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INDEFINITE MINKOWSKI SPACES

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The purpose of this article is to characterize Minkowski general G-spaces. The unit sphere K is shown to have at most four components.

Assume the space R is not reducible. If K has one component, R is an ordinary Minkowski G-space. If K has two components they are quadrics and R is nearly pseudoeuclidean. When K has three components, one is a quadric and the other two are strictly convex. The unit sphere has four components only in dimension two.

The axioms of a general G-space have been given in [4] and the interesting two dimensional spaces have been investigated in [1]. We will denote the indefinite distance from x to y by xy. We refer to xy as a metric even though it is not in general a true metric.

DEFINITION 1.1. The general G-space R is called a Minkowski space if R is the real n-dimensional affine space A^n, the family of Arcs A consists of the affine segments and $w = (1/2)(x + y)$ implies $wx = wy = (1/2)xy$.

If L^r is an r-dimensional flat in R, then L^r is an r-dimensional Minkowski space with the induced distance.

Let $e(x, y)$ be an associated euclidean metrization of A^n. Then for each line L in R there is a number $\phi(L)$ such that $xy = \phi(L)e(x, y)$ for all $x, y \in L$. If $\phi(L) = 0$, we call L a null line. The number $\phi(L)$ depends continuously on L and $\phi(L) = \phi(L_1)$ if L_1 is parallel to L, see [1]. It follows that the affine translations preserve the distance xy.

Let z always denote the origin in A^n. We call $C = \{x \mid xz = 0\}$ the *light cone* and $K = \{x \mid xz = 1\}$ the *unit sphere*. If K is given the distance xy is uniquely determined.

For $x \neq y$ let $L(x, y)$ denote the line through x and y and let $\alpha(x, y)$ denote the affine segment from x to y. When $S \subset A^n$ define $-S = \{x \mid -x \in S\}$. If $S = -S$ the set S is called symmetric about z or simply symmetric. The sets C and K are symmetric.

Two general G-spaces R_1 and R_2 are said to be *topologically isometric* if there exists a topological map of R_1 onto R_2 that preserves the indefinite distance xy.

It is easily seen that if R_1 and R_2 are Minkowski spaces defined on A^n with unit spheres K and K^* respectively, then R_1 and R_2 are topologically isometric if and only if there is an affinity mapping K onto K^*.

2. Two dimensional spaces. If R is A^2, then by [4, p. 241] one of the following must hold: (1) no null lines exist in R, (2) there is exactly one null line through each point of R, (3) there are exactly two null lines through each point of R, or (4) all lines in R are null.

In case (1) we call R a *spacelike plane*. By [4, p. 239], a spacelike plane is an ordinary Minkowski G-space with unit sphere a strictly convex closed curve.

In case (2) we call R a *neutral plane*. A neutral plane is topologically isometric to the (s, t) plane with distance from (s_1, t_1) to (s_2, t_2) given by $|t_1 - t_2|$.

When R has exactly two null lines through each point it is called a *doubly timelike* (Minkowski) *plane*, see [1]. The unit sphere has four components each of which is strictly convex and not compact.

If all lines in R are null, we call R a *null plane*.

3. Reducible spaces. Let R be an n-dimensional Minkowski space. Then R is reducible to $R^r \times N^{n-r}$ for $r < n$, provided affine coordinates $x_1, x_2, \cdots, x_n$ may be chosen such that

(1) R^r is given by $x_{r+1} = x_{r+2} = \cdots = x_n = 0$ and N^{n-r} is given by $x_1 = \cdots = x_r = 0$.

(2) The projection of R onto R^r preserves the metric xy.

The maximum possible value of $n - r$ is called the index of reducibility of R. A null plane has index 2 and a neutral plane index 1. Spacelike and doubly timelike planes are not reducible.

Nonreducible spaces often contain reducible subspaces. In the three dimensional Lorentz space any plane tangent to the light cone is neutral and hence reducible.

Given a line N the parallel to N through x will always be denoted by N_x.

DEFINITION 3.1. A line N through z is called a line of reduction of R if $x \in K$ implies $N_x \subset K$.

LEMMA 3.2. *The space R is reducible if and only if R has a line of reduction.*

Proof. If N is a line of reduction of R and L^{n-1} is a hyperplane with $L^{n-1} \cap N = z$, the projection of R onto L^{n-1} along parallels to N preserves the metric.

On the other hand if R is reducible to $R^r \times N^{n-r}$ any line N through z and in N^{n-r} is a line of reduction of R.

4. **The r-flat topology.** If $\{M_m\}$ is a sequence of closed subsets of R, we say M_m converges to the closed set M if $\lim M_m = M$ in the sense of Hausdorff's closed limit, see [2]. This limit induces a topology on the closed subsets of R. If L^r is an r-flat and $W(L^r)$ is a neighborhood of L^r in this topology, let $W_r(L^r)$ denote the r-flats in $W(L^r)$.

