)^r$. When $k = 1$, $\mu_k(n)$ is the usual Möbius function, $\mu_1(n) = \mu(n)$.

This paper discusses some of the relations that hold among the functions μ_k for various values of k. We use these to derive an asymptotic formula for the summatory function

$$M_k(x) = \sum_{n \leqq x} \mu_k(n)$$

for each $k \geqq 2$. Unfortunately, the analysis sheds no light on the behavior of the function $M_1(x) = \sum_{n \leqq x} \mu(n)$.

It is clear that $|\mu_k|$ is the characteristic function of the set Q_{k+1} of $(k+1)$-free integers (positive integers whose prime factors are all of multiplicity less than $k+1$). Further relations with Q_{k+1} are given in §'s 4 and 5.

The asymptotic formula for $M_k(x)$ is given in the following theorem.

THEOREM 1. *If $k \geqq 2$ we have*

$$\sum_{n \leqq x} \mu_k(n) = A_k x + O(x^{1/k} \log x)\,, \tag{1}$$

where

$$A_k = \frac{1}{\zeta(k)} \sum_{n=1}^{\infty} \frac{\mu(n)}{n^k} \prod_{p \mid n} \frac{1 - p^{-1}}{1 - p^{-k}}\,. \tag{2}$$

Note. In (2), $\zeta(k)$ is the Riemann zeta function. The formula for A_k can also be expressed in the form

$$A_k = \frac{1}{\zeta(k)} \sum_{n=1}^{\infty} \frac{\mu(n)\varphi(n)}{nJ_k(n)} \tag{3}$$

where $\varphi(n)$ and $J_k(n)$ are the totient functions of Euler and Jordan, given by

$$\varphi(n) = n \prod_{p \mid n} (1 - p^{-1}),\ J_k(n) = n^k \prod_{p \mid n} (1 - p^{-k}) .$$

We also have the Euler product representation

$$A_k = \prod_p \left(1 - \frac{2}{p^k} + \frac{1}{p^{k+1}}\right) . \tag{4}$$

2. Lemmas. The proof of Theorem 1 is based on a number of lemmas.

LEMMA 1. *If $k \geqq 1$ we have $\mu_k(n^k) = \mu(n)$.*

LEMMA 2. *Each function μ_k is multiplicative. That is,*

$$\mu_k(mn) = \mu_k(m)\mu_k(n) \quad \textit{whenever} \quad (m, n) = 1 .$$

LEMMA 3. *Let f and g be multiplicative arithmetical functions and let a and b be positive integers, with $a \geqq b$. Then the function h defined by the equation*

$$h(n) = \sum_{d^a \mid n} f\left(\frac{n}{d^a}\right) g\left(\frac{n}{d^b}\right)$$

is also multiplicative. (The sum is extended over those divisors d of n for which d^a divides n.)

The first two lemmas follow easily from the definition of the function μ_k. The proof of Lemma 3 is a straightforward exercise.

The next lemma relates μ_k to μ_{k-1}.

LEMMA 4. *If $k \geqq 2$ we have*

$$\mu_k(n) = \sum_{d^k \mid n} \mu_{k-1}\left(\frac{n}{d^k}\right) \mu_{k-1}\left(\frac{n}{d}\right) .$$

Proof. By Lemmas 2 and 3, the sum on the right is a multiplicative function of n. To complete the proof we simply verify that the sum agrees with $\mu_k(n)$ when n is a prime power.

LEMMA 5. *If $k \geqq 1$ we have*

$$| \mu_k(n) | = \sum_{d^{k+1} \mid n} \mu(d) .$$

Proof. Again we note that both members are multiplicative functions of n which agree when n is a prime power.

LEMMA 6. *If $k \geqq 2$ and $r \geqq 1$, let*

$$F_r(x) = \sum_{n \leqq x} \mu_{k-1}(n)\mu_{k-1}(r^{k-1}n) .$$

Then we have the asymptotic formula

$$F_r(x) = \frac{x}{\zeta(k)} \frac{\mu(r)\varphi(r)r^{k-1}}{J_k(r)} + O(x^{1/k}\sigma_{-s}(r)) ,$$

where $\sigma_\alpha(r)$ is the sum of the αth powers of the divisors of r, and s is any number satisfying $0 < s < 1/k$. (The constant implied by the O-symbol is independent of r.)

