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TANGENTIAL CAUCHY-RIEMANN EQUATIONS AND UNIFORM APPROXIMATION

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A smooth ($\mathscr{C}^\infty$) function on a smooth real submanifold M of complex Euclidean space $\mathbf{C}^n$ is a *CR function* if it satisfies the Cauchy-Riemann equations tangential to M. It is shown that each CR function admits an extension to an open neighborhood of M in $\mathbf{C}^n$ whose $\bar{z}$-derivatives all vanish on M to a prescribed high order, provided that the system of tangential Cauchy-Riemann equations has minimal rank throughout M. This result is applied to show that on a holomorphically convex compact set in M each CR fuction can be uniformly approximated by holomorphic functions.

1. Extension and approximation of CR functions.

Each point p of a smooth real submanifold M of $\mathbf{C}^n$ has a *complex tangent space* H_pM. It is the largest complex-linear subspace of the ordinary real tangent space T_pM; evidently $H_pM = T_pM \cap iT_pM$. Its complex dimension is the *complex rank* of M at p. The theorem of linear algebra relating the real dimensions of T_pM, iT_pM and their sum and intersection shows that if M has real codimension k its complex rank is not less than $n - k$.

DEFINITION 1.1. M is a *CR manifold* if its complex rank is constant. It is *generic* if in addition this rank is minimal; that is, equal to the larger of 0 and $n - k$. A smooth function f on M is a *CR function* if $\ker \bar{\partial}_p f \supset H_pM$ for each p in M.

Here f is assumed to be extended in a smooth manner to an open neighborhood of M and $\bar{\partial}_p f$ is regarded as the conjugate complex-linear part of the ordinary Fréchet differential d_pf. Since the condition on $\bar{\partial}_p f$ is independent of the extension chosen, the definition makes sense. Computational equivalents to it and some elaboration are given in § 2. A more comprehensive treatment of these ideas is found in the paper by S. Greenfield [1]. It should be mentioned that his definition [1, Definition II. A. 1] of CR manifolds also requires that the distribution $p \to H_pM$ be involutive. That assumption is not needed here.

If M is a complex submanifold of $\mathbf{C}^n$, then it is CR with complex rank equal to its complex dimension. It is not generic if it has positive codimension. Of course the CR functions on M are just its holomorphic functions.

At the other extreme, every real hypersurface is a generic CR

manifold of complex rank $n-1$. These frequently have no nontrivial complex submanifolds, which is true for example of the usual $2n-1$ sphere in $\mathbf{C}^n$.

M is a generic CR manifold if its complex rank is everywhere zero, which is the *totally real* [5] case.

An example of a proper generic CR submanifold which is neither totally real nor a hypersurface can of course only be found if $n \geqq 3$. There is one in $\mathbf{C}^3$, a 4-sphere S^4 given as the intersection of the usual 5-sphere and a real hyperplane transverse to it. Let

$$\rho_1 = |z_1|^2 + |z_2|^2 + |z_3|^2 - 1$$

and $\rho_2 = z_3 + \bar{z}_3$, where z_1, z_2, z_3 are the usual coordinates for $\mathbf{C}^3$, and let $S^4 = \{\rho_1 = \rho_2 = 0\}$. It follows from (2.2) below that S^4 has the requisite properties. Furthermore, S^4 has no nontrivial complex submanifolds (since the 5-sphere does not).

THEOREM 1.2. *If f is a CR function on a generic CR manifold M in $\mathbf{C}^n$ and m is a nonnegative integer, then there is an extension of f to a smooth function f_m on an open set $U \supset M$ such that $\bar{\partial} f_m$ vanishes on M to order m in all directions.*

This result is known [3, Lemma 4.3] and [5, Lemma 3.1] when M is totally real. It is also proved in [2, Th. 2.3.2′] when M is a real hypersurface. A local version which does not require that M be generic is proved in [5, Lemma 3.3].

Theorem 1.2 plays a key role in a program outlined by L. Hörmander for showing that CR functions can be uniformly approximated by holomorphic functions. The basic idea is to take a compact set K in M and a given CR function f on M and find a solution g of $\bar{\partial} g = \bar{\partial} f$ with $\sup_K |g|$ small. Then $u = f - g$ is holomorphic and approximates f uniformly on K with error no larger than $\sup_K |g|$.

