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ON k-SHRINKING AND k-BOUNDEDLY COMPLETE BASES IN BANACH SPACES

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We study the problems of existence of bases of various order of shrinkingness and boundedly completeness in Banach spaces with bases and we give a characterization of quasi-reflexive Banach spaces with bases.

For Banach spaces with bases, R. C. James has given the following characterization of reflexivity ([4], Th. 1):

THEOREM (J). *A Banach space E with a basis $\{x_j\}$ is reflexive if and only if* (a) *the basis $\{x_j\}$ is boundedly complete* [3] *i.e., for every sequence of scalars $\{a_j\}$ such that $\sup_m \|\sum_{i=1}^m a_i x_i\| < +\infty$ the series $\sum_{i=1}^\infty a_i x_i$ is convergent and* (b) *the basis $\{x_j\}$ is shrinking* [3], *i.e., $\lim_{m\to\infty} \|f\|_m = 0$ for all functionals $f \in E^*$, where $\|f\|_m$ denotes the norm of the restriction of f to the closed linear subspace $[x_i]_{i=m+1}^\infty$ of E spanned by $x_{m+1}, x_{m+2}, \cdots$.*

From this theorem it follows that if E is a reflexive Banach space with a basis then all bases of E are shrinking. In [5], the question was raised whether the converse is also true, i.e., whether this property characterizes the reflexivity of Banach spaces with bases and a similar question for boundedly complete bases. Recently, M. Zippin has proved that the answer to both questions is affirmative ([10], Theorems 1 and 2).

The notions of shrinking bases and boundedly complete bases have the following extension ([7], Definitions 2 and 1): Let $k \geqq 0$ be an integer. A basis $\{x_j\}$ of a Banach space E is called *k-shrinking* (or *shrinking of order k*) if (a) in every $(k+1)$-dimensional linear subspace B_{k+1} of E^* there exists a nonzero element $f \in B_{k+1}$ such that $\lim_{m\to\infty} \|f\|_m = 0$; (b) there exists a $(k+1)$-dimensional linear subspace B'_{k+1} of E^* for which the above element f is unique up to a homothety. A basis $\{x_j\}$ of a Banach space E is called *k-boundedly complete* (or *boundedly complete of order k*) if (a) in every $(k+1)$-dimensional linear subspace P_{k+1} of $\mathscr{E} \equiv \{\{a_j\} \mid \sup_m \|\sum_{i=1}^m a_i x_i\| < +\infty\}$ there exists a nonzero element $\{a_j\} \in P_{k+1}$ such that the series $\sum_{i=1}^\infty a_i x_i$ is convergent; (b) there exists a $(k+1)$-dimensional linear subspace P'_{k+1} of $\mathscr{E}$ for which the above element $\{a_j\}$ is unique up to a homothety.

It is natural to ask the following questions ([6], Chapter II, Problems 2.2 and 2.6): Let $k > 0$ be an integer. Does every Banach space with a k-shrinking (k-boundedly complete) basis have a non-k-shrinking (respectively, non-k-boundedly complete) basis? (For $k = 0$, the answer is negative by the above theorem of James.) From the results which we shall prove below it will follow as a consequence that the answer to both questions is affirmative (see Corollary 1). Moreover, we shall prove that in general the orders of shrinkingness and boundedly completeness of bases can be both increased and decreased by 1 (Theorems 1 and 3).

P. Civin and B. Yood have introduced [2], in connection with an example of R. C. James [4], the following generalization of reflexivity: A Banach space E is called quasi-reflexive (of order n) if $\operatorname{codim}_{E^{**}} \pi(E) < +\infty$ ($\operatorname{codim}_{E^{**}} \pi(E) = n$), where π denotes the canonical embedding of E into E^{**}; we recall that if X is a Banach space and G a closed linear subspace of X, then, by definition $\operatorname{codim}_X G = \dim X/G$. The above theorem of James on characterization of reflexivity of Banach spaces with bases admits the following extension ([7], Th. 3): A Banach space E with a basis $\{x_j\}$ is quasi-reflexive of order n if and only if there exists an integer k, $0 \leqq k \leqq n$, such that $\{x_j\}$ is k-shrinking and $(n-k)$-boundedly complete.

Hence, if a Banach space E with a basis is quasi-reflexive of order $\leqq n$, then all bases of E are shrinking of order $\leqq n$ and it is natural to ask whether the converse is also true and a similar question for boundedly complete bases of order $\leqq n$. From our results it will follow that the answer to both questions is affirmative (see Theorem 2).

