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INVOLUTIONS OF THE 3-SPHERE WHICH FIX 2-SPHERES

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We show in that the space of involutions of the 3-sphere whose fixed point sets are 2-spheres is pathwise and locally pathwise connected. From Smith theory it is known that these involutions are orientation reversing. The fixed point sets need not be tame 2-spheres; Bing and others have many examples of involutions of the 3-sphere whose fixed point sets are wild 2-spheres.

In order to prove the connectivity theorem (§ 6) just mentioned we develop an approximation theory for involutions of the 3-sphere in §'s 3, 4. Some of the results there are interesting in their own right. Corollary 3.1 states that involutions which fix wild 2-spheres can be approximated by involutions which fix tame 2-spheres. Theorem 4.6 states that if an involution g fixing a 2-sphere R approximates an involution f fixing a 2-sphere S very closely then R approximates S. We also make use of Theorem 5.2, a modified version of the Alexander deformation theorem, which states that if the boundary of a 3-cell C in the 3-sphere approximates a given 2-sphere very closely then very small homeomorphisms of C onto itself which fix Bd (C) can be deformed back to the identity by small isotopies of C which fix Bd (C).

NOTATION. Most of our notation conventions are discussed in [12]. We mention a few items here.

With one exception which we note later in this paragraph ρ denotes the metric on a metric space. In the case of Euclidean space E^n and its subspaces ρ is given by $\rho(x, y) = \{\Sigma(x_i - y_i)^2\}^{1/2}$ where $x = (x_1, \cdots, x_n)$, $y = (y_1, \cdots, y_n)$. For spaces X, Y, $\mathscr{H}(X, Y)$ denotes the space of homeomorphisms of X into Y with the compact open topology. If X is compact and Y metric $\mathscr{H}(X, Y)$ is a metric space with metric d given by $d(f, g) = \sup\{\rho(f(x), g(x)) \mid x \in X\}$.

An *isotopy* $H_t(a \leqq t \leqq b)$ of a space X into itself is a continuous, one parameter family of homeomorphisms of X into itself or alternately a parameterized path in $\mathscr{H}(X, X)$. In case X is a metric space we say H_t is an ε-*isotopy* provided the track under H_t of each point x of X —$\{H_t(x) \mid t \in [a, b]\}$— has diameter less than ε. An ε-*set* in a metric space X is a subset of X of diameter less than ε. If $x \in X$ then $N(X, x, \varepsilon)$ denotes $\{y \in X \mid \rho(x, y) < \varepsilon\}$. If $Y \subseteqq X$ then $N(X, Y, \varepsilon)$ denotes $\bigcup \{N(X, y, \varepsilon) \mid y \in Y\}$.

We denote the 3-sphere, the unit sphere in Euclidean 4-space E^4, by Σ. We denote by $\mathscr{F}$ the subspace of $\mathscr{H}(\Sigma, \Sigma)$ consisting of those involutions of Σ whose fixed point sets are 2-spheres and by $\mathscr{G}$ the subspace of $\mathscr{F}$ consisting of those involutions whose fixed point sets are tame 2-spheres. If S is a 2-sphere in Σ, $\mathscr{F}(S)$, $\mathscr{G}(S)$ denote the subspaces of $\mathscr{F}$, $\mathscr{G}$ consisting of elements which have S for a fixed point set. A *crumpled cube* is a space homeomorphic to the closure of the bounded component of the complement of a 2-sphere in E^3.

We assume the reader is familiar with the works of Moise and Bing on the triangulation theorem and Hauptvermutung for 3-manifolds [4, 6, 17, 18, 19] as well as some of the elementary consequences of these works—for example from [17, Corollary to Theorem 1] that tame 2-spheres bound pairs of 3-cells in Σ and that disjoint tame 2-spheres in Σ cobound annuli.

The following theorem which combines special cases of [12, § 9] and [13, § 8] will be applied in several places in this paper:

THEOREM 1.1. *Suppose M is a* (pwl) *3-manifold without boundary, K is a compact polyhedron with no local cut points, f is a homeomorphism of K into M, and $\varepsilon > 0$.*

There is a $\delta > 0$ such that:

(a) *if f_0 and f_1 are* (pwl) *homeomorphisms of K onto tame sets in M with $d(f, f_e) < \delta (e = 0, 1)$, then there is a* (pwl) *ε-isotopy $H_t (0 \leqq t \leqq 1)$ of M onto itself such that $H_0 = I$ (Identity), $H_t = I$ on $M - N(M, f(K), \varepsilon)$, and $H_1 f_0 = f_1$, and*

(b) *if K is a 2-sphere and if f_0 and f_1 are homeomorphisms of K onto disjoint tame sets in M so that $d(f, f_e) < \delta\ (e = 0, 1)$, then there is a homeomorphism g of $K \times [0, 1]$ into M such that $g(x, e) = f_e(x)\ (x \in K, e = 0, 1)$ and $\rho(f(x), g(x, t)) < \varepsilon (x \in K, t \in [0, 1])$.*

We wish to thank Dean Montgomery for pointing out certain elementary facts about equivalences of involutions.

2. Converting isotopies of Σ and crumpled cubes into paths in $\mathscr{F}$.

Here we introduce an isotopy construction to be used in § 6. We omit proofs of Lemmas 2.1 and 2.2.

LEMMA 2.1. *Suppose $f \in \mathscr{F}(S)$ and S bounds crumpled cubes C and D in Σ.*

Then the following statements hold:

(1) *If $g \in \mathscr{F}(S)$ with $g | C = f | C$, then $g = f$,*

(2) *If h is a homeomorphism of Σ, then $hfh^{-1} \in \mathscr{F}(h(S))$, and*

(3) *If $g \in \mathscr{F}(S)$ and h is a homeomorphism of Σ given by $h | C = gf$ and $h | D = I$, then $g = hfh^{-1}$.*

LEMMA 2.2. *Suppose $f \in \mathscr{F}(S)$ and $H_t (0 \leqq t \leqq 1)$ is an isotopy of Σ.*

Then $h_t(0 \leqq t \leqq 1)$ *given by* $h_t = H_t f H_t^{-1}$ *is a path in* $\mathscr{F}$ *with* $h_t \in \mathscr{F}(H_t(S))$ $(t \in [0, 1])$.

LEMMA 2.3. *Suppose* $f \in \mathscr{F}$ *and* $\varepsilon > 0$.

There is a $\delta > 0$ *such that if* $g \in N(\mathscr{F}, f, \delta)$ *and* $H_t(0 \leqq t \leqq 1)$ *is a* δ*-isotopy of* Σ *with* $H_0 = I$, *then* $h_t(0 \leqq t \leqq 1)$ *given by* $h_t = H_t g H_t^{-1}$ *is a path in* $N(\mathscr{F}, f, \varepsilon)$.

Proof. Choose $\delta < \varepsilon/3$ so that the image under f of each δ-subset of Σ has diameter less than $\varepsilon/3$.

