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UNKNOTTING UNIONS OF CELLS

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In this note we consider the problem of determining whether the union of cells is nicely embedded in the n-sphere if each of the cells is nicely embedded. This question is related to many embedding problems. For instance, the n-dimensional Annulus Conjecture (now known to be true for $n \neq 4$) is a special case. Cantrell and Lacher have shown that an affirmative answer implies local flatness of certain submanifolds. Also, this question is related to the conjecture that an embedding of a complex into the n-sphere which is locally flat on open simplexes is ε-tame in codimension three.

The problem mentioned above was first investigated by Doyle [9] [10] in the three dimensional case and by Cantrell [2] in high dimensions and later by Lacher [15] Cantrell and Lacher [3][4], Kirby [13], Černavskii [5][6] and the author [17]. Also, Sher [21] has generalized a construction of Debrunner and Fox [8] to obtain counterexamples in certain cases. Since the n-dimensional Annulus Conjecture, $n \neq 4$, is now known to be true [14], only two results of § 7 of [17] remain of interest. First we will prove a strengthened form of one of those results and we greatly simplify the proof by employing the powerful tools now available. In particular we prove the following theorem.

THEOREM 1. *If $D_1^{m_1}$ and $D_2^{m_2}$ are cells in S^n, $n > 5$, of dimensions m_1 and m_2, respectively, and if $D_1^{m_1} \cap D_2^{m_2} = \partial D_1^{m_1} \cap \partial D_2^{m_2} = D$ is a k-cell (possibly empty), $n - k \geqq 4$, which is locally flat in $\partial D_1^{m_1}$, in $\partial D_2^{m_2}$ and in S^n and is such that $D_1^{m_1} - D$ and $D_2^{m_2} - D$ are locally flat, then there is an ambient isotopy e_t of S^n such that $e_1(D_1^{m_1})$ and $e_1(D_2^{m_2})$ are simplexes and $e_1(D_1^{m_1} \cap D_2^{m_2})$ is a face of each.*

REMARK. If the above theorem is modified by requiring $n - k = 3$, then counterexamples can be constructed for any m_1 and m_2 (see [21]).

Proof of Theorem 1. Every orientation preserving homeomorphism of S^n, $n \geqq 5$, is stable [14], hence isotopic to the identity. It will then suffice to construct an orientation preserving homeomorphism e_1 satisfying the conclusion of the theorem. By Theorem 5.2 of [1], we may assume that $D_1^{m_1}$ and $D_2^{m_2}$ are locally flat. For $i = 1, 2$, it is easy to construct a homeomorphism $f_i: S^n \rightarrow S^n$ such that $f_i(D_i^{m_i}, D) = (\Delta^{m_i}, \Delta^k)$ where Δ^{m_i} is an m_i-simplex and Δ^k is a k-face. Thus, by using f_i, $i = 1, 2$, and Lemma 3.6 of [18], we can construct locally flat n-cells D_1^n and

D_2^n satisfying the following conditions,

(1) $D_1^n \cap D_2^n = \partial D_1^n \cap \partial D_2^n = D$,

(2) D is locally flat in ∂D_1^n and ∂D_2^n, and

(3) $(D_i^n, D_i^{m_i})$ is a trivial cell pair, $i = 1, 2$.

Let Δ_1^n and Δ_2^n be n-simplexes in S^n such that $\Delta_1^n \cap \Delta_2^n = \Delta$ is a k-face of each. We will now construct an orientation preserving homeomorphism h of S^n such that $h((D_1^n, D_2^n, D)) = (\Delta_1^n, \Delta_2^n, \Delta)$. It is easy to obtain an orientation preserving homeomorphism h_1 of S^n such that $h_1((D_2^n, D)) = (\Delta_2^n, \Delta)$. Let Δ_0 be the n-simplex having as vertices the midpoints of the segments which join the vertices of Δ_2^n with the barycenter of Δ_2^n. Let $f: I^k \twoheadrightarrow \Delta$ be a PL-homeomorphism and define $F: I^k \times I \rightarrow \Delta_2^n$ by extending linearly on each segment $\{x\} \times I$, $x \in I^k$, the map which takes $(x, 0)$ to $f(x)$ and $(x, 1)$ to the midpoint of the segment joining $f(x)$ and the barycenter of Δ_2^n. Then, $E = F(I^k \times \{1\})$ is a k-face of Δ_0. Now, by using the Annulus Theorem, it is easy to get an orientation preserving homeomorphism h_2 of S^n such that

(1) $h_2((\Delta_0, h_1(D_1^n)) = (\Delta_0, \Delta_1^n)$, and

(2) $h_2 \mid \Delta \cup E = 1$.