LEMMA 4.1. *Let $\{L^2_m\}$ be a sequence of doubly timelike planes, each containing z, such that $\{L^2_m\}$ converges to the two flat L^2. Assume $x^m_i \in K \cap L^2_m$ and $x^m_i \to x_i$ for $i = 1, 2$.*

(1) *Let L^2 be doubly timelike and let x_1, x_2 lie on the same component [opposed components] of K. Then for sufficiently large m the points x^m_1 and x^m_2 always lie on the same component [opposed components] of $K \cap L^2_m$.*

(2) *If L^2 is neutral, then for sufficiently large m the points x^m_1 and x^m_2 are always on the same or else always on opposed components of $K \cap L^2_m$.*

Proof. The proofs are similar and consequently we only consider statement (2) in which L^2 is neutral.

Without loss of generality assume x_1 and x_2 are on the same component of $K \cap L^2$ since if $x^m_1 \to x_1$ then $-x^m_1 \to -x_1$.

If $y \in \alpha(x_1, x_2)$ then $y \in K$ and $zy = 1$. Therefore, there exists an open set V containing the set $\alpha(x_1, x_2)$ such that all $p \in V$ have $zp > 0$. For sufficiently large m all points of $\alpha(x^m_1, x^m_2)$ lie in V and have positive distance from z. It follows that x^m_1 and x^m_2 lie on the same component of $K \cap L^2_m$ for large m.

The components of K are arcwise connected since they are connected and locally arcwise connected.

LEMMA 4.2. *Let x_1 and x_2 lie on the same component of K and let L^2 be a two flat containing z, x_1 and x_2. If S_1 and S_2 are the components of $K \cap L^2$ containing x_1 and x_2 respectively then either $S_1 = S_2$ or else $S_1 = -S_2$.*

Proof. Let $x(t)$ for $0 \leqq t \leqq 1$ be a curve on K connecting x_1 and x_2 with $x(0) = x_1$ and $x(1) = x_2$.

Call the two flat $L^2(t)$ admissible if $z, x_1, x(t) \in L^2(t)$ and $K \cap L^2(t)$ has components S_1 and $S(t)$ containing x_1 and $x(t)$ respectively such that either $S_1 = S(t)$ or else $S_1 = -S(t)$. For sufficiently small t there must exist admissible $L^2(t)$. Set $M = \{t \in [0, 1] \mid \text{there exists an admissible } L^2(t)\}$.

We now show M is closed. If $\{L^2(t_m)\}$ is a sequence of admissible planes and $t_m \to t_0$, then there is a convergent subsequence $\{L^2(t_k)\} \subset \{L^2(t_m)\}$ such that $L^2(t_k) \to L^2_0$. Clearly $z, x_1, x(t_0) \in L^2(t_0)$. Statement (1) of Lemma 4.1 implies L_0^2 cannot be doubly timelike with x_1 and $x(t_0)$ neither on the same nor on opposed components of $K \cap L_0^2$. Therefore, $t_0 \in M$.

To show M is open let $\tau \in M$ and $L^2(\tau)$ be admissible. If $L^2(\tau)$ is spacelike there must exist a neighborhood $W_2(L^2)$ containing only spacelike planes. But this implies the existence of a neighborhood $U(\tau)$ of the number τ with $U(\tau) \subset M$. If $L^2(\tau)$ is a doubly timelike plane statement (1) of Lemma 4.1 implies the existence of a neighborhood $U(\tau) \subset M$. In case $L^2(\tau)$ is a neutral plane first construct a neighborhood $W_2(L^2(\tau))$ in which no null planes exist. If only spacelike and neutral planes exist in $W_2(L^2(\tau))$ there is nothing to show. If there is a sequence of doubly timelike planes $L^2(t_m)$ converging to $L^2(\tau)$, statement (2) of Lemma 4.1 guarantees that for large m the planes $L^2(t_m)$ are admissible. It follows that there is a neighborhood $U(\tau) \subset M$. Therefore, M is open as well as closed. Since $M \neq \phi$, $M = [0, 1]$ and the lemma is established.