Let g and H_t be given as in the hypothesis. For each $x \in \Sigma$, $t \in [0, 1]$, dia $(x \cup H_t^{-1}(x)) < \delta$ so $g(x \cup H_t^{-1}(x)) \subseteqq N(\Sigma, f(x \cup H_t^{-1}(x)), \delta) \subseteqq N(\Sigma, f(x), \varepsilon/3 + \delta)$ and

$$H_t(g(x \cup H_t^{-1}(x))) \subseteqq N(\Sigma, f(x), \varepsilon/3 + 2\delta) \subseteqq N(\Sigma, f(x), \varepsilon) .$$

Thus $\rho(f(x), h_t(x)) < \varepsilon$ $(x \in \Sigma, t \in [0, 1])$ and $h_t(0 \leqq t \leqq 1) \subseteqq N(\mathscr{F}, f, \varepsilon)$.

3. $\mathscr{G}$ **is dense in** $\mathscr{F}$. Bing and Wu [3, 9, 22] have shown that there are uncountably many inequivalent involutions in $\mathscr{F} - \mathscr{G}$. (See also [2].) In fact Bing's methods in [9, § 2] can be used to show that $\mathscr{F} - \mathscr{G}$ is dense in $\mathscr{F}$. We use Bing's approximation theorem for spheres to show that $\mathscr{G}$ is also dense in $\mathscr{F}$.

THEOREM 3.1. *Suppose* $f \in \mathscr{F}(S)$ *and* $\varepsilon > 0$.

There is a $\delta > 0$ *such that if* R *is a tame* 2*-sphere in* Σ *homeomorphically within* δ *of* S, *then there is an element* $g \in \mathscr{F}(R)$ *such that* $d(f, g) < \varepsilon$.

Proof. Let $\varepsilon_1 > 0$ be sufficiently small so that $\varepsilon_1 < \varepsilon$ and $d(f, fh) < \varepsilon$ for every ε_1-homeomorphism h of Σ. Let ϕ be a homeomorphism of a polyhedral 2-sphere K onto S. Let $\varepsilon_2 < \varepsilon_1$ correspond to δ in Theorem 1.1 for the substitution $(\Sigma \to M, K \to K, \phi \to f, \varepsilon_1 \to \varepsilon)$. Choose a positive number $\delta < \varepsilon_2/2$ so small that $\rho(x, f(x)) < \varepsilon_2/2$ $(x \in N(\Sigma, S, \delta))$.

Let R be a tame 2-sphere homeomorphically within δ of S. There is a homeomorphism ϕ_1 of K onto R such that $d(\phi, \phi_1) < \delta$. Set $\phi_0 = f\phi_1$. From the conditions on δ we have $d(\phi_1, \phi_0) < \varepsilon_2/2$ so $d(\phi, \phi_0) < \varepsilon_2$. From Theorem 1.1 there is an ε_1-isotopy $H_t(0 \leqq t \leqq 1)$ of Σ such that $H_0 = I$ and $H_1\phi_0 = \phi_1$. That is, $H_1 f \mid R = I$. Now R bounds 3-cells C and D, and $H_1 f$ switches these 3-cells.

Define g by $H_1 f$ on D and by fH_1^{-1} on C. Clearly $g \in \mathscr{F}(R)$. For $x \in D$ we have $\rho(f(x), H_1 f(x)) < \varepsilon_1 < \varepsilon$. For $x \in C$ we have

$$\rho(f(x), fH_1^{-1}(x)) < \varepsilon$$

by the definition of ε_1.

From [10, Th. 1.1] we get:

COROLLARY 3.1. *The subspace* $\mathscr{G}$ *is dense in* $\mathscr{F}$.

4. Homeomorphic closeness of fixed point sets.

LEMMA 4.1. *Suppose* $f \in \mathscr{F}(S)$ *and* $\varepsilon > 0$.

There is a $\delta > 0$ *such that if* $g \in \mathscr{F}(R)$ *with* $d(f, g) < \delta$ *then* (i) $R \subseteqq N(\Sigma, S, \varepsilon)$, (ii) *R separates two points of* $\Sigma - N(\Sigma, S, \varepsilon)$ *if and only if S separates them, and* (iii) $\operatorname{dia}(R) > \operatorname{dia}(S) - \varepsilon$.

Proof. Let S separate Σ into components U and V. Let Y_1, Z_1 be nonempty, compact subsets of U, V such that $\Sigma - N(\Sigma, S, \varepsilon) \subseteqq Y_1 \cup Z_1$ and for each $x \in S$, $N(\Sigma, x, \varepsilon/6) \cap Y_1 \neq \emptyset$, $N(\Sigma, x, \varepsilon/6) \cap Z_1 \neq \emptyset$. Set $\varepsilon_1 = \inf\{\rho(x, f(x)) \mid x \in Y_1 \cup Z_1\}$. Choose $\varepsilon_2 > 0$ so that

$$\sup\{\rho(x, f(x)) \mid x \in N(\Sigma, S, \varepsilon_2)\} < \varepsilon_1/4 .$$

Let Y, Z be compact, connected subsets of U, V such that

$$(\Sigma - N(\Sigma, S, \varepsilon_2)) \subseteqq Y \cup Z .$$

Set $\delta = 1/4 \inf\{\rho(x, f(x)) \mid x \in Y \cup Z\}$.

Let $g \in \mathscr{F}(R)$ with $d(f, g) < \delta$. For each $x \in Y \cup Z$, $\rho(x, g(x)) \geqq 4\delta - \delta$. Thus $R \subseteqq \Sigma - (Y \cup Z) \subseteqq N(\Sigma, S, \varepsilon_2) \subseteqq N(\Sigma, S, \varepsilon)$. Now R bounds crumpled cubes C and D with $Y \subseteqq C$, and g switches C and D. Suppose R does not separate Y from Z. Then

$$Y \cup Z \subseteqq C,\ D \subseteqq N(\Sigma, S, \varepsilon_2) ,$$

and $\rho(x, g(x)) < \varepsilon_1/4 + \delta < \varepsilon_1/2\ (x \in D)$. Let $c \in Y_1 \cup Z_1$, and let $d = g(c) \in D$. Then $c = g(d)$, and we have $\rho(c, d) \geqq \varepsilon_1 - \delta \geqq 3/4\varepsilon_1$ because $c \in Y_1 \cup Z_1$, but $\rho(d, c) < \varepsilon_1/2$ because $d \in D$. From the contradiction we conclude that R separates Y from Z. Thus R separates two points of $\Sigma - N(\Sigma, S, \varepsilon)$ if and only if S separates them.

Let p, q be points of S such that $\rho(p, q) = \operatorname{dia}(S)$, and let $y_1, y_2 \in Y$, $z_1, z_2 \in Z$ be points such that $\rho(p, y_1 \cup z_1) < \varepsilon/6$, $\rho(q, y_2 \cup z_2) < \varepsilon/6$. We have $\rho(y_1, z_1) < \varepsilon/3$, $\rho(y_2, z_2) < \varepsilon/3$. Since R separates Y from Z there are points p', q' of R on the shorter segments of the great circles through y_1, z_1 and y_2, z_2. We have $\rho(p, p') < \varepsilon/2$, $\rho(q, q') < \varepsilon/2$ so $\operatorname{dia}(R) \geqq \rho(p', q') > \operatorname{dia}(S) - \varepsilon$.