Proof. In the sum defining $F_r(x)$ the factor $\mu_{k-1}(r^{k-1}n) = 0$ if r and n have a prime factor in common. Therefore we need consider only those n relatively prime to r. But if $(r, n) = 1$ the multiplicative property of μ_{k-1} gives us

$$\mu_{k-1}(n)\mu_{k-1}(r^{k-1}n) = \mu_{k-1}(n)^2\mu_{k-1}(r^{k-1}) = | \mu_{k-1}(n) | \mu(r) ,$$

where in the last step we used Lemma 1. Therefore we have

$$F_r(x) = \mu(r) \sum_{\substack{n \leqq x \\ (n,r)=1}} | \mu_{k-1}(n) | .$$

Using Lemma 5 we rewrite this in the form

$$\begin{aligned} F_r(x) &= \mu(r) \sum_{\substack{n \leqq x \\ (n,r)=1}} \sum_{d^k | n} \mu(d) = \mu(r) \sum_{\substack{d^k \leqq x \\ (d,r)=1}} \mu(d) \sum_{\substack{q \leqq x/d^k \\ (q,r)=1}} 1 \\ &= \mu(r) \sum_{\substack{d^k \leqq x \\ (d,r)=1}} \mu(d) \sum_{t|r} \mu(t) \left[\frac{x}{td^k}\right] \\ &= \mu(r) \sum_{t|r} \mu(t) \sum_{\substack{d^k \leqq x \\ (d,r)=1}} \mu(d) \left[\frac{x}{td^k}\right] . \end{aligned}$$

At this point we use the relation $[x] = x + O(x^s)$, valid for any fixed s satisfying $0 \leqq s < 1$, to obtain

$$\begin{aligned} F_r(x) &= \mu(r) \sum_{t|r} \mu(t) \sum_{\substack{d^k \leqq x \\ (d,r)=1}} \mu(d) \left\{\frac{x}{td^k} + O\left(\frac{x^s}{t^s d^{ks}}\right)\right\} \\ &= x\mu(r) \sum_{t|r} \frac{\mu(t)}{t} \sum_{\substack{d^k \leqq x \\ (d,r)=1}} \frac{\mu(d)}{d^k} + O\left(x^s \sum_{t|r} \frac{1}{t^s} \sum_{d \leqq x^{1/k}} \frac{1}{d^{ks}}\right) . \end{aligned}$$

If we choose s so that $0 < ks < 1$ we have

$$\sum_{d \leq x^{1/k}} \frac{1}{d^{ks}} = O\left(\int_1^{x^{1/k}} \frac{dt}{t^{ks}}\right) = O(x^{-s+1/k}) ,$$

and the O-term in the last formula for $F_r(x)$ is $O(x^{1/k}\sigma_{-s}(r))$. To complete the proof of Lemma 6 we use the relations

$$\sum_{t|r} \frac{\mu(t)}{t} = \frac{\varphi(r)}{r}$$

and

$$\begin{aligned}\sum_{\substack{d^k \leq x \\ (d,r)=1}} \frac{\mu(d)}{d^k} &= \sum_{\substack{d=1 \\ (d,r)=1}}^{\infty} \frac{\mu(d)}{d^k} + O\Big(\sum_{d > x^{1/k}} d^{-k}\Big) \\ &= \frac{1}{\zeta(k)} \prod_{p|r} \frac{1}{1-p^{-k}} + O(x^{(1-k)/k}) \\ &= \frac{1}{\zeta(k)} \frac{r^k}{J_k(r)} + O(x^{(1-k)/k}) .\end{aligned}$$

3. Proof of Theorem 1. In the sum defining $M_k(x)$ we use Lemma 4 to write

$$\begin{aligned}M_k(x) &= \sum_{n \leq x} \mu_k(n) = \sum_{n \leq x} \sum_{d^k | n} \mu_{k-1}\Big(\frac{n}{d^k}\Big)\mu_{k-1}\Big(\frac{n}{d}\Big) \\ &= \sum_{d^k \leq x} \sum_{m \leq x/d^k} \mu_{k-1}(m)\mu_{k-1}(d^{k-1}m) \\ &= \sum_{d^k \leq x} F_d(x/d^k) = \sum_{r \leq x^{1/k}} F_r(x/r^k) .\end{aligned}$$

Using Lemma 6 we obtain

$$M_k(x) = \frac{x}{\zeta(k)} \sum_{r \leq x^{1/k}} \frac{\mu(r)\varphi(r)}{rJ_k(r)} + O\Big(x^{1/k} \sum_{r \leq x^{1/k}} \frac{\sigma_{-s}(r)}{r}\Big) . \tag{5}$$