In Hörmander's implementation of this idea, Theorem 1.2 implies that a certain bound on an L^2 norm of the Sobolev type is imposed on $\bar{\partial} g$. The existence of solutions to $\bar{\partial} g = \bar{\partial} f$ subject to the same a priori bound [2] and a Sobolev inequality are used to estimate $\sup_K |g|$. This proof appears in [3] and [5] for the cases cited above. Since the only step of it which depends on the complex rank of M is the conclusion of Theorem 1.2, this proof will, without further modification, yield a result on uniform approximation.

THEOREM 1.3. *If M is a closed generic CR submanifold of a domain of holomorphy U in $\mathbf{C}^n$ and K is a compact subset of M holomorphically convex with respect to U, then each smooth CR func-*

tion on M is a uniform limit on K of functions holomorphic on U.

In fact, the same method in conjunction with Theorem 1.2 will prove the stronger statement that approximation holds in the $\mathscr{C}^\infty$ topology; c.f. [5, Th. 6.1]. One merely replaces $\sup_K |g|$ by a $\mathscr{C}^k$ norm of g on K.

In the totally real case, it is known that the holomorphic convexity of any given compact subset K with respect to *some* domain of holomorphy is a consequence of the absence of complex tangent vectors. This follows from the fact [3, Th. 3.1] and [5, Corollary 4.2] that each K has arbitrarily small tubular neighborhoods which are domains of holomorphy. However, the case of the $2n-1$ sphere in $\mathbf{C}^n$ shows that in the presence of complex tangent vectors holomorphic convexity must be assumed. When there is complex tangency, the problem of determining holomorphic convexity of a given compact subset of M is very difficult, even for the examples mentioned above.

It should be remarked that in Definition 1.1 and Theorem 1.2 $\mathbf{C}^n$ may be replaced by any complex manifold, and if this manifold is Stein [2], it may replace U in Theorem 1.3. No significant modification of the exposition is required.

2. *CR* **manifolds and functions.** Each real-linear map $L: \mathbf{C}^n \to \mathbf{C}^k$ is uniquely expressible as a sum $L = S + T$ where $S, T: \mathbf{C}^n \to \mathbf{C}^k$, S is complex linear, and T is conjugate complex linear. If $J: v \to iv$, a direct computation shows that $S = \frac{1}{2}(L - JLJ)$ and $T = \frac{1}{2}(L + JLJ)$. Applying this result to the Fréchet differential $d_p\rho$ of a smooth map $\rho: \mathbf{C}^n \to \mathbf{C}^k$ at p there results

$$d_p\rho = \partial_p\rho + \bar{\partial}_p\rho$$

in which $\partial_p\rho$ is the complex linear part of $d_p\rho$ and $\bar{\partial}_p\rho$ the conjugate complex linear part.

Each point of M has an open neighborhood U in $\mathbf{C}^n$ on which there exists a smooth map $\rho = (\rho_1, \cdots, \rho_k): U \to \mathbf{R}^k$ with maximal rank k on U and satisfying

$$M \cap U = \{z \in U: \rho(z) = 0\} . \tag{2.1}$$

Regarding $\mathbf{R}^k$ as contained in $\mathbf{C}^k$ in the usual way, and applying the remarks above to Definition 1.1, it follows that M is *CR* if and only if $\bar{\partial}\rho$ has constant complex rank on $M \cap U$, and is generic exactly when this rank is maximal. When $k \geqq n$ this means that $H_pM = 0$, which is the totally real case. The case of interest here is $k \leqq n$, when M is generic if and only if $\bar{\partial}\rho$ has complex rank k on $M \cap U$. Henceforth, it is assumed that $k \leqq n$. Since it is clear that $\bar{\partial}\rho = (\bar{\partial}\rho_1, \cdots, \bar{\partial}\rho_k)$ it follows that the condition

$$\bar{\partial}\rho_1 \wedge \cdots \wedge \bar{\partial}\rho_k \quad \text{has no zeros on } M \cap U \tag{2.2}$$

is necessary and sufficient that M be a generic CR manifold.