LEMMA 4.2. *Suppose* $f \in \mathscr{F}(S)$ *and* $\varepsilon > 0$.

There is a $\delta > 0$ *such that if* $g \in \mathscr{F}(R)$ *with* $d(f, g) < \delta$, *then every simple closed curve of diameter less than* δ *on R bounds an* ε*-disk on R.*

Proof. We suppose $\operatorname{dia}(S) > 4\varepsilon$. Choose $\varepsilon_1 > 0$ so that every $5\varepsilon_1$-subset of Σ is contained in the interior of a 3-cell of diameter less than $\varepsilon/3$. Choose $\varepsilon_2 > 0$ so that $\rho(x, f(x)) < \varepsilon_1 (x \in N(\Sigma, S, \varepsilon_2))$. Choose $\varepsilon_3 > 0$ so that $\varepsilon_3 < \varepsilon_2/2$ and

$$\varepsilon_3 < \inf \{\rho(x, f(x)) \mid x \in \Sigma - N(\Sigma, S, \varepsilon_2)\} .$$

Choose δ from Lemma 4.1 for f and ε_3 sufficiently small so that every 3δ-subset of Σ is contained in the interior of a 3-cell of diameter less than $\varepsilon_2/2$.

Let $g \in \mathscr{F}(R)$ with $d(f, g) < \delta$, and let J be a simple closed curve on R with $\operatorname{dia}(J) < \delta$. Use Theorem 3.1 and [10, Th. 1.1] to get a tame 2-sphere R' in Σ, a δ-homeomorphism ϕ of R onto R', and an element $g' \in \mathscr{F}(R')$ such that $d(f, g') < \delta$. From Lemma 4.1, $\operatorname{dia}(R') > 3\varepsilon$. Use [4, 6, 19] to give Σ a triangulation T in which R' is a polyhedron. The set $J' = \phi(J)$ has diameter less than 3δ and so lies in the interior of a 3-cell C of diameter less than $\varepsilon_2/2$. From [6] we can suppose that $\operatorname{Bd}(C)$ is a polyhedron in T and is in general position with respect to R'. Each component of $\operatorname{Bd}(C) \cap R'$ is a simple closed curve which we claim bounds an $\varepsilon/3$-disk on R'.

Let L be a component of $\operatorname{Bd}(C) \cap R'$. Now L bounds an $\varepsilon_2/2$-disk D on $\operatorname{Bd}(C)$. Let $D_1, \cdots, D_m$ denote the closures of the components of $\operatorname{Int}(D) - R'$. From Lemma 4.1, $R' \subseteqq N(\Sigma, S, \varepsilon_3)$ so $D \subseteqq N(\Sigma, S, \varepsilon_2)$, $\rho(x, g'(x)) < \varepsilon_1 + \delta \ (x \in D)$, and $\operatorname{dia}(D \cup g'(D)) < \varepsilon_2 + 2\varepsilon_1 + 2\delta < 5\varepsilon_1$. From the choice of ε_1, $D \cup g'(D)$ is contained in an open 3-cell U of diameter less than $\varepsilon/3$. Because $\operatorname{Int}(D)$ is in general position with respect to R', each $D_i \cup g'(D_i)$ is a surface which bounds a 3-manifold Q_i in U. Furthermore $Q_i \cap R'$ is a surface F_i whose boundary is $\operatorname{Bd}(D_i)$. We show that L is contained in a disk in $R' \cap U$.

Some D_i, say D_k, is a disk so Q_k is a 3-cell and F_k is a disk. Either $L \subseteqq F_k$ or $L \cap F_k = \varnothing$. In the first case L bounds a subdisk of F_k in R' which has diameter less than $\varepsilon/3$. In the second case locate an inner most simple closed curve L_j of $D \cap F_k$ in F_k, cut out the disk D_{kj} it bounds in D, replace that disk by the disk F_{kj} which L_j bounds in F_k, and push a neighborhood of F_{kj} in the adjusted disk slightly to one side of R' to obtain a new polyhedral disk $D(1)$ with boundary L such that $\operatorname{Int}(D(1))$ is in general position with respect to R', $\operatorname{Int}(D(1)) \cap R'$ consists of a proper subcollection of the simple closed curves of $\operatorname{Int}(D) \cap R'$, and $D(1) \cup g'(D(1)) \subseteqq U$. After a sufficient number of repetitions of this process we arrive at a polyhedral disk $D(n)$ such that $D(n) \cap R' = \operatorname{Bd}(D(n))$, $D(n) \cup g'(D(n))$ bounds a 3-manifold P in U, and $L \subseteqq P$. But then $F(n) = R' \cap P$ is an $\varepsilon/3$-disk, and L bounds a subdisk of $F(n)$ in R'.

Let $K_1, \cdots, K_r$ denote the $\varepsilon/3$-disks on R' which the simple closed curves on $\operatorname{Bd}(C) \cap R'$ bound. Since $\operatorname{dia}(C \cup (\cup K_i)) < \varepsilon$ and $\operatorname{dia}(R') >$

3ε, the set $R' - \cup K_i$ is connected, has diameter greater than ε, and thus does not intersect C. This shows that J' is contained in one of the disks K_i and so bounds an $\varepsilon/3$-disk K on R'. Then J bounds the $\varepsilon/3 + 2\delta < \varepsilon$-disk $\phi^{-1}(K)$ on R.

LEMMA 4.3. *Suppose $f \in \mathscr{F}(S)$, A is a tame arc in Σ which pierces S at a point p and otherwise fails to intersect S, and $\varepsilon > 0$.*

There is an $\eta > 0$ such that if C is a tame 3-cell of diameter less than η whose interior contains p and which intersects A in an unknotted arc A' that spans Bd (C), *then the following statement is valid:*

There is a $\delta > 0$ such that if $g \in \mathscr{G}(R)$ with $d(f, g) < \delta$ and $R \cap$ Bd (C) *is a finite collection of simple closed curves at which R crosses* Bd (C), *then $R \cap A \subseteqq$* Int (A'), *there is a component U of $R - C$ such that each component of $R - U$ is an ε-disk, and exactly one component of* Cl $(U) \cap$ Bd (C) *separates the two endpoints of* Bd (A') *on* Bd (C).

Proof. We suppose dia $(S) \geqq 4\varepsilon$. Complete A to a tame, unknotted simple closed curve L in Σ. Let $\varepsilon_1 < \varepsilon$ be a positive number such that $N(\Sigma, p, 2\varepsilon_1) \cap L \subseteqq A$. Let η correspond to δ in Lemma 4.2 for the substitution $f \rightarrow f$, $\varepsilon_1 \rightarrow \varepsilon$.