1. We begin with the following lemma, due essentially to M. Zippin ([10], Lemma 3).

LEMMA 1. *Let $\{x_n\}$ be a basis of a Banach space E and let*

$$y_n = \sum_{i=m_{n-1}+1}^{m_n} \alpha_i x_i \neq 0 \qquad (n = 1, 2, \cdots) , \tag{1}$$

where $0 = m_0 < m_1 < m_2 < \cdots$. Then there exists a basis $\{z_n\}$ of E such that

$$z_{m_n} = y_n \qquad (n = 1, 2, \cdots) , \tag{2}$$

$$[h_n] = [f_n] , \tag{3}$$

where $\{f_n\}, \{h_n\} \subset E^*$, $f_i(x_j) = h_i(z_j) = \delta_{ij}$.

Proof. By ([10], Lemma 3), there exists a basis $\{z_n\}$ of E satisfying (2) and such that

$$[z_i]_{i=m_{n-1}+1}^{m_n} = [x_i]_{i=m_{n-1}+1}^{m_n} \qquad (n = 1, 2, \cdots) . \tag{4}$$

Now, for $\{h_n\} \subset E^*$ with $h_i(z_j) = \delta_{ij}$ we have, by (4), $h_i(x_j) = 0$

($i = m_{n-1} + 1, \cdots, m_n$; $j \neq m_{n-1} + 1, \cdots, m_n$; $n = 1, 2, \cdots$), that is,

$$(5) \qquad [h_i]_{i=m_{n-1}+1}^{m_n} \subset ([x_j]_{j \neq m_{n-1}+1,\cdots,m_n})^{\perp} \qquad (n = 1, 2, \cdots) .$$

On the other hand, for $\{f_n\} \subset E^*$ with $f_i(x_j) = \delta_{ij}$ we have

$$(6) \qquad [f_i]_{i=m_{n-1}+1}^{m_n} = ([x_j]_{j \neq m_{n-1}+1,\cdots,m_n})^{\perp} \qquad (n = 1, 2, \cdots)$$

(see, e.g., [5], proof of Proposition 4).

From (5) and (6) we infer

$$[h_i]_{j=m_{n-1}+1}^{m_n} = [f_i]_{i=m_{n-1}+1}^{m_n} \qquad (n = 1, 2, \cdots) ,$$

whence (3), which completes the proof of Lemma 1.

REMARK 1. From Lemma 1 it follows that every block basic sequence $\{y_n\}$ with respect to a basis $\{x_n\}$ of a Banach space E can be extended to a basis $\{z_n\}$ of E such that the orders of shrinkingness and of boundedly completeness of $\{z_n\}$ remain the same as the corresponding orders of the initial basis $\{x_n\}$.

LEMMA 2. *Let $k \geqq 0$ be an integer. If a Banach space E has a k-boundedly complete basis $\{x_n\}$, then every subspace of finite codimension in E also has a k-boundedly complete basis.*

Proof. Let us first prove that for each $m \geqq 1$ the sequence $\{x_n\}_m^\infty$ is a k-boundedly complete basis of the subspace $[x_n]_m^\infty$ of codimension $m - 1$ of E.

Indeed, let

$$(7) \qquad \mathscr{E}_m = \{\{\alpha_n\}_m^\infty \mid \sup_{m \leqq n < \infty} \|\sum_{i=1}^{n} \alpha_i x_i\| < +\infty\}$$

and let Q_{k+1} be a $(k + 1)$-dimensional linear subspace of $\mathscr{E}_m$. Then

$$P_{k+1} = \{\{\underbrace{0, \cdots, 0}_{m-1}, \alpha_m, \alpha_{m+1}, \alpha_{m+2}, \cdots\} \mid \{\alpha_n\}_m^\infty \in Q_{k+1}\}$$

is a $(k + 1)$-dimensional linear subspace of

$$\mathscr{E} = \mathscr{E}_0 = \{\{\alpha_n\}_1^\infty \mid \sup_{1 \leqq n < \infty} \|\sum_{i=1}^{n} \alpha_i x_i\| < +\infty\} .$$

Hence, since $\{x_n\}$ is a k-boundedly complete basis of E, there exists a nonzero element $\{\underbrace{0, \cdots, 0}_{m-1}, \alpha_m, \alpha_{m+1}, \alpha_{m+2}, \cdots\} \in P_{k+1}$ such that the series $\sum_{i=m}^\infty \alpha_i x_i$ is convergent. Then $\{\alpha_n\}_m^\infty \in Q_{k+1}$ and the series $\sum_{i=m}^\infty \alpha_i x_i$ is convergent. Similarly, $\{x_n\}_m^\infty$ also satisfies condition (b) of the definition of k-boundedly complete bases and thus $\{x_n\}_m^\infty$ is a k-boundedly complete basis of $[x_n]_m^\infty$.