From Definition 1.1 and (2.2) it follows that a smooth function f on M is CR if and only if

$$\bar{\partial}f \wedge \bar{\partial}\rho_1 \wedge \cdots \wedge \bar{\partial}\rho_k = 0 \quad \text{on } M. \tag{2.3}$$

Equivalently, since $\{\bar{\partial}\rho_1, \cdots, \bar{\partial}\rho_k\}$ is, at points of M, by virtue of (2.2) part of a basis for the space of conjugate-linear functionals on $\mathbf{C}^n$, there exist smooth functions $h_1, \cdots, h_k$ on U such that

$$\bar{\partial}f = \sum_{j=1}^{k} h_j\bar{\partial}\rho_j + O(\rho) . \tag{2.4}$$

Here $O(\rho)$ denotes a form which vanishes on $M \cap U$. It is a standard result [4, Lemma 2.1] that if g is a smooth $O(\rho)$-form there exist smooth forms $g_1, \cdots, g_k$ such that

$$g = \sum_{j=1}^{k} \rho_j g_j . \tag{2.5}$$

More generally, $O(\rho^m)$ will denote a smooth form on U which vanishes on $M \cap U$ to order m. Induction on m using (2.5) shows that if g is such a form there are smooth forms g_α on U satisfying

$$g = \sum_{|\alpha|=m} \rho^\alpha g_\alpha , \tag{2.6}$$

in which the standard multi-index notation has been used. Thus $\alpha = (\alpha_1, \cdots, \alpha_k)$ is a k-tuple of nonnegative integers, $|\alpha| = \alpha_1 + \cdots + \alpha_k$, and $\rho^\alpha = \rho_1^{\alpha_1}\cdots\rho_k^{\alpha_k}$. The coefficients g_α are not unique on U, but the fact that they are determined on $M \cap U$ will be essential.

LEMMA 2.1. *If smooth forms g, g_α are related on U by*

$$g = \sum_{|\alpha|=m} \rho^\alpha g_\alpha + O(\rho^{m+1})$$

then for each α, $D^\alpha g \mid M \cap U = \alpha! g_\alpha \mid M \cap U$. In particular, if $g = 0$ on U then each $g_\alpha \mid M \cap U = 0$.

Here $D^\alpha = D_1^{\alpha_1}\cdots D_k^{\alpha_k}$, where D_j denotes differentiation with respect to ρ_j and $\alpha! = \alpha_1! \cdots \alpha_k!$.

Proof. The statement is local and since ρ has rank k, the proof can be reduced to the case where each $\rho_j = x_j$, the jth ordinary Euclidean coordinate function. Then the lemma follows from the general Leibniz formula

$$D^{\alpha}(fg) = \sum_{\gamma \leqq \alpha} \binom{\alpha}{\gamma} D^{\gamma} f \cdot D^{\alpha-\gamma} g$$

with $f = x^{\alpha}$, noting that $D^{\gamma} x^{\alpha} = 0$ on $M \cap U$ if $\gamma < \alpha$ and $D^{\alpha} x^{\alpha} = \alpha!$. Here $\binom{\alpha}{\gamma} = \alpha!/\gamma!(\alpha - \gamma)!$ and $\gamma < \alpha$ means that $\gamma_j < \alpha_j$ for some j.

3. Proof of Theorem 1.2. The proof is an induction on m in which f_{m+1} is obtained by subtraction of an $O(\rho^{m+1})$ function from f_m. Similar procedures have been used in [2, Th. 2.3.2′], [3, Lemma 4.3], and [5, Lemmas 3.1 and 3.3]. The one used here borrows ideas from all of these. Since the totally real generic cases where $k \geqq n$ are treated in [3] and [5], it will be assumed that $k \leqq n$. However, the proof below can be read with $k \geqq n$, with some slight modifications.

In the presence of complex tangent vectors, the only known result is local in nature [5, Lemma 3.3]. Its proof refers to a particular local coordinate system for $\mathbf{C}^n$ and uses an initial extension f_0 which is independent of the coordinates normal to M. This feature is clearly not preserved by the patching construction intended here, so an arbitrary extension of f must be admitted at each step. This introduces remainder terms of the form $O(\rho^m)$, and it is necessary to keep an accurate account of their effects.

To begin the induction, extend a given CR function f from M to a smooth function f_0 on an open set $U \supset M$.

First assume that the representation (2.1) holds on U. Then $\bar{\partial} f_0$ is of the form (2.4) and if $u = \sum_{j=1}^{k} \rho_j h_j$ it is clear that $\bar{\partial}(f_0 - u) = O(\rho)$.

In general U has a locally finite cover by open sets U_ι on each of which there exists a defining function ρ_ι presenting $M \cap U_\iota$ as in (2.1) and a $O(\rho_\iota)$ function u_ι satisfying $\bar{\partial}(f_0 - u_\iota) = O(\rho_\iota)$ on U_ι. If $\{\varphi_\iota\}$ is a partition of unity subordinate to $\{U_\iota\}$ and

$$u = \sum_\iota \varphi_\iota u_\iota \tag{3.1}$$

then

$$\bar{\partial}(f_0 - u) = \sum_\iota \varphi_\iota \bar{\partial}(f_0 - u_\iota) - \sum_\iota u_\iota \bar{\partial} \varphi_\iota \,. \tag{3.2}$$

By construction each term of either sum in (3.2) vanishes on M. Therefore so does $\bar{\partial} f_1$ if $f_1 = f_0 - u$.