Let C be a tame 3-cell with the properties described in the hypothesis of the lemma. Let $\varepsilon_2 > 0$ be a number such that

$$N(\Sigma, S, 5\varepsilon_2) \cap A \subseteqq \mathrm{Int}\,(A')$$

where $A' = C \cap A$ and $N(\Sigma, p, 5\varepsilon_2) \subseteqq \mathrm{Int}\,(C)$. Choose $\varepsilon_3 > 0$ less than ε_2 so that $\rho(x, f(x)) < \varepsilon_2\,(x \in N(\Sigma, S, \varepsilon_3))$. Choose $\varepsilon_4 > 0$ so that $\varepsilon_4 < \varepsilon_3$ and $N(\Sigma, S, \varepsilon_4) \cap A$ is contained in an arc A'' in $N(\Sigma, p, \varepsilon_3)$. Choose $\delta < \varepsilon_4$ from Lemma 4.1 for f, ε_4.

Let g be given as in the hypothesis of the lemma. It follows from the conditions on η that each component of $R \cap \mathrm{Bd}\,(C)$ bounds an ε_1-disk on R. By throwing away disks contained in the interiors of others we find mutually exclusive ε_1-disks $F_1, \cdots, F_m$ such that each $\mathrm{Bd}\,(F_i) \subseteqq \mathrm{Bd}\,(C)$ and $R \cap \mathrm{Bd}\,(C) \subseteqq \bigcup F_i$. From the conditions on ε_1 no F_i intersects $L - A$, and from the conditions on δ, $R \cap A \subseteqq \mathrm{Int}\,(A')$. Set $U = R - \bigcup F_i$. Now U is connected so either $U \subseteqq \mathrm{Int}\,(C)$ or $U \cap C = \varnothing$. In the first case $\mathrm{dia}\,(R) < 2\varepsilon_1 + \eta < 3\varepsilon_1 < \mathrm{dia}\,(S) - \varepsilon$. But from the choice of δ, $\mathrm{dia}\,(R) > \mathrm{dia}\,(S) - \varepsilon$ so $U \cap C = \varnothing$. Because A pierces S the endpoints of A' lie in different components of $\Sigma - S$. Lemma 4.1 shows that the endpoints of A' also lie in different components of $\Sigma - R$.

Suppose no $\mathrm{Bd}\,(F_i)$ separates the endpoints of A' on $\mathrm{Bd}\,(C)$. From

[7, Th. 7.3] there is a homeomorphism of Σ onto itself which is the identity on $L - A'$ and pulls R entirely off C. But this contradicts the fact that R separates the endpoints of A'. Suppose $\mathrm{Bd}(F_i)$ and $\mathrm{Bd}(F_j)$ $(i \neq j)$ both separate the endpoints of A' on $\mathrm{Bd}(C)$. Both $\mathrm{Bd}(F_i)$ and $\mathrm{Bd}(F_j)$ link L, and both F_i and F_j fail to intersect $L - \mathrm{Int}(A')$ so they both intersect $\mathrm{Int}(A')$. Thus there are distinct components H and K of $R \cap C$ and there is a subarc B of A' with endpoints $r \in H$ and $s \in K$ so that $\mathrm{Int}(B)$ fails to intersect R. The conditions on δ show that $B \subseteqq A'' \subseteqq N(\Sigma, p, \varepsilon_3)$. Furthermore

$$\rho(x, g(x)) \leqq \delta + \varepsilon_2 \leqq 2\varepsilon_2 (x \in N(\Sigma, p, \varepsilon_3))$$

so $g(A'') \subseteqq N(\Sigma, p, 5\varepsilon_2) \subseteqq \mathrm{Int}(C)$. Now H separates C into components U and V. Suppose $K \subseteqq U$. Because g switches the 3-cells bounded by R, $g(B - R) \subseteqq V$. But this is nonsense for $g(s) = s$. From the contradiction we conclude that exactly one $\mathrm{Bd}(F_i)$ separates the endpoints of A' on $\mathrm{Bd}(C)$.

The following lemma is essentially Theorem 6.1 of [11]. It is obtained by changing E^3 to Σ, the disk D to a 2-sphere S, introducing a triangulation of Σ, and making a few small adjustments in the proof of the theorem—one of them is pushing the triangulation of Σ keeping S fixed rather than the other way around.

LEMMA 4.4. *Suppose S is a 2-sphere in Σ and $\varepsilon > 0$.*

There is a triangulation T of Σ with i-skeleton T_i and mesh less than ε, there is a tame Sierpinski curve X on S, and there is an ε-homeomorphism g of S onto a tame sphere S' so that

(1) *each component of $S - X$ has diameter less than ε,*

(2) *g is the identity on X,*

(3) *S misses T_0 and $S \cap T_1$ is a finite collection of points in $I(X, S)$ (the inaccessible points of X in S) where 1-simplexes of T pierce S,*

(4) *$g(S)$ is a polyhedron in T which is in general position with respect to T_2, and*

(5) *$g(S) \cap T_2 = X \cap T_2 = I(X, S) \cap T_2$.*

Following [11] we say, for a 2-sphere S in Σ, a tame Sierpinski curve X in S, and a triangulation T of Σ with i-skeleton T_i and mesh less than ε, (S, X, T_2, ε) has *Property Q* provided there is an ε-homeomorphism g of S onto a tame sphere S' so that the five conditions are satisfied in the conclusion of Lemma 4.4.

Bing [7] defines a *stable graph* as a finite, planar graph such that each homeomorphism between two embeddings of it into 2-spheres can be extended to a homeomorphism between the 2-spheres. The following lemma about stable graphs is similar to Theorem 3.2 of [7].

LEMMA 4.5. *Suppose* $f \in \mathscr{F}(S)$ *and* $\varepsilon > 0$.
There is a stable graph $G = \bigcup \mathrm{Bd}\,(D_i)$ *where* $D_1, \cdots, D_m$ *are* ε-*disks filling up* S *and having mutually exclusive interiors, and there is a* $\delta > 0$ *such that if* $g \in \mathscr{F}(R)$ *with* $d(f, g) < \delta$, *then there is an* ε-*homeomorphism* π *of* G *into* R.

Proof. From Lemma 4.4 and [7, § 9] it follows as in [12, Lemmas 6.2 and 6.3] that there is a triangulation T of Σ with i-skeleton T_i, a tame Sierpinski curve X on S, and an $\varepsilon_1 > 0$ so that $(S, X, T_2, \varepsilon_1)$ has Property Q, and if G' denotes the graph which consists of the sum of the components of $X \cap T_2$ containing points of T_1, then G' contains a stable subgraph $G = \bigcup \mathrm{Bd}\,(D_i)$ where $D_1, \cdots, D_m$ are $\varepsilon/3$-disks filling up S and having mutually exclusive interiors. Let $t_1, \cdots, t_j, \cdots$ denote the arcs which are the closures of the components of $G - T_1$, and let $p_1, \cdots, p_k, \cdots$ denote the points of $G \cap T_1$.