Since by a remark of C. Bessaga and A. Pelczynski [1] all subspaces of codimension m of E are isomorphic to each other it follows that every subspace of codimension m of E has a k-boundedly complete basis, which completes the proof of Lemma 2.

In particular, from Lemma 2 it follows (for $k = 0$) that every subspace of finite codimension of a Banach space E with a boundedly complete basis is isomorphic to a conjugate Banach space.

REMARK 2. A result similar to Lemma 2 for k-shrinking bases is also valid, but we shall not need it in the sequel. Let us also mention that if E has a k-boundedly complete (k-shrinking) basis $\{x_n\}$ then every superspace of E in which E is of finite codimension also has a k-boundedly complete (respectively, k-shrinking) basis ([7], proof of Theorem 5).

2. Now we can prove

THEOREM 1. *Let $k \geqq 0$ be an integer and let E be a Banach space which is not quasi-reflexive of order k.*

$1°$. *If E has a k-shrinking basis $\{x_n\}$, then E has a $(k+1)$-shrinking basis.*

$2°$. *If E has a k-boundedly complete basis $\{x_n\}$, then E has a $(k+1)$-boundedly complete basis.*

Proof. $1°$. Assume that $\{x_n\}$ is k-shrinking and E is not quasi-reflexive of order k. Then by ([7], Th. 3), $\{x_n\}$ is nonboundedly complete, whence by ([5], Th. 1, $2°$) $\{x_n\}$ admits a block basic sequence $\{y_n\}$ of type P. By Lemma 1 there exists a basis $\{z_n\}$ of E satisfying (2) and (3), where $\{f_n\}, \{h_n\} \subset E^*, f_i(x_j) = h_i(z_j) = \delta_{ij}$. Then, as observed by M. Zippin ([10], p. 77, proof of Theorem 1), the sequence

$$u_i = \begin{cases} z_i & \text{for } i \neq m_n \quad (n = 1, 2, \cdots) \\ \sum_{j=1}^{n} z_{m_j} & \text{for } i = m_n \quad (n = 1, 2, \cdots) \end{cases} \tag{8}$$

is a basis of E, with the associated coefficient functionals

$$g_i = \begin{cases} h_i & \text{for } i \neq m_n \quad (n = 1, 2, \cdots) \\ h_{m_n} - h_{m_{n+1}} & \text{for } i = m_n \quad (n = 1, 2, \cdots)\,. \end{cases} \tag{9}$$

We shall show that $\{u_n\}$ is a $(k+1)$-shrinking basis of E, which will complete the proof of $1°$.

Since $\{z_n\}$ is a basis of E and $\sup_n \|\sum_{j=1}^{n} z_{m_j}\| < +\infty$ (because $\{z_{m_n}\}$ is of type P), there exists by [9], a functional $\Phi \in E^{**}$ such that

$$\Phi(h_i) = \begin{cases} 0 & \text{for } i \neq m_n \quad (n = 1, 2, \cdots) \\ 1 & \text{for } i = m_n \quad (n = 1, 2, \cdots) . \end{cases}$$

Hence $\Phi(h_{m_1}) = 1$, $\Phi(g_i) = 0$ $(i = 1, 2, \cdots)$ and therefore $h_{m_1} \notin [g_i]_1^\infty$. Consequently, by (9),

$$[f_n]_1^\infty = [h_n]_1^\infty = [g_i]_1^\infty \oplus [h_{m_1}]$$

and hence $\operatorname{codim}_{E^*} [g_i]_1^\infty = k + 1$, i.e., $\{u_n\}$ is a $(k + 1)$-shrinking basis of E.