THEOREM 4.3. *Let K_1 and K_2 be distinct components of K that are opposed (i.e., $K_2 = -K_1$). Then K_1 and K_2 are convex hypersurfaces.*

Proof. Let $K_1^0 = \{y \mid \alpha(z, y) \cap K_1 \neq \phi\}$. Then K_1^0 has boundary K_1 and $y \in K_1^0$ implies $zy \geqq 1$. If $y_1, y_2 \in K_1^0$ let L^2 be a two flat through z, y_1 and y_2. Then L^2 must either be neutral or doubly timelike. In either case $\alpha(y_1, y_2) \subset K_1^0$ if y_1 and y_2 lie on the same component of $K_1 \cap L^2$. Clearly y_1 and y_2 lie on the same component for L^2 neutral. If L^2 is doubly timelike, then $K_1 \neq K_2$ and Lemma 4.2 imply y_1 and y_2 lie on the same component of $K_1 \cap L^2$. It follows that K_1^0 is convex and that its boundary K_1 is a convex hypersurface. In the same fashion one may show K_2 is a convex hypersurface.

LEMMA 4.4. *Let K have a component K_1 that is symmetric about z. Then for each $x \in K_1$ there is a two flat L^2 through z and x that is spacelike.*

Proof. Assume the statement is false. Any two flat containing $L(z, x)$ is then either neutral or doubly timelike. Orient $L(z, x)$ to get $L^+(z, x)$. If L_1 is a line parallel to $L^+(z, x)$, orient L_1^+ in the same direction. This gives an ordering $<$ on each line parallel to $L(z, x)$.

Let $x(t)$ for $0 \leqq t \leqq 1$ be a curve on K_1 with $x(0) = x$, $x(1) = -x$ and $x(t) \notin L(x, -x)$ for $0 < t < 1$. Let $L^+(t)$ be the oriented line containing $x(t)$ and parallel to $L^+(z, x)$. The line $L^+(t)$ is never a null line.

In the ordering $<$ along $L^+(t)$ let $p(t)$ be the first element in $\{y \mid y \in L^+(t) \text{ and } zy = 0\}$. Let $f(t)$ be the signed euclidean distance from $x(t)$ to $p(t)$ where $f(t) < 0$ if $x(t) < p(t)$. If $z < x$ then $f(0) < 0$

and $f(1) > 0$.

The function $f(t)$ is continuous at 0 and 1 since $p(t) \to z$ for $t \to 0$ and $t \to 1$. To show $f(t)$ is continuous on $(0, 1)$ let $0 < t_0 < 1$ and $t_m \to t_0$. For $0 < t < 1$ let $L^2(t)$ denote the unique plane containing $L^+(t)$ and z. Clearly if $L(t_0)$ is neutral we have $L(z, p(t_m)) \to L(z, p(t_0))$. If $L^2(t_0)$ is doubly timelike, one can show using (1) of Lemma 4.1 that $L(z, p(t_m)) \to L(z, p(t_0))$. In either case $p(t_m) \to p(t_0)$ and $f(t)$ is continuous. But then $f(\tau) = 0$ for some $0 < \tau < 1$ which implies $x(\tau) = p(\tau)$. This is impossible since $zx(\tau) = 1$ and $zp(\tau) = 0$.

5. **Three dimensional spaces.** In this section we only consider three dimensional Minkowski spaces.

LEMMA 5.1. *Let K have three components K_1, K_2 and K_3 with $K_3 = -K_3$. Then $K_1 = -K_2$ and K_1 (hence also K_2) is strictly convex.*

Proof. By Lemma 4.4 there is a two flat L^2 through z that is spacelike with $L^2 \cap K_3 \neq \phi$. This flat separates A^3 and does not intersect K_2. Hence $K_2 \neq -K_2$. Consequently, $K_1 = -K_2$.

To see that K_1 is strictly convex let $x, y \in K_1$. If L_0^2 is a two flat through x, y and z it must be doubly timelike since $L_0^2 \cap L^2 \neq \phi$. Then $L_0^2 \cap K_1$ is a strictly convex curve. It follows that $u \in \alpha(x, y) - x - y$ implies $zu > 1$. Therefore, K_1 must be strictly convex.

If K_i is a component of K then so is $-K_i$. Consequently, if K has exactly three components there is always one, say K_3, that is symmetric about z.

Extend A^3 to the real three dimensional projective space P^3 by adding a plane L^2_∞ at ∞. The projective lines that the light cone C determine intersect L^2_∞ in a curve C_∞. Let K have exactly three components. Since spacelike planes exist in this case, there is a line $L_0 \subset L^2_\infty$ with $L_0 \cap C_\infty = \phi$. The set $L^2_\infty - L_0$ is an affine plane with L_0 the line at ∞.

Let $p, q \in C_\infty$ with $p \neq q$. Let L^2 be two flat in P^3 that contains z, p, q. Then $L^2 \cap A^3$ cannot be a null plane, since if it were it would separate A^3 and K_3 could not be symmetric. Consequently, $L^2 \cap A^3$ must be a doubly timelike plane.