Let $\varepsilon_2 > 0$ be so small that each dia $(N(\Sigma, D_i, \varepsilon_2)) < \varepsilon/3$. Because the accessible points of X fail to intersect T_2 there is a homeomorphism λ of $G \times [-1, 1]$ into T_2 so that (1) for each simplex s of T,

$$\lambda((\mathrm{Int}\,(s) \cap G) \times [-1, 1]) \subseteqq \mathrm{Int}\,(s) ,$$

(2) $\lambda((G \cap T_1) \times [-1, 1]) \cap G = G \cap T_1$, (3) for each D_i,

$$\lambda(\mathrm{Bd}\,(D_i) \times [-1, 1]) \subseteqq N(\Sigma, D_i, \varepsilon_2) ,$$

and (4) $G_{-1} = \lambda(G \times -1)$ and $G_1 = \lambda(G \times 1)$ lie in different components of $\Sigma - S$. One obtains G_{-1}, G_1 satisfying (4) in much the same way that one finds the piercing arcs in [8, § 4]. For each p_k set $A_k = \lambda(p_k \times [-1, 1])$.

Choose $\varepsilon_3 > 0$ so that $\varepsilon_3 < \varepsilon_2$, $N(\Sigma, S, \varepsilon_3) \cap (G_{-1} \cup G_1) = \varnothing$, the sets $N(\Sigma, p_k, \varepsilon_3)$ are mutually exclusive, and each $N(\Sigma, p_k, \varepsilon_3) \cap T_1 \subseteqq \mathrm{Int}\,(A_k)$. Choose $\varepsilon_4 < \varepsilon_3/2$ so that it corresponds to η in Lemma 4.3 for f and $\varepsilon_3/2$. For each p_k let C_k be a 3-cell of diameter less than ε_4 whose interior contains p_k, which is polyhedral in T and in general position with respect to T_2, and whose intersection with each simplex s of T is either empty or a cell of the dimension of s. For each C_k let A'_k denote the subarc $C_k \cap T_1$ of A_k. Finally choose δ so that it is subject to the conditions on δ in Lemma 4.1 for f and ε_3 and subject to the conditions on δ in Lemma 4.3 for each substitution ($f \to f$, $A_k \to A$, $C_k \to C$, $\varepsilon_3/2 \to \varepsilon$).

Let $g \in \mathscr{F}(R)$ with $d(f, g) < \delta$. Use [5, Th. 7] and Theorem 3.1 to find a polyhedral 2-sphere R' in T which is in general position with respect to T_2 and each C_k so that there is a δ-homeomorphism ϕ of R onto R' and an involution $g' \in \mathscr{F}(R')$ with $d(f, g') < \delta$.

From the conditions on δ, ε_2, and ε_3 we find that (1) G_{-1} and G_1 lie in different components of $\Sigma - R'$, (2) each $A_k \cap R' \subseteqq \mathrm{Int}\,(A'_k)$, (3) there is a component U of $R' - \bigcup C_k$ such that each component of $R' - U$ is an $\varepsilon_3/2$-disk — denote these disks by F_{kj} so that $\mathrm{Bd}\,(F_{kj}) \subseteqq \mathrm{Bd}\,(C_k)$—and (4) for each k exactly one $\mathrm{Bd}\,(F_{kj})$, say $\mathrm{Bd}\,(F_{k1})$, separates the endpoints of A'_k on $\mathrm{Bd}\,(C_k)$.

Following Step 2 in § 4 of [7] we define a homeomorphism h of Σ which is the identity on $\mathrm{Cl}\,(U) \cup (T_1 - \bigcup C_k) \cup (\Sigma - \bigcup N(\Sigma, p_k, \varepsilon_3))$ so that each $h(F_{kj})$ $(j > 1)$ fails to intersect T_1 and each $h(\mathrm{Int}\,(F_{k1})) \subseteqq \mathrm{Int}\,(C_k)$. We can suppose that $h(R')$ is in general position with respect to T_2 and that each $h(F_{k1}) \cap T_1$ is the single point p_k. Because $h(R')$ separates G_{-1} from G_1 there must be an arc t'_j in each disk $\lambda(t_j \times [-1, 1])$ which spans $\mathrm{Bd}\,(\lambda(t_j \times [-1, 1]))$ and has

$$\mathrm{Bd}\,(t_j) = h(R') \cap \mathrm{Bd}\,(\lambda(t_j \times [-1, 1]))$$

for its endpoints. Define a homeomorphism π' of G into $h(R')$ so that each $\pi'(p_k) = p_k$ and each $\pi'(t_j) = t'_j$.

Define the homeomorphism π by $\pi = \phi^{-1}h^{-1}\pi'$. Each of π', h^{-1}, and ϕ^{-1} is an $\varepsilon/3$-homeomorphism so π is an ε-homeomorphism of G into R.

THEOREM 4.6. *Suppose $f \in \mathscr{F}(S)$ and $\varepsilon > 0$.*

There is a $\delta > 0$ such that if $g \in \mathscr{F}(R)$ with $d(f, g) < \delta$ then there is an ε-homeomorphism of S onto R.

Proof. We suppose $\mathrm{dia}\,(S) > 5\varepsilon$. Let $3\varepsilon_1 < \varepsilon/2$ correspond to δ in Lemma 4.2 for f and $\varepsilon/2$. Choose δ from Lemma 4.5 for f and ε_1.

Let $g \in \mathscr{F}(R)$ with $d(f, g) < \delta$. From Lemma 4.5 there are ε_1-disks $D_1, \cdots, D_m$ which fill up S and have mutually exclusive interiors so that $G = \bigcup \mathrm{Bd}\,(D_i)$ is a stable graph, and there is an ε_1-homeomorphism π of G into R. Since G is stable we can extend π to a homeomorphism of S onto R which we also call π. Each $\pi(\mathrm{Bd}\,(D_i))$ has diameter less than $3\varepsilon_1$ so by the choice of ε_1 it bounds an $\varepsilon/2$-disk F_i in R. Suppose for some D_i, $\pi(D_i) \neq F_i$. Then $\pi(G) \subseteqq F_i$. But $\mathrm{dia}\,(G) > 3\varepsilon$ and $\pi \mid G$ is an ε_1-homeomorphism so $\mathrm{dia}\,(\pi(G)) > \varepsilon > \mathrm{dia}\,(F_i)$. Thus each $\pi(D_i) = F_i$ and π is a $3\varepsilon_1 + \varepsilon/2 < \varepsilon$-homeomorphism of S onto R.

5. Small deformations of cells whose boundaries approximate a given sphere.

We omit a proof of Lemma 5.1. The proof is straight forward but involves a tedious pasting together of small isotopies.

LEMMA 5.1. *Suppose M is a 3-manifold, F is a compact surface, f is a homeomorphism of F into M, and $\varepsilon > 0$.*

There is a $\delta > 0$ *such that if* g *is a homeomorphism of* $F \times [0, 1]$ *onto a solid* P *in* M *where* $\rho(f(x), g(x, t)) < \delta$ $(x \in F, t \in [0, 1])$, *and if* h *is a* δ*-homeomorphism of* P *onto itself which is the identity on* $\mathrm{Bd}(P)$, *then there is an* ε*-isotopy* $H_t(0 \leqq t \leqq 1)$ *of* P *onto itself such that* $H_0 = I$, $H_t \mid \mathrm{Bd}(P) = I$, *and* $H_1 h = I$.