For the inductive step assume that $m > 0$ and f has an extension f_m to U such that $\bar{\partial} f_m$ vanishes on M to order m. A global modification of f_m will again be obtained by patching local ones, so the construction is again begun by assuming that M is globally presented by (2.1).

Then by (2.6) there are smooth $(0, 1)$ forms g_α such that

$$\bar{\partial} f_m = \sum_{|\alpha|=m} \rho^\alpha g_\alpha . \tag{3.3}$$

Hence

$$0 = \bar{\partial}^2 f_m = \sum_{|\alpha|=m} \sum_{j=1}^{k} \alpha_j \rho^{\alpha-j} \bar{\partial}\rho_j \wedge g_\alpha + O(\rho^m) , \tag{3.4}$$

in which $\alpha - j$ denotes $(\alpha_1, \cdots, \alpha_j - 1, \cdots, \alpha_k)$ if $\alpha_j > 0$. Wedge this equation with $\bar{\partial}\rho_1 \wedge \cdots \wedge \widehat{\bar{\partial}\rho_j} \wedge \cdots \wedge \bar{\partial}\rho_k$ ($\bar{\partial}\rho_j$ is missing) to show that for each j

$$0 = \sum_{|\alpha|=m} \alpha_j \rho^{\alpha-j} \bar{\partial}\rho_1 \wedge \cdots \wedge \bar{\partial}\rho_k \wedge g_\alpha + O(\rho^m) . \tag{3.5}$$

Now for fixed j, the map $\alpha \to \alpha - j$ is a one-to-one correspondence of $\{\alpha : |\alpha| = m \text{ and } \alpha_j > 0\}$ with $\{\beta : |\beta| = m - 1\}$. Therefore (3.5) may be rewritten as

$$0 = \sum_{|\beta|=m-1} (\beta_j + 1) \rho^\beta \bar{\partial}\rho_1 \wedge \cdots \wedge \bar{\partial}\rho_k \wedge g_{\beta+j} + O(\rho^m)$$

and Lemma 2.1 applied to deduce that $g_{\beta+j} \wedge \bar{\partial}\rho_1 \wedge \cdots \wedge \bar{\partial}\rho_k = 0$ on M. Since this holds for every j and β, it follows from the linear independence of $\bar{\partial}\rho_1, \cdots, \bar{\partial}\rho_k$ on M that for each α, $|\alpha| = m$, and each j, $1 \leqq j \leqq k$, there is a function $h_{\alpha j}$ such that

$$g_\alpha = \sum_{j=1}^{k} h_{\alpha j} \bar{\partial}\rho_j + O(\rho) . \tag{3.6}$$

When substituted for g_α in (3.3) and (3.4) this relation yields

$$\bar{\partial} f_m = \sum_{|\alpha|=m} \sum_{j=1}^{k} \rho^\alpha h_{\alpha j} \bar{\partial}\rho_j + O(\rho^{m+1}) \tag{3.7}$$

and

$$0 = \sum_{|\alpha|=m} \sum_{i,j=1}^{k} \alpha_j \rho^{\alpha-j} h_{\alpha i} \bar{\partial}\rho_j \wedge \bar{\partial}\rho_i + O(\rho^m) . \tag{3.8}$$

The expression (3.7) suggests modifying f_m by

$$u = \frac{1}{n+1} \sum_{|\alpha|=m} \sum_{j=1}^{k} \rho^\alpha \rho_j h_{\alpha j}$$

(the need for the constant $1/(n + 1)$ will appear as a consequence of (3.11)). Now

$$(n+1)\bar{\partial} u = \sum_{\alpha,j} \rho^\alpha h_{\alpha j} \bar{\partial}\rho_j + \sum_{\alpha,j} \sum_{i=1}^{k} \rho_j \alpha_i \rho^{\alpha-i} h_{\alpha j} \bar{\partial}\rho_i + \sum_{\alpha,j} \rho^\alpha \rho_j \bar{\partial} h_{\alpha j} . \tag{3.9}$$

The first term of this is $\bar{\partial} f_m$. The second is

$$\sum_{i,j=1}^{k} \rho_j \Big(\sum_{|\alpha|=m} \alpha_i \rho^{\alpha-i} h_{\alpha j} \Big) \bar{\partial} \rho_i \,, \tag{3.10}$$

which will be shown to equal $n\bar{\partial} f_m + O(\rho^{m+1})$.