It follows that $L^2 \cap (L^2_\infty - L_0)$ is an affine line in $L^2_\infty - L_0$ that intersects C_∞ in only the two points p and q. But C_∞ is a closed curve. Hence, C_∞ is a strictly convex curve in $L^2_\infty - L_0$.

THEOREM 5.2. *Let* $\dim R = 3$. *If K has three components K_1, K_2 and K_3 with $K_3 = -K_3$, then K_3 is a hyperboloid of one sheet.*

Proof. Let $u \in L^2_\infty - L_0$ and let u be exterior to the convex set in $L^2_\infty - L_0$ whose boundary is C_∞. Then there are lines L_1 and L_2 through u that are supporting lines of C_∞. Let L^2_i be the projective plane containing z and L_i for $i = 1, 2$. Then $L^2_i \cap C_\infty$ is a single point and hence $L^2_i \cap A^3$ is a neutral plane.

The set $L^2_i \cap A^3 \cap K$ consists of two parallel lines which must be on K_3 since K_1 and K_2 are strictly convex. For any $q \in K_3$ let $u = L(z, q) \cap L^2_\infty$ and without loss of generality assume $u \notin L_0$. Then u must be exterior to C_∞. By the above arguments there must be two straight lines on K_3 through q. By [5, p. 272] the set K_3 is a hyperboloid of one sheet.

Notice that the above theorem gives the additional information that C is elliptic and C_∞ is an ellipse in $L^2_\infty - L_0$.

LEMMA 5.3. *K can have at most four components. If K does have four components, R is reducible and no component of K is symmetric about z.*

Proof. Let K_1 be a component of K. Assume $K_1 = -K_1$, then there is a spacelike plane L^2_0 through z with $L^2_0 \cap K_1 \neq \phi$. Take $K_2 \neq K_1$ and $x \in K_2$. Let $L^2(\theta)$ be a two flat containing $L(z, x)$ that revolves continuously in θ and sweeps out A^3 for $0 \leqq \theta \leqq \pi$. Each $L^2(\theta)$ intersects L^2_0 in a line through z so that $L^2(\theta) \cap K_1 \neq \phi$ for all θ. Therefore, each $L^2(\theta)$ is doubly timelike and intersects K in four components. Two of these components lie on K_1, and the other two are subsets of K_2 and $-K_2$. Since this holds for all $\theta \in [0, \pi]$, K can have at most three components. Therefore, $K_1 \neq -K_1$ if K has four components.

By the above, it must be possible to find at least two components K_1 and K_2 of K with $K_1 \neq -K_1$, $K_2 \neq -K_2$ and $K_1 \neq -K_2$. Set $K_3 = -K_1$ and $K_4 = -K_2$. Let $y \in K_1$ and let $L^2(\psi)$ be a two flat through $L(z, y)$ that sweeps out A^3 continuously for $0 \leqq \psi \leqq \pi$. It can be assumed without loss of generality that $L^2(0) \cap K_2 \neq \phi$. Therefore, let x_2 belong to $L^2(0) \cap K_2$. $L^2(\psi)$ cannot be doubly timelike for all ψ or else x_2 and $-x_2$ would be on the same component of K. Therefore, there is a first ψ_0 with $L^2(\psi_0)$ neutral. Let $N \subset L^2(\psi_0)$ be the null line through z. Claim N is a line of reduction of R.

It is clear that if $x \in K_1 \cup K_3$ then $N_x \subset K_1 \cup K_3$ since these are convex surfaces and $N_y \subset K_1$ as well as $N_{-y} \subset K_3$. For $x \in K_2 \cup K_4$ consider the following argument. Let $L^2(\gamma)$ be a plane through $L(z, x_2)$ sweeping out A^3 continuously for $0 \leqq \gamma \leqq \pi$ with $y \in L^2(0)$. By the same reasoning as before, there is a first γ_0 with $L^2(\gamma_0)$ neutral. The above N must be in $L^2(\gamma_0)$ since $N_y \subset K_1$ and K_1 is not flat. This implies $N_x \subset K_2 \cup K_4$ whenever $x \in K_2 \cup K_4$.

It is now possible to show K has at most, four components. If L^2_1 is a two flat containing the above N either L^2_1 is neutral or null.

If it is null, it intersects $L^2(\gamma)$ for $\gamma = 0$ in a null line. If it is neutral, it intersects either K_1 and K_3 or else K_2 and K_4. In any case it cannot contain a point of K not on $K_1 \cup K_2 \cup K_3 \cup K_4$.

An immediate consequence is that if K has four components $R = R^2 \times N'$ where R^2 is a doubly timelike plane.

Consider now the case of K having one component. If R has no null lines, then by [4, p. 239] it is a Minkowski G-space and K must be strictly convex.