Let A denote $Cl(S^n - (\Delta_0 \cup \Delta_1^n))$. Then, the embedding $h_2F: I^k \times I \rightarrow A$ satisfies the hypotheses of Theorem 1 of [19]; hence, by that theorem there is a homeomorphism h_3 of A such that $h_3 \mid \partial\Delta_0 \cup \partial\Delta_1^n = 1$ and $h_3h_2F: I^k \times I \rightarrow A$ is PL. Extend h_3 to all of S by way of the identity. Consider the two PL embeddings $F \mid \partial I^k \times I: \partial I^k \times I \rightarrow A$ and $h_3h_2F \mid \partial I^k \times I: \partial I^k \times I \rightarrow A$. These two embeddings clearly satisfy the hypotheses of Theorem 4 of [11]; therefore, by that theorem there is a PL homeomorphism h_4 of A such that $h_4h_3h_2F \mid \partial I^k \times I = F \mid \partial I^k \times I$ and $h_4 \mid \partial\Delta_0 \cup \partial\Delta_1^n = 1$. Extend h_4 to S^n by the identity. Now, the PL embeddings $h_4h_3h_2F: I^k \times I \rightarrow A$ and $F: I^k \times I \rightarrow A$ satisfy the hypothesis of Theorem 4 of [11] and so by another application of that theorem we get a PL homeomorphism h_5 of A such that $h_5h_4h_3h_2F = F$ and $h_5 \mid \partial\Delta_0 \cup \partial\Delta_1^n = 1$. Extend, h_5 to S^n by the identity.

Let $p: S^n \twoheadrightarrow S^n$ be a map such that

(1) $p(\Delta_0) = \Delta_2^n$,

(2) $p \mid h_1(D_1^n) \cup \Delta_1^n = 1$, and

(3) $p \mid S^n - F(I^k \times I)$ is one-to-one, and $p(F(\{x\} \times I)) = F(x, 0)$ for each $x \in I^k$.

It is now easy to check that $h = ph_5h_4h_3h_2p^{-1}h_1$ is the desired homeomorphism that flattens the pair $D_1^n \cup D_2^n$.

Let $\Delta_i^{m_i-1}$ be a face of Δ_i^n of dimension $m_i - 1$ which has Δ as a face. Let δ_i denote the face of Δ_i^n dual to $\Delta_i^{m_i-1}$ and let $\hat{\delta}_i$ denote the barycenter of δ_i. Now, let $\Delta_i^{m_i}$ be the m_i-simplex $\Delta_i^{m_i-1} * \hat{\delta}_i$. Then, it is easy to get a homeomorphism $g_i: \Delta_i^n \twoheadrightarrow \Delta_i^n$ such that $g_ih(D_i^{m_i}) = \Delta_i^{m_i}$ and $g_i \mid \Delta = 1$. Furthermore, we may assume that $g_i \mid \partial\Delta_i^n$ is orientation preserving for if it is not we may follow g_i by an appropriate reflection of Δ_i^n.

Let A_i, $i = 1, 2$, be an annulus pinched at Δ, in particular, $A_i = (\partial\Delta_i^n \times I)/\sim$ where $(x, t) \sim (x, 0)$ if $x \in \Delta$, $t \in I$. Let $C_i\colon A_i \to S^n$, $i = 1, 2$, be homeomorphisms satisfying the following conditions:

(1) $C_i(A_i) \subset S^n - (\operatorname{int}\Delta_1^n \cup \operatorname{int}\Delta_2^n)$,

(2) $C_i((x, 1)) = x$ for $x \in \partial\Delta_i^n$, and

(3) $C_1(A_1) \cap C_2(A_2) = \Delta$.

(Thus, $C_i(A_i)$ is a certain pinched collar of $\partial\Delta_i^n$.)

It follows from [20] that $g_i\colon \Delta_i^n \twoheadrightarrow \Delta_i^n$ can be extended to $\Delta_i^n \cup C_i(A_i)$ such that $g_i \mid \partial(\Delta_i^n \cup C_i(A_i)) = 1$. Let g be the homeomorphism taking $\bigcup_{i=1,2}(\Delta_i^n \cup C_i(A_i))$ onto itself which is g_i on $\Delta_i^n \cup C_i(A_i)$. Then, g can be extended to S^n by way of the identity and it is clear that $e_1 = gh$ is the desired orientation preserving homeomorphism which flattens the pair $D_1^{m_1} \cup D_2^{m_2}$ since $gh(D_1^{m_1} \cup D_2^{m_2}) = \Delta_1^{m_1} \cup \Delta_2^{m_2}$.

THEOREM 2. *Let $\{\Delta_i^{m_i}\}$, $i = 1, 2, \cdots, p$ be simplexes such that $\Delta_i^{m_i}$ is of dimension m_i and such that $\bigcap_{i=1}^p \Delta_i^{m_i} = \Delta$ is a k-face of each $\Delta_i^{m_i}$. Let $f, g\colon \bigcup_{i=1}^p \Delta_i^{m_i} \to \operatorname{int} Q^n$ be PL embeddings into the connected n-dimensional PL manifold Q^n, $n \geqq m_i + 3$, $i = 1, 2, \cdots, p$. Then, there is a PL isotopy e_t of Q such that $e_0 = 1$ and $e_1 f = g$.*

If one can tame certain clusters of cells, then Theorem 2 can be used to unknot them. For instance, the following corollary follows from Theorem 1′ of [7].