2°. Assume that $\{x_n\}$ is k-boundedly complete and E is not quasi-reflexive of order k. Then by ([7], Th. 4), $\{f_n\}$ is a k-shrinking basis of $[f_n]$. Observe now that $[f_n]$ is not quasi-reflexive of order k since otherwise, by ([7], Th. 3), $\{f_n\}$ would be k-shrinking and boundedly complete whence by ([7], Th. 4), $\{x_n\}$ would be shrinking and k-boundedly complete and hence by ([7], Th. 3), E would be quasi-reflexive of order k, in contradiction with our hypothesis. Therefore, by part 1° proved above, $[f_n]$ has a $(k + 1)$-shrinking basis $\{g_n\}$ and hence, the coefficient functionals associated to $\{g_n\}$ constitute a $(k + 1)$-boundedly complete basis of a subspace of codimension $k + 1$ in $[f_n]$.* Consequently, by Remark 2 (second part), $[f_n]^*$ has a $(k + 1)$-boundedly complete basis. Since $\{x_n\}$ is k-boundedly complete, by ([7], Th. 1) we have $\operatorname{codim}_{[f_n]^*} \varphi(E) = k < +\infty$, where φ denotes the canonical mapping of E into $[f_n]^*$ (i.e., $[\varphi(x)](f) = f(x)$ for all $x \in E, f \in [f_n]$) and hence, by Lemma 2, the subspace $\varphi(E)$ of $[f_n]^*$ also has a $(k + 1)$-boundedly complete basis. Since by ([6], Th. 1.13), φ is an isomorphism of E into $[f_n]^*$ it follows that E has a $(k + 1)$-boundedly complete basis, which completes the proof.

REMARK 3. In the particular case when $k = 0$ the above proof of Theorem 1, 1° reduces essentially to ([10], proof of Theorem 1); we establish here somewhat more, namely, that the basis $\{u_n\}$ is 1-shrinking (in [10] it was only proved that $\{u_n\}$ is nonshrinking). Let us also observe that in the particular case when $k = 0$, the above proof of Theorem 1, 2° gives a simpler proof of ([10], Th. 2) (namely, if $\{x_n\}$ is a boundedly complete nonshrinking basis of E then by ([5], Proposition 5), the sequence of coefficient functionals $\{f_n\}$ is a shrinking non-boundedly complete basis of $[f_n]$, whence by ([10], Th. 1), sharpened as above, $[f_n]$ has a 1-shrinking basis $\{g_n\}$ and consequently $E \cong [f_n]^*$ has 1-boundedly complete basis, which is, of course, nonboundedly complete).

REMARK 4. Both hypotheses in Theorem 1, 1° (respectively, Theorem 1, 2°) are essential. Indeed, the spaces l^1 and c_0 are not quasi-reflexive of any order k but the space $E = l^1$ has no k-shrinking basis for any k and the space $E = c_0$ has no k-boundedly complete basis for any k. Furthermore, if E is quasi-reflexive of order k then by ([7], Corollary 1), every basis of E is k_1-shrinking and k_2-boundedly complete with $k_1, k_2 \leqq k$.

COROLLARY 1. *Let $k > 0$ be an integer and let E be a Banach space.*

1°. *If E has a k-shrinking basis then E has a non-k-shrinking basis as well.*

2°. *If E has a k-boundedly complete basis then E has a non-k-boundedly complete basis as well.*

Proof. 1°. If E is quasi-reflexive of order k and if all bases of E are k-shrinking, then by ([7], Th. 3), all bases of E are boundedly complete, whence by ([10], Th. 2), E is reflexive. Consequently by ([4], Th. 1), all bases of E are shrinking and hence non-k-shrinking (because $k > 0$). If E is not quasi-reflexive of order k, then by Theorem 1, 1° it must have a $(k + 1)$-shrinking basis.

The proof of 2° is similar.

REMARK 5. Corollary 1 gives the answer to ([6], Problems 2.2 and 2.6). Corollary 1 may be also regarded as a sharpening of the following results of L. Sternbach ([8], Theorems 3.11, 15): If $k > 0$ and E has a k-shrinking (k-boundedly complete) basis, then E has a $(k - 1)$-shrinking (respectively, 1-boundedly complete) basic sequence.

COROLLARY 2. *Let $k \geqq 0$ be an integer and let E be a Banach space which is not quasi-reflexive.*

1°. *If E has a k-shrinking basis then E has an n-shrinking basis for all $n \geqq k$.*

2°. *If E has a k-boundedly complete basis then E has an n-boundedly complete basis for all $n \geqq k$.*

THEOREM 2. *Let $n \geqq 0$ be an integer and let E be a Banach space with a basis. Then*

1°. *E is quasi-reflexive of order n if and only if all bases of E are k-shrinking with $k \leqq n$ depending on the basis and there exists an n-shrinking basis $\{x_n\}$ of E.*

2°. *E is quasi-reflexive of order n if and only if all bases of E are k-boundedly complete with $k \leqq n$ depending on the basis and* *there exists an n-boundedly complete basis of E.*

Moreover, in this case, for each k with $0 \leqq k \leqq n$, E has a k-shrinking basis and a k-boundedly complete basis.