LEMMA 5.4. *Let K have one component and not be strictly convex. Then K is a cylinder and $R = R^2 \times N^1$ where R^2 is a spacelike plane.*

Proof. Let K contain a segment α and consider the two flat L_0^2 through z and α. L_0^2 must be neutral, hence the line containing α must lie on K. Let N be the null line in L_0^2 through z. Since K has only one component, there is a spacelike plane L^2 through z. Any two flat L_1^2 containing N must intersect L^2 in a line through z.

The plane L_1^2 cannot be a doubly timelike because of Lemma 4.2 and the fact that K has only one component. Therefore, L_1^2 is neutral and contains two lines on K parallel to N. It follows K must be a cylinder with generators parallel to N.

Projecting R onto L^2 along parallels to N gives $R = R^2 \times N^1$ for R^2 the spacelike plane L^2.

If K has two components K_1 and K_2 in dimension three, then $K_1 = -K_2$ since otherwise there would be a spacelike plane L^2 through z intersecting only one component of K yet separating A^3. Both K_1 and K_2 must be flat since if $x, y \in K_1$ with $x \neq y$, the two flat L_1^2 containing x, y and z would have to be neutral.

It can easily be shown that for K having two components, the space is always topologically isometric to (x_1, x_2, x_3)-space with the distance from (a_1, a_2, a_3) to (b_1, b_2, b_3) given by $|a_1 - b_1|$. K consists of two parallel planes and $R = R^1 \times N^2$ for R^1 the real line.

6. Higher dimensional spaces.

The n dimensional situation is now investigated by the use of r-flats.

LEMMA 6.1. *K_1, K_2, K_3 be three distinct components of K, then two are reflections through z of each other.*

Proof. Consider $p_i \in K_i$ for $i = 1, 2, 3$ and let L^3 be a three flat containing z, p_1, p_2, and p_3. Let $S_i = K_i \cap L^3$, then S_1, S_2, and S_3 are disjoint components of $K \cap L^3$. By the last section $K \cap L^3$ has either three or four components, and in any case, any three of the components of $K \cap L^3$ contain a pair that are symmetric to each other. If we assume $S_1 = -S_2$ then clearly $K_1 = -K_2$.

LEMMA 6.2. *K has at most four components. If K does have four components K_1, K_2, K_3 and K_4, without loss of generality, one may assume $K_1 = -K_3$ and $K_2 = -K_4$.*

Proof. Assume K has five components K_1, K_2, K_3, K_4 and K_5. Then lemma 6.1 applied to K_1, K_2 and K_3 allows the assumption $K_3 = -K_1$. Applying Lemma 6.1 to K_1, K_2 and K_4 yields $K_2 = -K_4$.

Let $p_1 \in K_1$, $p_2 \in K_2$ and $p_5 \in K_5$, then let L_3 be a three flat containing p_1, p_2, p_5 and z. $K \cap L^3$ then contains five disjoint components, which is impossible by Lemma 5.3.

LEMMA 6.3. *Let $N_x \subset K$ then if one of the following holds, N_z is a line of reduction.*

(1) *K has exactly one component.*

(2) *K has exactly two components K_1 and K_2 that are symmetric to each other.*

(3) *K has exactly three components K_1, K_2, K_3 with $K_3 = -K_3$ and $N_x \subset K_1 \cup K_2$.*

(4) *K has four components.*

Proof. The proofs of the above four cases all follow the same general pattern. Therefore, the first case is the only one discussed.

If $N_x \subset K$ and K has one component, consider $y \in R$ and let L^3 be a three flat containing z, y and N_x. Either $N_y \subset K$ or else $K \cap L^3$ has three components. If $K \cap L^3$ has three components, there is a two flat $L^2 \subset L^3$ through z that is doubly timelike. But then $K \cap L^2$ has four components, and Lemma 4.2 would imply K had more than one component.

For convenience the following notation is adopted. If $k, p, \cdots, m$ are r distinct integers from the set $1, 2, \cdots, n$ let $L^r_{kp\cdots m}$ be the unique r-flat through the $x_k, x_p, \cdots, x_m$ axes. If L_0 is a line with $L_0 \not\subset L^r_{kp\cdots m}$ let $L^{r+1}_{0kp\cdots m}$ be the $r+1$ flat containing L_0 and $L^r_{kp\cdots m}$. Here we assume $L_0 \cap L^r_{kp\cdots m} \neq \phi$.

Repeated application of the last lemma gives the following partial description of the nonreducible spaces:

THEOREM 6.4. *In all cases K has at most four components. Let R be nonreducible.*

(1) *If K has one component, then R is a Minkowski G-space.*

(2) *If K has two components that are opposed to each other then R is isometric to the real line.*

(3) *If K has three components, then one is symmetric about z*

and the other two are strictly convex.