The following theorem is the key to establishing the connectivity properties of $\mathscr{F}$.

THEOREM 5.2. *Suppose* S *is a 2-sphere in* Σ *and* $\varepsilon > 0$.
There is a $\delta > 0$ *such that if* C *is a 3-cell in* Σ *whose boundary* R *is homeomorphically within* δ *of* S, *and if* h *is a* δ*-homeomorphism of* C *onto itself which is the identity on* R, *then there is an* ε*-isotopy* $H_t(0 \leqq t \leqq 1)$ *of* C *onto itself such that* $H_0 = I$, $H_t \mid R = I$, *and* $H_1 h = I$.

Proof. If S were tame the problem would be relatively easy. With the help of Theorem 1.1 and Lemma 0 of [15] we could construct the isotopy essentially as the Alexander isotopy is constructed in [1]. However, in order to deal with 2-spheres which are possibly wild we have to reach our goal by a devious route.

It is easily seen that an equivalent theorem is obtained if in the hypothesis Σ is replaced by E^3. It is this equivalent version which we prove. Except for item (5) we suggest that on first reading one skip the epsilonics which follow in the next paragraph and concentrate on the geometry in the proof.

Consider then a 2-sphere S in E^3 and a number $\varepsilon > 0$. We suppose for convenience that $\mathrm{dia}(S) > 10\varepsilon$. Let f be a homeomorphism of a polyhedral 2-sphere K onto S. We obtain in succession seven positive numbers—$\varepsilon_1, \cdots, \varepsilon_6$, and δ.

(1) *Conditions on* ε_1: Substitute $(E^3 \to M, K \to F, f \to f, \varepsilon/4 \to \varepsilon)$ in Lemma 5.1 to get $\varepsilon_1 > 0$ corresponding to δ there.

(2) *Conditions on* ε_2: Substitute $(E^3 \to M, K \to K, f \to f, \varepsilon_1 \to \varepsilon)$ in Theorem 1.1 to get $\varepsilon_2 > 0$ corresponding to δ there.

(3) *Conditions on* ε_3: Choose $\varepsilon_3 > 0$ so that $\varepsilon_3 < \varepsilon_2/8$ and $\varepsilon_1/400$.

(4) *Conditions on* ε_4: Choose $\varepsilon_4 > 0$ so that every $3\varepsilon_4$-subset of S is contained in a disk on S of diameter less than $\varepsilon_3/3$.

(5) *A special polyhedral neighborhood* M *of* S: Use [16] to find a pwl homeomorphism g of $K \times [0, 1]$ onto a polyhedron P in E^3 with boundary components $S_0 = g(K \times 0)$ and $S_1 = g(K \times 1)$ so that $\rho(g(y, e), f(y)) < \varepsilon_4$ $(y \in K, e = 0, 1)$ and to find mutually exclusive, polyhedral cubes-with-handles $G_1, \cdots, G_m, K_1, \cdots, K_n$ so that each $\mathrm{dia}(G_i) < \varepsilon_4$, each $\mathrm{dia}(K_i) < \varepsilon_4$, each $G_i \cap P$ is a polyhedral disk on S_1, and each $K_i \cap P$ is a polyhedral disk on S_0 and so that $M = P \cup (\bigcup G_i) \cup (\bigcup K_i)$ contains a neighborhood of S in E^3. We suppose that $S_0 \subseteqq \mathrm{Int}(S_1)$.

Let F denote the component of $\mathrm{Bd}(M)$ which intersects S_0, and let Q denote the polyhedral 3-manifold which F bounds in $\mathrm{Int}(S)$.

(6) *Conditions on* ε_5: Choose ε_5 less than one fourth the distance from Q to S and so small that any 2-sphere in E^3 which is homeomorphically within ε_5 of S contains $N(E^3, Q, 4\varepsilon_5)$ in its interior. Theorem VI 10 of [14] guarantees that the second condition can be met.

(7) *Conditions on* ε_6: Substitute ($E^3 \rightarrow M, Q \rightarrow K, I$ (Identity) $\rightarrow f, \varepsilon_5 \rightarrow \varepsilon$) in Theorem 1.1 to get ε_6 corresponding to δ there.

(8) *Conditions on* δ: Choose $\delta < \varepsilon_6/2$.

Now let C, R, and h be given as in the hypothesis of the theorem. We construct in succession isotopies $H_t^i (0 \leqq t \leqq 1)$ $(i = 1, \cdots, 4)$ of C onto itself such that each $H_0^i = I$.

Use [6, 18] together with [15, Lemma 0] to obtain a δ-isotopy H_t^1 which is the identity on $\mathrm{Bd}(C)$ so that $H_1^1 h$ is locally pwl on $\mathrm{Int}(C)$. Now $H_1^1 h \mid Q$ is an ε_6-homeomorphism so from items (6) and (7) there is a pwl ε_5-isotopy H_t^2 of C which is the identity on $\mathrm{Bd}(C)$ such that $H_1^2 H_1^1 h \mid Q = I$.

In each K_i there are mutually exclusive, polyhedral ε_4-disks D_{ij} spanning $\mathrm{Bd}(K_i)$ such that the closure of K_i minus thin, disjoint regular neighborhoods of the D_{ij}'s is a 3-cell. We can suppose that the D_{ij}'s fail to intersect S_0. Use [5, Th. 7] and the fact that R is collared in C to find a δ-homeomorphism θ of S onto a polyhedral 2-sphere R' in $\mathrm{Int}(C) \cap \mathrm{Int}(M)$ which is in general position with respect to $\bigcup D_{ij}$. Each component of $\theta^{-1}((\bigcup D_{ij}) \cap R')$ has diameter less than $\varepsilon_4 + 2\delta < 3\varepsilon_4$ and so by (4) bounds an $\varepsilon_3/3$-disk on S; thus each component of $(\bigcup D_{ij}) \cap R'$ bounds a disk on R' of diameter less than $\varepsilon_3/3 + 2\delta < \varepsilon_3$. By cutting away some closures of components of $(D_{ij} - R')$'s, replacing them by closures of components of $(R' - \bigcup D_{ij})$, and then pushing these modified disks slightly into $\mathrm{Int}(R')$ we obtain a new collection of mutually exclusive, polyhedral $3\varepsilon_3$-disks $\{E_{ij}\}$ in $M \cap \mathrm{Int}(C)$ which span F and have the same boundaries as the D_{ij}'s. Choose mutually exclusive regular neighborhoods N_{ij} of the E_{ij}'s in $M \cap \mathrm{Int}(C)$ so that each N_{ij} has diameter less than $3\varepsilon_3$ and intersects $\mathrm{Bd}(M)$ in a regular neighborhood of $\mathrm{Bd}(E_{ij})$ missing S_0. Then in each N_{ij} choose a smallar regular neighborhood N'_{ij} of E_{ij} so that $N'_{ij} \cap \mathrm{Bd}(N_{ij})$ is a regular neighborhood of $\mathrm{Bd}(E_{ij})$ in $\mathrm{Int}(N_{ij} \cap \mathrm{Bd}(M))$.