The sum in the first term is equal to

$$\begin{aligned}\sum_{r \leq x^{1/k}} \frac{\mu(r)}{r^k} \prod_{p|r} \frac{1-p^{-1}}{1-p^{-k}} &= \sum_{r=1}^{\infty} \frac{\mu(r)}{r^k} \prod_{p|r} \frac{1-p^{-1}}{1-p^{-k}} + O\Big(\sum_{r > x^{1/k}} \frac{1}{r^k}\Big) \\ &= \sum_{r=1}^{\infty} \frac{\mu(r)\varphi(r)}{rJ_k(r)} + O(x^{(1-k)/k}) .\end{aligned}$$

The sum in the O-term in (5) is equal to

$$\begin{aligned}\sum_{r \leq x^{1/k}} \frac{\sigma_{-s}(r)}{r} &= \sum_{r \leq x^{1/k}} r^{-1} \sum_{d\delta = r} d^{-s} = \sum_{\delta \leq x^{1/k}} \delta^{-1} \sum_{d \leq x^{1/k}/\delta} d^{-1-s} \\ &= O\Big(\sum_{\delta \leq x^{1/k}} \delta^{-1}\Big) = O(\log x) .\end{aligned}$$

Therefore (5) becomes

$$M_k(x) = \frac{x}{\zeta(k)} \sum_{r=1}^{\infty} \frac{\mu(r)\varphi(r)}{rJ_k(r)} + O(x^{1/k} \log x) ,$$

which completes the proof of Theorem 1.

To deduce (4) from (2) we note that (2) has the form

$$A_k = \frac{1}{\zeta(k)} \sum_{n=1}^{\infty} f(n)$$

where $f(n)$ is multiplicative and $f(p^a) = 0$ for $a \geqq 2$. Hence we have the Euler product decomposition: [see 3, Th. 286]

$$\begin{aligned} A_k &= \frac{1}{\zeta(k)} \prod_p \{1 + f(p)\} = \prod_p (1 - p^{-k}) \prod_p \left\{1 - \frac{1}{p^k} \frac{1 - p^{-1}}{1 - p^{-k}}\right\} \\ &= \prod_p \left\{1 - p^{-k} - \frac{1 - p^{-1}}{p^k}\right\} = \prod_p \left\{1 - \frac{2}{p^k} + \frac{1}{p^{k+1}}\right\} . \end{aligned}$$

4. **Relations to k-free integers.** Let Q_k denote the set of k-free integers (positive integers whose prime factors are all of multiplicity less than k), and let q_k denote the characteristic function of Q_k:

$$q_k(n) = \begin{cases} 1 & \text{if } \ n \in Q_k, \\ 0 & \text{otherwise.} \end{cases}$$

Gegenbauer [2, p. 47] has proved that the number of k-free integers $\leqq x$ is given by

$$(6) \qquad \sum_{n \leqq x} q_k(n) = \frac{x}{\zeta(k)} + O(x^{1/k}) , \qquad (k \geqq 2) .$$

From the definition of μ_k it follows that $q_{k+1}(n) = |\mu_k(n)|$, so Gegenbauer's theorem implies the asymptotic formula

$$(7) \qquad \sum_{n \leqq x} |\mu_k(n)| = \frac{x}{\zeta(k+1)} + O(x^{1/(k+1)}) , \qquad (k \geqq 1) .$$

From our Theorem 1 we have

$$(8) \qquad \sum_{n \leqq x} \mu_k(n) = A_k x + O(x^{1/k} \log x) \qquad (k > 1) .$$

The two formulas (7) and (8) show that among the $(k+1)$-free integers, $k > 1$, those for which $\mu_k(n) = 1$ occur asymptotically more frequently than those for which $\mu_k(n) = -1$; in particular, these two sets of integers have, respectively, the densities

$$\frac{1}{2}\left(\frac{1}{\zeta(k+1)} + A_k\right) \quad \text{and} \quad \frac{1}{2}\left(\frac{1}{\zeta(k+1)} - A_k\right) .$$

This is in contrast to the case $k = 1$ for which it is known that

$$\sum_{n \leq x} | \mu(n) | = \frac{x}{\zeta(2)} + O(x^{1/2}), \quad \text{but} \quad \sum_{n \leq x} \mu(n) = o(x) ,$$

so the square-free integers with $\mu(n) = 1$ occur with the same asymptotic frequency as those with $\mu(n) = -1$ [see 3, p. 270].