(4) *If K has four components, then R is a doubly timelike plane.*

The case where K has two components which are not opposed is discussed in Theorem 6.13 and additional information on the case of three components is found in Theorem 6.8.

LEMMA 6.7. *Let $n = 3$ and K have three components. Assume coordinates x_1, x_2, x_3 are chosen such that the light cone is given by $x_1^2 + x_2^2 = x_3^2$. Then the plane $x_3 = 0$ intersects K_3 in a set $x_1^2 + x_2^2 = a^2$ for some $a > 0$.*

Proof. Let p lie on K_3 and in the plane $x_3 = 0$. For some $a > 0$ the point p lies on $x_1^2 + x_2^2 - x_3^2 = a^2$. We claim that the only hyperboloid of one sheet containing p that has C as light cone is $x_1^2 + x_2^2 - x_3^2 = a^2$.

Since p is contained in exactly two planes tangent to C, the two lines on K_3 through p are determined. For any q on one of these two lines, the same argument yields that the two lines on K_3 through q are determined. It follows K_3 is determined by p and C.

Consider now $n > 3$ and extend A^n to P^n by adding a hyperplane L_∞^{n-1} at ∞. Let the projective lines that contain the lines of the light cone C intersect L_∞^{n-1} in a set C_∞.

If R is nonreducible and K has three components, let L_0^{n-1} be a supporting hyperplane to K_1. If L^{n-1} is the hyperplane parallel to L_0^{n-1} through z, then $L^{n-1} \cap C = z$. Otherwize $L^{n-1} \cap C$ would contain a line N. For $p \in L_0^{n-1} \cap K_1$ then the two flat L^2 through p and N would be neutral or doubly timelike. It could not be neutral because of Lemma 6.3. It could not be doubly timelike since then N_p would not be a supporting line of K_1.

Set $L^{n-1} \cap L_\infty^{n-1} = L_\infty^{n-2}$ an $n - 2$ dimensional flat. By taking L_∞^{n-2} as the $n - 2$ flat at ∞ of L_∞^{n-1} the set $L_\infty^{n-1} - L_\infty^{n-2}$ becomes an $n - 1$ dimensional affine space. Let $x, y \in C_\infty$ for $x \neq y$ and let L_1^2 be the two flat containing x, y and z. Then $L_1^2 \cap A^n$ is a doubly timelike plane. In the same manner as the argument after Lemma 5.1, we conclude C_∞ is a strictly convex $n - 2$ dimensional surface in the space $L_\infty^{n-1} - L_\infty^{n-2}$.

LEMMA 6.6. *C_∞ is an ellipsoid in $L_\infty^{n-1} - L_\infty^{n-2}$.*

Proof. Let L_∞^2 be a two flat in L_∞^{n-1} with $L_\infty^2 \cap C_\infty$ containing more than one point. Let L^3 be the three flat containing z and L_∞^2. Then $L^3 \cap A^n$ is an indefinite metric space whose unit sphere has three components. By Theorem 5.2, $L_\infty^2 \cap C_\infty$ is an ellipse and hence by [2, p. 91] C_∞ is an ellipsoid.

Take now coordinates $x_1, x_2, \cdots, x_n$ in A^n such that C has the form

$x_n^2 = x_1^2 + \cdots + x_{n-1}^2$ and let L_1^{n-1} be the hyperplane $x_n = 0$.

LEMMA 6.7. *$L_1^{n-1} \cap K$ has the form $x_1^2 + \cdots + x_{n-1}^2 = a^2$ for $a > 0$.*

Proof. Let L^2 be any two flat in L_1^{n-1} passing through z. Let L^3 be the three flat containing L^2 and the x_n axis. Since $L^3 \cap K$ always has three components, $L^2 \cap K$ is always an ellipse of center z. Therefore, $L_1^{n-1} \cap K$ is an ellipsoid in L_1^{n-1} of center z.

If L^2 contains the x_i and x_j axis Lemma 6.5 implies $L^2 \cap K_3$ has the form $x_i^2 + x_j^2 = a_{ij}^2$. If p_i and p_j are points of $L^2 \cap K_3$ that lie on the i^{th} and j^{th} axes respectively, $|p_i|^2 = |p_j|^2 = a_{ij}^2$. Therefore, a_{ij} is independent of i and j. Setting $a = a_{ij}$ yields the desired result.

THEOREM 6.8. *Let R be nonreducible and K have three components. If K_3 is the components of K symmetric about z it is a quadric. In proper affine coordinates K_3 is given by*

$$x_1^2 + \cdots + x_{n-1}^2 - x_n^2 = a^2 .$$

Proof. Using the same notation as in Lemma 6.9 define

$$S = \{(x_1, x_2, \cdots, x_n) \mid x_1^2 + \cdots + x_{n-1}^2 - x_n^2 = a^2\} .$$

If L^3 contains the x_n axis then $L^3 \cap S = L^3 \cap K_3$. The result follows by letting L^3 sweep out A^n.