Now $H_1^2 H_1^1 h$ is a $\delta + \delta + \varepsilon_5 < \varepsilon_3$-homeomorphism so each $H_1^2 H_1^1 h(N'_{ij})$ has diameter less than $5\varepsilon_3$. Pushing each $H_1^2 H_1^1 h(N'_{ij}) \cap \mathrm{Int}(M)$ slightly so it is in general position with respect to $\bigcup N_{ij}$ and then using [7, §7] (see also [12, Lemma 2.9]) we define a $65\varepsilon_3$-isotopy $H_t^3 (0 \leqq t \leqq 1)$ of C so that $H_t^3 \mid Q \cup \mathrm{Bd}(C) = I$ and $H_1^3 H_1^2 H_1^1 h \mid Q \cup (\bigcup N'_{ij}) = I$.

Consider the 3-manifold $T = \mathrm{Cl}(C - (Q \cup (\bigcup N'_{ij})))$. Its boundary components are the 2-spheres $R = \mathrm{Bd}(C)$ and R'' which is obtained from F by cutting out annuli and replacing them by pairs of disks. More specifically, for each K_i, $\bigcup_j (N'_{ij} \cap F)$ is replaced by

$$\mathrm{Cl}\,((\bigcup_j \mathrm{Bd}\,(N'_{ij})) - F)\ .$$

Each $\mathrm{dia}\,(K_i \cup (\bigcup_j N'_{ij})) < \varepsilon_4 + 2(3\varepsilon_3) < 7\varepsilon_3$. Define a homeomorphism f_0 of K onto R'' such that $f_0(y) = g(y, 0)(y \notin g^{-1}(\bigcup K_i))$ and for each K_i, $f_0(g^{-1}(K_i \cap S_0)) \subseteqq K_i \cup (\bigcup_j N'_{ij})$. We have $d(f, f_0) < \varepsilon_4 + 7\varepsilon_3 < 8\varepsilon_3 < \varepsilon_2$. Let f_1 be a homeomorphism of K onto R such that $d(f, f_1) < \delta$.

From (2) and the fact that R is collared in C there is a homeomorphism ϕ of $K \times [0, 1]$ onto T such that $\phi(y, e) = f_e(y)(y \in K, e = 0, 1)$ and $\rho(\phi(y, t), f(y)) < \varepsilon_1\,(y \in K, t \in [0, 1])$. Furthermore by (3) $H_1^3H_1^2H_1^1h$ is an ε_1-homeomorphism. Thus by (1) there is an $\varepsilon/4$-isotopy $H_t^4(0 \leqq t \leqq 1)$ of C such that $H_t^4 \mid C - \mathrm{Int}\,(T) = I$ and $H_1^4H_1^3H_1^2H_1^1h = I$.

The promised isotopy H_t is given by $H_0 = I$ and

$$H_t = H^i_{4(t-(i-1)/4}H_{(i-1)/4}((i-1)/4 \leqq t \leqq i/4,\ i = 1, \cdots, 4)\ .$$

Each H_t^i is an $\varepsilon/4$-isotopy so H_t is an ε-isotopy.

6. Pathwise and local pathwise connectivity of $\mathscr{F}$.

THEOREM 6.1. *The space $\mathscr{F}$ is pathwise and locally pathwise connected.*

Proof. The proof is divided into four parts.

(1) *The subspace $\mathscr{G}$ is locally pathwise connected at each point of $\mathscr{F}$.* That is, if $f \in \mathscr{F}$ and $\varepsilon > 0$, there is a $\delta > 0$ such that if $g_0, g_1 \in N(\mathscr{F}, f, \delta) \cap \mathscr{G}$, then there is a path $h_t(0 \leqq t \leqq 1)$ in $N(\mathscr{F}, f, \varepsilon) \cap \mathscr{G}$ with endpoints $h_0 = g_0$ and $h_1 = g_1$.

Proof of (1). Let ε_1 correspond to δ in Lemma 2.3 for f and ε. Let ε_2 correspond to δ in Theorem 5.2 for S and ε_1. Let $\varepsilon_3 < \varepsilon_2$ be a positive number so small that $d(f_1f_2, I) < \varepsilon_2$ for each pair of elements $f_1, f_2 \in N(\mathscr{F}, f, \varepsilon_3)$. Let ε_4 correspond to δ in Lemma 2.3 for f, ε_3. Let ϕ be a homeomorphism of a polyhedral 2-sphere K onto S. Let ε_5 correspond to δ in Theorem 1.1 for the substitution $(\Sigma \rightarrow M, K \rightarrow K, \phi \rightarrow f, \varepsilon_4 \rightarrow \varepsilon)$. Finally choose $\delta < \varepsilon_5$ from Theorem 4.6 for f and ε_5.

Let $g_0, g_1 \in N(\mathscr{F}, f, \delta) \cap \mathscr{G}$ with $g_0 \in \mathscr{G}(S_0)$ and $g_1 \in \mathscr{G}(S_1)$. From the conditions on δ and ε_5 there are homeomorphisms ϕ_0, ϕ_1 of K onto S_0, S_1 so that $d(\phi, \phi_e) < \varepsilon_5\,(e = 0, 1)$ and there is an ε_4-isotopy $H_t^1(0 \leqq t \leqq 1)$ of Σ such that $H_0^1 = I$ and $H_1^1\phi_0 = \phi_1$. Define $h_t(0 \leqq t \leqq 1/2)$ by $h_t = H_{2t}^1g_0(H_{2t}^1)^{-1}$. From Lemmas 2.2 and 2.3, $h_t(0 \leqq t \leqq 1/2) \subseteqq N(\mathscr{F}, f, \varepsilon_3) \cap \mathscr{G}$ and $h_{1/2} \in \mathscr{G}(S_1)$. The conditions on ε_3 show that $d(h_{1/2}g_1^{-1}, I) < \varepsilon_2$.

Let S_1 bound 3-cells C and D in Σ. From Theorem 5.2 there is an ε_1-isotopy $H_t^2(0 \leqq t \leqq 1)$ of Σ onto itself such that $H_0^2 = I$, $H_t^2 \mid D = I$, and $H_1^2 \mid C = g_1 h_{1/2}$. Define $h_t(1/2 \leqq t \leqq 1)$ by $H_{2t-1}^2 h_{1/2}(H_{2t-1}^2)^{-1}$. Lemmas 2.1 and 2.3 show that $h_t(1/2 \leqq t \leqq 1) \subseteqq N(\mathscr{F}, f, \varepsilon) \cap \mathscr{G}(S_1)$ and $h_1 = g_1$. Thus $h_t(0 \leqq t \leqq 1) \subseteqq N(\mathscr{F}, f, \varepsilon) \cap \mathscr{G}$ with $h_0 = g_0$ and $h_1 = g_1$.