Our Theorem 1 can also be derived very simply from an asymptotic formula of Cohen [1, Th. 4.2]. Following the notation of Cohen, let Q_k^* denote the set of positive integers n with the property that the multiplicity of each prime divisor of n is not a multiple of k. Let q_k^* denote the characteristic function of Q_k^*. Then $q_k^*(1) = 1$, and for $n > 1$ we have

$$q_k^*(n) = \begin{cases} 1 & \text{if} \quad n = \prod_{i=1}^{r} p_i^{a_i}, \quad \text{with each} \quad a_i \not\equiv 0 \pmod{k} , \\ 0 & \text{otherwise.} \end{cases}$$

The functions q_k^* and μ_k are related by the following identity:

$$q_k^*(n) = \sum_{d^k | n} \mu_k\left(\frac{n}{d^k}\right) . \tag{9}$$

This is easily verified by noting that both members are multiplicative functions of n that agree when n is a prime power, or by equating coefficients in the Dirichlet series identity (14) given below in § 5. Inversion of (9) gives us

$$\mu_k(n) = \sum_{d^k | n} \mu(d) q_k^*\left(\frac{n}{d^k}\right) . \tag{10}$$

Cohen's asymptotic formula states that for $k \geqq 2$ we have

$$\sum_{n \leq x} q_k^*(n) = A_k \zeta(k) x + O(x^{1/k}) , \tag{11}$$

where A_k is the same constant that appears in our Theorem 1. To deduce Theorem 1 from (11) we use (10) to obtain

$$\begin{aligned} \sum_{n \leq x} \mu_k(n) &= \sum_{n \leq x} \sum_{d^k | n} \mu(d)\, q_k^*\left(\frac{n}{d^k}\right) = \sum_{d^k \leq x} \mu(d) \sum_{m \leq x/d^k} q_k^*(m) \\ &= \sum_{d^k \leq x} \mu(d) \left\{ A_k \zeta(k) \frac{x}{d^k} + O\left(\frac{x^{1/k}}{d}\right) \right\} \\ &= A_k \zeta(k) x \sum_{d \leq x^{1/k}} \frac{\mu(d)}{d^k} + O\left(x^{1/k} \sum_{d^k \leq x} \frac{1}{d} \right) \\ &= A_k \zeta(k) x \sum_{d=1}^{\infty} \frac{\mu(d)}{d^k} + O\left(\sum_{d > x^{1/k}} d^{-k} \right) + O(x^{1/k} \log x) \\ &= A_k x + O(x^{1/k} \log x) . \end{aligned}$$

Conversely, if we start with equation (9) and use Theorem 1 we can deduce Cohen's asymptotic formula (11) but with an error term $O(x^{1/k}\log x)$ in place of $O(x^{1/k})$.

5. Generating functions. The generating function for the k-free integers is known to be given by the Dirichlet series

$$\sum_{n=1}^{\infty}\frac{q_k(n)}{n^s}=\frac{\zeta(s)}{\zeta(ks)}\qquad(s>1) \tag{12}$$

[see 3, Th. 303, p. 255]. It is not difficult to determine the generating functions for the functions μ_k and q_k^* as well. Straightforward calculations with Euler products show that we have

$$\sum_{n=1}^{\infty}\frac{\mu_k(n)}{n^s}=\zeta(s)\prod_p\left\{1-\frac{2}{p^{ks}}+\frac{1}{p^{(k+1)s}}\right\} \tag{13}$$

and

$$\sum_{n=1}^{\infty}\frac{q_k^*(n)}{n^s}=\zeta(ks)\sum_{n=1}^{\infty}\frac{\mu_k(n)}{n^s} \tag{14}$$

for $s>1$. Equation (14) is also equivalent to equations (9) and (10). From (12) and (14) we obtain the following identity relating μ_k, q_k, and q_k^*:

$$\zeta(s)\sum_{n=1}^{\infty}\frac{\mu_k(n)}{n^s}=\left(\sum_{n=1}^{\infty}\frac{q_k(n)}{n^s}\right)\left(\sum_{n=1}^{\infty}\frac{q_k^*(n)}{n^s}\right).$$

This shows [see 3, §17.1] that the numerical integral of μ_k is the Dirichlet convolution of q_k and q_k^*:

$$\sum_{d\mid n}\mu_k(d)=\sum_{d\mid n}q_k(d)q_k^*\left(\frac{n}{d}\right).$$

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CALIFORNIA INSTITUTE OF TECHNOLOGY