COROLLARY. *Let $\{\Delta_i^{m_i}\}$, $i = 1, 2, \cdots, p$ be simplexes in the interior of the connected n-dimensional PL manifold Q^n, $m_i < (2/3)n - 1$, $i = 1, 2, \cdots, p$, such that $\bigcap_{i=1}^p \Delta_i^{m_i} = \Delta$ is a k-face of each $\Delta_i^{m_i}$. Let $f\colon \bigcup_{i=1}^p \Delta_i^{m_i} \to \operatorname{int} Q$ be an embedding which is locally flat on the open faces of $\Delta_i^{m_i}$, $i = 1, 2, \cdots, p$. Then, there is an isotopy e_t of Q such that $e_0 = 1$ and $e_1 f$ is the inclusion of $\bigcup_{i=1}^p \Delta_i^{m_i}$ into Q.*

Proof of Theorem 2. Let $\{v_j^i\}_{j=0}^{m_i}$ denote the vertices of $\Delta_i^{m_i}$ and let $\{v_j\}_{j=0}^k$ denote the vertices of Δ. Let $\Delta_i^{m_i-q}$ be the face of $\Delta_i^{m_i}$ spanned by the vertices $\{v_j^i\}_{j=0}^{m_i} - \{v_j\}_{j=k-q+1}^k$ and let Δ^{k-q} be the face of Δ spanned by $\{v_j\}_{j=0}^{k-q}$. Thus, for $0 \leqq q \leqq k$, Δ^{k-q} and $\Delta_i^{m_i-q}$, $i = 1, 2, \cdots, p$, are cones over $\Delta^{k-(q+1)}$ and $\Delta_i^{m_i-(q+1)}$, $i = 1, 2, \cdots, p$, respectively, with vertex v_{k-q}.

We will work with the following inductive statement.

q^{th} INDUCTIVE STATEMENT. *Let $f, g\colon \bigcup_{i=1}^p \Delta_i^{m_i-q} \to \operatorname{int} Q^n$ (n arbitrary) be PL embeddings. Then, there is a PL isotopy e_t of Q^n such* *that* $e_0 = 1$ *and* $e_1 f = g$.

The case $q = k + 1$ can be proved easily by using uniqueness of regular neighborhoods. Now we assume the $(q+1)$-inductive statement, where $0 \leqq q \leqq k$, and will establish the q^{th} inductive statement. Let N be a regular neighborhood of $f(\bigcup_{i=1}^{p} \Delta_i^{m_i - q}) \bmod f(\bigcup_{i=1}^{p} \Delta_i^{m_i-(q+1)})$ in Q (see [12]), and let N_* be a regular neighborhood of $g(\bigcup_{i=1}^{p} \Delta_i^{m_i-q}) \bmod g(\bigcup_{i=1}^{p} \Delta_i^{m_i-(q+1)})$ in Q. Then, there is a PL isotopy e_t^1 of Q such that $e_0^1 = 1$ and $e_1^1(N) = N_*$. But, $\partial(N_*)$ is a PL $(n-1)$-sphere and $e_1^1 f \mid \bigcup_{i=1}^{p} \Delta_i^{m_i-(q+1)}$ and $g \mid \bigcup_{i=1}^{p} \Delta_i^{m_i-(q+1)}$ are PL embeddings into $\partial(N_*)$. Hence, by the inductive assumption, there is a PL isotopy e_t^2 of $\partial(N_*)$ such that $e_0^2 = 1$ and $e_1^2 e_1^1 f \mid \bigcup_{i=1}^{p} \Delta_i^{m_i-(q+1)} = g \mid \bigcup_{i=1}^{p} \Delta_i^{m_i-(q+1)}$. It is now easy to extend e_t^2 over Q so that it is the identity at the zero level by using a PL bicollar of $\partial(N_*)$ in Q. Then,

$$e_1^2 e_1^1 f \colon \bigcup_{i=1}^{p} \Delta_i^{m_i-q} \longrightarrow N_* \quad \text{and} \quad g \colon \bigcup_{i=1}^{p} \Delta^{m_i-q} \longrightarrow N_*$$

are proper embeddings (in the sense of [16]) which agree on $\bigcup_{i=1}^{p} \Delta_i^{m_i-(q+1)}$ and so by Theorem 2 of [16] there is a PL isotopy e_t^3 of N_* which is the identity on $\partial(N_*)$ such that $e_0^3 = 1$ and $e_1^3 e_1^2 e_1^1 f = g$. Hence, we can extend e_t^3 to Q by way of the identity and we see that $e_t = e_t^3 e_t^2 e_t^1$ is the desired isotopy of Q.

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