In order to investigate nonreducible spaces in which K has two components, we first consider nondegenerate central quadrics that have z as a center. The general form in affine space is

$$\sum_{i,j=1}^{n} a_{ij}x_ix_j = 1 \text{ where } a_{ij} = a_{ji} \text{ and } \det(a_{ij}) \neq 0 .$$

If two such quadrics E_1 and E_2 are given respectively by

$$\sum a_{ij}\, x_ix_j = 1 \text{ and } \sum a_{ij}x_ix_j = -\lambda^2 \text{ for } \lambda > 0 ,$$

they will be called *semiconjugate.* We will refer to E_1 as the λ semiconjugate to E_2. For $\lambda = 1$ the quadrics are conjugate in the usual sense. Notice that one of the quadrics does not have a real locus if the quadric form is definite.

LEMMA 6.9. *Suppose the nonempty sets B_1 and B_2 contained in $\bigcup_{i \neq j} L_{ij}^2$ are such that the locus $B_2 \cap L_{ij}^2$ is always the λ semiconjugate quadric to $B_1 \cap L_{ij}^2$ for fixed λ. Then there are exactly two central quadrics E_1 and E_2 such that $E_1 \cap L_{ij}^2 = B_1 \cap L_{ij}^2$ and $E_2 \cap L_{ij}^2 = B_2 \cap L_{ij}^2$ for all $i \neq j$. Furthermore, E_2 is the λ semiconjugate to E_1.*

LEMMA 6.10. *Let $n = 4$ and K have two components K_1 and K_2*

each symmetric about z. Let L^3 be a three flat through z such that $L^3 \cap K$ has three components. Then $L^3 \cap K$ consists of two semiconjugate quadrics.

Proof. By Theorem 5.2 one component of $L^3 \cap K$ must be a hyperboloid of one sheet. Choose coordinates x_1, x_2, x_3 in L^3 such that $L^3 \cap C$ takes the form $x_1^2 + x_2^2 = x_3^2$. Let $L^3 \cap K$ have components S_1, S_2, S_3 with $S_3 = -S_3$. For some $a > 0$, S_3 is given by $x_1^2 + x_2^2 - x_3^2 = a^2$. Let L_0 be a line through z in L_{12}^2.

In R let L^2 be a spacelike plane containing the x_3 axis, so $L^2 \not\subset L^3$. Choose the x_4 axis in L^2. Assume K has components K_1 and K_2 with $S_3 \subset K_1$, then $L_{034}^3 \cap K_2$ is a hyperboloid of one sheet in L_{034}^3. Consequently, $L_{03}^2 \cap K_2$ is a hyperbola. This hyperbola is determined given only the intersection of K_2 with the x_3 axis and the intersection of L_{03}^2 with the surface $x_1^2 + x_2^2 = x_3^2$ in L^3.

Revolving L_0 in the plane L_{12}^2 shows $L^3 \cap K_2$ consists of a hyperboloid of two sheets that is a semiconjugate of $L^3 \cap K_1$.

LEMMA 6.11. *If $n = 4$ and K has two symmetric components, they are semiconjugate quadrics.*

Proof. Let the notation and coordinates be the same as in the last proof. Set $B_1 = \bigcup_{i \neq j} (L_{ij}^2 \cap K_1)$ and $B_2 = \bigcup_{i \neq j} (L_{ij}^2 \cap K_2)$.

If $L^3 \cap K_2$ is the λ semiconjugate to $L^3 \cap K_1$ in L^3, then $L_{034}^3 \cap K_2$ is the λ semiconjugate to $L_{034}^3 \cap K_1$ in L_{034}^3 for the same λ. This follows since L_{03}^2 is common to both three flats and intersects both components of K. Therefore, B_1 and B_2 satisfy the hypothesis of Lemma 6.9. Let E_1 and E_2 be the semiconjugate quadrics determined by B_1 and B_2.

$L^3 \cap E_1 = L^3 \cap K_1$ since each are quadrics in L^3 determined by $B_1 \cap L^3$ and $B_2 \cap L^3$. By the same reasoning, $L^3 \cap E_2 = L^3 \cap K_2$. Also $L_{124}^3 \cap K_i = L_{124}^3 \cap K_i$ for $i = 1, 2$.

Therefore, $L_{0j}^2 \cap K_i = L_{0j}^2 \cap E_i$ for $i = 1, 2$ and $j = 3, 4$. But then using Lemma 6.11 one last time, we find $L_{034}^3 \cap E_i = L_{034}^3 \cap K_i$. By revolving L_0 in L_{12}^2 it follows $E_i = K_i$ for $i = 1, 2$.