(2) *If $f \in \mathscr{F}$ and $\varepsilon > 0$ there is a path $h_t(0 \leqq t \leqq 1)$ in $N(\mathscr{F}, f, \varepsilon)$ such that $h_1 = f$ and $h_t \in \mathscr{G}$ $(t < 1)$.*

Proof of (2). From (1) there is for each $j \geqq 1$ a δ_j such that any two points in $N(\mathscr{F}, f, \delta_j) \cap \mathscr{G}$ can be joined by a path in $N(\mathscr{F}, f, \varepsilon/j) \cap \mathscr{G}$. We suppose that $\delta_1 > \delta_2 > \cdots > \delta_n > \cdots$. Use Corollary 3.1 to find for each j an element $f_j \in N(\mathscr{F}, f, \delta_j) \cap \mathscr{G}$. For each j there is a path $h_t(1 - 1/j \leqq t \leqq 1 - 1/(j+1))$ in $N(\mathscr{F}, f, \varepsilon/j) \cap \mathscr{G}$ such that $h_{1-1/j} = f_j$ and $h_{1-1/(j+1)} = f_{j+1}$. Because $\lim f_j = f$ we can set $h_1 = f$ to get the promised path.

(3) *The space $\mathscr{F}$ is locally pathwise connected.*

Proof of (3). Let $f \in \mathscr{F}$ and $\varepsilon > 0$. Choose δ from (1) for f and ε.

Let $f_0, f_1 \in N(\mathscr{F}, f, \delta)$. From (2) there are paths $h_t(0 \leqq t \leqq 1/4)$ and $h_t(3/4 \leqq t \leqq 1)$ in $N(\mathscr{F}, f, \delta)$ such that $h_0 = f_0$, $h_1 = f_1$, and $h_{1/4}$, $h_{3/4} \in \mathscr{G}$. Then (1) enables us to define $h_t(1/4 \leqq t \leqq 3/4)$ in $N(\mathscr{F}, f, \varepsilon)$ so it connects up $h_{1/4}$ and $h_{3/4}$.

(4) *The space $\mathscr{F}$ is pathwise connected.*

Proof of (4). In view of (2) it is sufficient to show that $\mathscr{G}$ is pathwise connected.

Let $f, g \in \mathscr{G}$ with $f \in \mathscr{G}(S)$ and $g \in \mathscr{G}(R)$. Let T be a tame 2-sphere in Σ disjoint from both S and R so that T bounds a 3-cell B containing both S and R. The pairs (T, S) and (T, R) both bound annuli and S, R bound 3-cells C, D in Int (B). Thus there is a homeomorphism h of B onto itself which is the identity on Bd (B) and takes C onto D. From [1] there is an isotopy $H_t^1(0 \leqq t \leqq 1)$ of Σ which is the identity on $\Sigma - \text{Int}(B)$ so that $H_0^1 = I$ and $H_1^1 \mid B = h$. From Lemma 2.2, $h_t(0 \leqq t \leqq 1/2)$ given by $h_t = H_{2t}^1 f (H_{2t}^1)^{-1}$ is a path in $\mathscr{G}$ with $h_0 = f$ and $h_{1/2} \in \mathscr{G}(R)$.

Use [1] to define an isotopy $H_t^2(0 \leqq t \leqq 1)$ of Σ onto itself such that $H_0^2 = I$, $H_t^2 \mid \Sigma - D = I$, and $H_1^2 \mid D = g h_{1/2}$. Define $h_t(1/2 \leqq t \leqq 1)$ by $h_t = H_{2t-1}^2 h_{1/2}(H_{2t-1}^2)^{-1}$. As in the proof of (1) h_t is a path in $\mathscr{G}$ with $h_1 = g$.

The following corollary shows that pseudo isotopies like the one Bing uses in [3] can be used to obtain all elements of $\mathscr{F} - \mathscr{G}$ from $\mathscr{G}$.

COROLLARY 6.1. *For each pair of involutions $f \in \mathscr{F}$, $g \in \mathscr{G}$ there is a pseudo isotopy $H_t(0 \leqq t \leqq 1)$ of Σ such that $H_0 = I$ and $f = H_1 g (H_1)^{-1}$.*

Proof. Let $f \in \mathscr{F}$ and $g \in \mathscr{G}$ be given. From (2) there is an element $g_2 \in \mathscr{G}$ and there is a path $h_t(1/2 \leqq t \leqq 1)$ in $\mathscr{F}$ such that $h_{1/2} = g_2$ and $h_1 = f$. From the proof of (2) we can assume that $h_t(1/2 \leqq t \leqq 1)$ is made up of pieces $h_t(1 - 1/j \leqq t \leqq 1 - 1/(j+1)))\,(j = 2, 3, \cdots)$ where $h_t(1 - 1/j \leqq t \leqq 1 - 1/(j+1)) \subseteqq N(\mathscr{F}, f, 1/j) \cap \mathscr{G}$. From the proof of (1) each piece of h_t can be chosen to have the form

$$H_t^j h_{1-1/j} (H_t^j)^{-1} (1 - 1/j \leqq t \leqq 1 - 1/(j+1))$$

where $H_t^j(1 - 1/j \leqq t \leqq 1 - 1/(j+1))$ is a $1/j$-isotopy of Σ with $H_{1-1/j}^j = I$. Define a pseudo isotopy $H_t^1(1/2 \leqq t \leqq 1)$ of Σ by the rule $H_{1/2}^1 = I$, $H_t^1 = H_t^j H_{1-1/j}^1 (1 - 1/j \leqq t \leqq 1 - 1/(j+1), j = 2, 3, \cdots)$, and

$$H_1^1 = \lim H_t^1 (t \to 1) .$$

Note that $H_1^1 h_{1/2} (H_1^1)^{-1} = \lim h_t(t \to 1) = f$.

By a similar argument we obtain from the proofs of (4) and (1) an isotopy $H_t^0(0 \leqq t \leqq 1/2)$ of Σ such that $H_0^0 = I$ and $g_2 = H_{1/2}^0 g (H_{1/2}^0)^{-1}$. Define the pseudo isotopy H_t by the rule $H_t = H_t^0(0 \leqq t \leqq 1/2)$ and $H_t = H_t^1 H_{1/2}(1/2 \leqq t \leqq 1)$. For $1/2 \leqq t < 1$ we have $H_t g (H_t)^{-1} = H_t^1 H_{1/2} g (H_t^1 H_{1/2})^{-1} = H_t^1 g_2 (H_t^1)^{-1} = h_t$; thus $f = H_1 g H_1^{-1}$.

Both Bob Daverman and the referee suggested the following alternative way to obtain H_t^1: Let S bound crumpled cubes C and D. Split open Σ along S and add $S \times [-1, 1]$ so that Σ is represented as the sum $C \cup S \times [-1, 1] \cup D$. Define $g_2 = f$ on C and D, and set $g_2((x, t)) = (x, -t)$ for $(x, t) \in S \times [-1, 1]$. A result of Price's [20] provides a pseudo isotopy $H_t^2(1/2 \leqq t \leqq 1)$ of Σ which shrinks the fibers of $S \times [-1, 1]$ back to points and transforms g_2 to f by the conjugation $f = H_1^1 g_2 (H_1^1)^{-1}$.

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