To that end, for each $i < j$, wedging (3.8) with

$$\bar{\partial}\rho_1 \wedge \cdots \wedge \widehat{\bar{\partial}\rho_i} \wedge \cdots \wedge \widehat{\bar{\partial}\rho_j} \wedge \cdots \wedge \bar{\partial}\rho_k$$

($\bar{\partial}\rho_i$ and $\bar{\partial}\rho_j$ are missing) gives the symmetry relation

$$0 = \sum_{|\alpha|=m} (\alpha_j \rho^{\alpha-j} h_{\alpha i} - \alpha_i \rho^{\alpha-i} h_{\alpha j}) + O(\rho^m) \,. \tag{3.11}$$

Using this in (3.10) it becomes

$$\sum_{i,j=1}^{k} \rho_j \Big(\sum_{|\alpha|=m} \alpha_j \rho^{\alpha-j} h_{\alpha i} \Big) \bar{\partial} \rho_i + O(\rho^{m+1})$$

which when the summation over j is performed first is

$$\sum_{|\alpha|=m} \sum_{i=1}^{k} \Big(\sum_{j=1}^{k} \alpha_j \Big) \rho^\alpha h_{\alpha i} \bar{\partial}\rho_i + O(\rho^{m+1}) \,.$$

Noting that $\sum_{j=1}^{k} \alpha_j = n$ completes the argument that the second term of (3.9) is $n\bar{\partial} f_m + O(\rho^{m+1})$. Therefore $\bar{\partial} u = \bar{\partial} f_m + O(\rho^{m+1})$.

Thus on each U_ι there is a function $u_\iota = O(\rho_\iota^{m+1})$ such that $\bar{\partial}(f_m - u_\iota) \,|\, U_\iota = O(\rho_\iota^{m+1})$. With u defined again by (3.1) and $f_{m+1} = f_m - u$ it follows as before from (3.2) that $\bar{\partial} f_{m+1}$ vanishes on M to order $m+1$. This completes the proof.

4. Remarks.

We know of no nongeneric examples where Theorem 1.2 fails. However, when M is not generic, the above proof breaks down at the inductive step from $m = 1$ to $m = 2$: Since $\bar{\partial}\rho$ does not have maximal rank it may be assumed that there is an integer $l < k$ such that $\bar{\partial}\rho_1 \wedge \cdots \wedge \bar{\partial}\rho_l$ has no zeros on M but $\bar{\partial}\rho_1 \wedge \cdots \wedge \bar{\partial}\rho_j = 0$ on M if $j > l$. Thus there are more unknowns g_α than equations available from (3.4). There are very simple cases where this occurs:

EXAMPLE 4.1. If the usual coordinates of $\mathbf{C}^2$ are denoted z_1, z_2 and $M = \{z \colon z_2 = 0\}$ then the function $f = z_2 \bar{z}_1$ is CR, for $\bar{\partial} f = z_2 d\bar{z}_1$. The most general function u vanishing to second order on M is by (the complex analogue of (2.5)) of the form

$$u = z_2^2 g_1 + z_2 \bar{z}_2 g_2 + \bar{z}_2^2 g_3$$

for suitable smooth functions g_1, g_2, and g_3. Therefore

$$\bar{\partial} u = z_2^2 \bar{\partial} g_1 + z_2 g_2 d\bar{z}_2 + z_2 \bar{z}_2 \bar{\partial} g_2 + 2\bar{z}_2 g_3 d\bar{z}_2 + \bar{z}_2^2 \bar{\partial} g_3 \,.$$

Each of these terms either vanishes to second order on M or is linearly independent of $\bar{\partial} f$. Therefore no such u will satisfy $\bar{\partial}(f - u) = O(\rho^2)$.

However since f is zero on M, it obviously satisfies the conclusion of Theorem 1.2. In fact, if M is a complex manifold, each CR function f is holomorphic, so if U is a domain of holomorphy Theorem 1.2 for U and $M \cap U$ follows from Cartan's Theorem B [2], which implies that f has a holomorphic extension to U. Moreover, standard results in several complex variables show that Theorem 1.3 is true for any complex manifold M. Thus Theorem 1.2 and a consequent Theorem 1.3 may still hold in the nongeneric case, but some new ideas for proof are necessary.

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