LEMMA 6.12. *Let $n = 5$ and K have two components K_1 and K_2 symmetric about z. If R is not reducible, K_1 and K_2 are semiconjugate quadrics.*

Proof. Two cases are considered.

Case 1. Let there exist a three flat L^3 through z such that $L^3 \cap K$ has one component. Assume $L^3 \cap K_2 \neq \phi$. Choose coordinates x_1, x_2, x_3 in L^3. We may assume that $L_{12}^2, L_{13}^2, L_{23}^2$ are spacelike planes. Choose coordinates x_4, x_5 such that L^2_{45} is spacelike and intersects K_1. By arguments as in Lemma 6.10 and Lemma 6.11, it is possible to show $L^2_{ij} \cap K_1$ and $L^2_{ij} \cap K_2$ are always semiconjugate quadrics for fixed λ. Therefore, $B_1 = \bigcup_{i \neq j} (L^2_{ij} \cap K_1)$ and $B = \bigcup_{i \neq j} (L^2_{ij} \cap K_2)$ satisfy the hypothesis of Lemma 6.9.

Let E_1 and E_2 be the quadrics determined by B_1 and B_2. Let L_0 be a line through z in L^2_{12}. Since $L^2_{12j} \cap E_i = L^3_{12j} \cap K_i$, clearly $L^2_{0j} \cap E_i = L^2_{0j} \cap K_1$ for $i = 1, 2$ and $j = 3, 4, 5$. Therefore $L^4_{0345} \cap E_i = L^4_{0345} \cap K_i$. By revolving L_0 in L^2_{12} it follows that $E_i = K_i$.

Case 2. Assume no L^3 through z exists with $L^3 \cap K$ having only one component. We will show this leads to a contradiction.

Choose coordinates x_1, x_2, x_3, x_4, x_5 such that L^2_{12} and L^2_{34} are spacelike planes intersecting respectively K_1 and K_2. By Theorem 6.8, the set $K \cap L^4_{2345}$ cannot have exactly three components. Consequently, $L^3_{2345} \cap K$ consists of two symmetric components. The same must also be true of $L^4_{1235} \cap K$.

By Lemma 6.11 the sets $L^4_{1234} \cap K$, $L^4_{2345} \cap K$ and $L^4_{1235} \cap K$ each consists of two quadrics. In each of the three sets one quadric is the semiconjugate of the other for some fixed λ. Define

$$B_1 = \bigcup_{i \neq j} (L^2_{ij} \cap K_1) \quad \text{and} \quad B_2 = \bigcup_{i \neq j} (L^2_{ij} \cap K_2) .$$

Let E_1 and E_2 be the quadrics determined.

Let L_0 be a line through z in L^2_{12}. Then $L^2_{0j} \cap K_i = L^2_{0j} \cap E_i$ for $j = 3, 4, 5$ and $i = 1, 2$. Therefore, $L^4_{0345} \cap E_i = L^4_{0345} \cap K_i$ and revolving L_0 in L^2_{12} gives $E_i = K_i$ for $i = 1, 2$.

Then in proper affine coordinates y_1, y_2, y_3, y_4, y_5 the components of K are given by $y_1^2 + y_2^2 + y_3^2 - y_4^2 - y_5^2 = 1$ and $y_1^2 + y_2^2 + y_3^2 - y_4^2 - y_5^2 = -\lambda^2$. This contradicts the assumption of Case 2.

The n dimensional case now follows using induction.

THEOREM 6.13. *If R is not reducible and K has two components which are not opposed, then $n \geqq 4$ and the components are semiconjugate quadrics.*

Proof. Assume $n \geqq 6$. Take L^{n-1} to be a hyperplane containing L^2_1 and L^2_2, which are spacelike two flats through z with $L^2_i \cap K_i \neq \phi$. Then $L^{n-1} \cap K$ has exactly two symmetric components. Because of Lemma 6.12, there exists an L^3 through z and contained in L^{n-1} with $L^3 \cap K$ having one component. Take the x_1, x_2, x_3 affine coordinates in L^3 and $x_1, x_2, \cdots, x_{n-1}$ affine coordinates in L^{n-1}. For $p \in K - L^{n-1}$ let the x_n axis be $L(z, p)$. Take L_0 to be a line through z in L^2_{12}. By induction $L^{n-1}_{0345\cdots n} \cap K_i$ must consist of two semiconjugate quadrics. The argument is the same as before, letting L_0 revolve in L^2_{12}.

An interesting result of this section is the following.

COROLLARY 6.14. *If R is a nonreducible Minkowski space and not a G-space, then any spacelike plane in R is euclidean.*

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