Proof. 1°. If E is quasi-reflexive of order n then, by ([7], Corollary 1), every basis of E is k-shrinking with $k \leqq n$. Conversely, assume now that the conditions of Theorem 2 are satisfied but E is not quasi-reflexive of order n. Then, since E has an n-shrinking basis, by Theorem 1, 1°, E also has an $(n+1)$-shrinking basis, in contradiction with the assumption that all bases of E are k-shrinking with $k \leqq n$.

The proof of 2° is similar.

Assume now again that E is quasi-reflexive of order n and let $\{x_j\}$ be an arbitrary basis of E. Then by ([7], Corollary 1), $\{x_j\}$ is k_0-shrinking where $k_0 \leqq n$. Hence, by Theorem 1, 1°, E has an n-shrinking basis. Consequently by virtue of ([7], Th. 5), E^* has an n-boundedly complete basis, say $\{h_j\}$. Since E^* is also quasi-reflexive of order n ([2], Lemma 3.4), by ([7], Corollary 1), $\{h_n\}$ is shrinking. Therefore, by Theorem 1, 1°, E^* has a k-shrinking basis for every k with $0 \leqq k \leqq n$ and hence by ([7], Th. 5), E has a k-boundedly complete basis for every k with $k \leqq n$. Consequently, by ([7], Corollary 1), E has an $(n-k)$-shrinking basis for every k with $0 \leqq k \leqq n$, which completes the proof of Theorem 2.

Let us observe that the last part of Theorem 2 is also a consequence of Theorem 1 and of Theorem 3 below.

In the particular case when $k = 0$, the necessity part of Theorem 2 reduces to results of R. C. James [4] and the sufficiency part reduces to results of M. Zippin ([10], Theorems 1 and 2).

THEOREM 3. *Let $k > 0$ be an integer and let E be a Banach space.*

1°. *If E has a k-shrinking basis, then E has a $(k-1)$-shrinking basis and hence also a k_1-shrinking basis for every k_1 with $0 \leqq k_1 \leqq k$.*

2°. *If E has a k-boundedly complete basis, then E has a $(k-1)$-boundedly complete basis and hence also a k_1-boundedly complete basis for every k_1 with $0 \leqq k_1 \leqq k$.*

Proof. 1°. Let $\{x_n\}$ be a k-shrinking basis of E. Then, since $k > 0$, $\{x_n\}$ is nonshrinking and hence there exist an $f \in E^*$ and a block basic sequence $\{y_n\}$ such that $0 < \inf_n \| y_n \| \leqq \sup_n \| y_n \| < \infty$, $f(y_j) = 1$ $(j = 1, 2, \cdots)$. By Lemma 1 there exists a basis $\{z_n\}$ of E satisfying (2) and (3), where $\{f_n\}, \{h_n\} \subset E^*$, $f_i(x_j) = h_i(z_j) = \delta_{ij}$. Moreover, as observed by M. Zippin ([10], proof of Theorem 2), $\{z_n\}$ can be chosen so that

$$f(z_i) = 0 \qquad \text{for } i \neq m_n \ (n = 1, 2, \cdots) \tag{10}$$

and in this case the sequence

$$u_i = \begin{cases} z_i & \text{for } i \neq m_n \ (n = 1, 2, \cdots) \\ z_{m_n} - z_{m_{n-1}} & \text{for } i = m_n \ (n = 1, 2, \cdots) \end{cases} \tag{11}$$

is a basis of E, with the associated coefficient functionals

$$g_i = \begin{cases} h_i & \text{for } i \neq m_n \ (n = 1, 2, \cdots) \\ f & \text{for } i = m_1 \\ f - \sum_{j=1}^{n-1} h_{m_j} & \text{for } i = m_n \ (n = 2, 3, \cdots) . \end{cases} \tag{12}$$

Then, by (12) and (3) we have $[g_n] = [f] + [h_n]_1^\infty = [f] + [f_n]_1^\infty$, whence, since $f \notin [f_n]_1^\infty$ (by the choice of f and $\{y_n\}$), we infer $\operatorname{codim}_{E^*} [g_n] = \operatorname{codim}_{E^*} [f_n] - 1 = k - 1$, and thus $\{u_n\}$ is a $(k-1)$-shrinking bosis of E.

2°. Let $\{x_n\}$ be a k-boundedly complete basis of E. Then, by ([7], Th. 4), $\{f_n\}$ is a k-shrinking basis of $[f_n]$. Therefore, by part 1° proved above, $[f_n]$ has a $(k-1)$-shrinking basis $\{g_n\}$. Hence, as in the final part of the above proof of Theorem 1, it follows that E has a $(k-1)$-boundedly complete basis, which completes the proof.

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