PACIFIC JOURNAL OF MATHEMATICS
Vol. 33, No. 3, 1970

COHOMOLOGY OF NONASSOCIATIVE ALGEBRAS

NEAL D. GLASSMAN

A cohomology theory is constructed for an arbitrary non-associative (not necessarily associative) algebra satisfying a set of identities, within which the associative and Lie theories are special cases.

1. Exactness of the fundamental sequence through H^3. Let T be a set of identities, $\mathscr{A}$ a T-algebra over a commutative ring K with unit, M a T-bimodule for $\mathscr{A}$. When T is clear we call M an $\mathscr{A}$-bimodule. Let $(U(\mathscr{A}), \lambda_{\mathscr{A}}, \rho_{\mathscr{A}})$ be the universal T-multiplication envelope of $\mathscr{A}$ with $\lambda_{\mathscr{A}}, \rho_{\mathscr{A}}$ the canonical maps. When $\lambda_{\mathscr{A}}, \rho_{\mathscr{A}}$ are obvious, we write $U(\mathscr{A})$. Let $D(\mathscr{A}, M)$ be the K-module (under pointwise addition and scalar multiplication) of derivations from $\mathscr{A}$ to M. $\nu \in \mathrm{Hom}_{U(\mathscr{A})}(M_1, M_2)$ induces $D(\mathscr{A}, \nu) \in \mathrm{Hom}_K(D(\mathscr{A}, M_1), D(\mathscr{A}, M_2))$ in the obvious fashion. For further details of these objects see Jacobson [16].

Regarding $U(\mathscr{A})$ as the free $\mathscr{A}$-bimodule on one generator, we define, for $u \in U(\mathscr{A}), f_u: U(\mathscr{A}) \to U(\mathscr{A})$ such that $1_{U(\mathscr{A})} f_u = u$. $D(\mathscr{A}, U(\mathscr{A}))$ is a left $U(\mathscr{A})$-module under the multiplication $ud = dD(\mathscr{A}, f_u)$.

DEFINITION. An *inner derivation functor* is an epimorphism preserving subfunctor of $D(\mathscr{A}, \ )$.

For example, suppose $\mathscr{A}$ is Jordan. Define $J(\mathscr{A}, M)$ to be the K-module generated by all mappings of the form $\sum_i [R_{a_i} R_{m_i}]$ where $a_i \in \mathscr{A}$ and $m_i \in M$. Then $J(\mathscr{A}, M) \subseteqq D(\mathscr{A}, M)$ and J is an inner derivation functor.

THEOREM 1. *There is a one-to-one correspondance between the set of inner derivation functors and the set of left $U(\mathscr{A})$ submodules of $D(\mathscr{A}, U(\mathscr{A}))$.*

Proof. If $J(\mathscr{A}, \ ) \subseteqq D(\mathscr{A}, \ )$ is an inner derivation functor, define $\theta(J) = J(\mathscr{A}, U(\mathscr{A}))$. We need to define an inverse $\psi = \theta^{-1}$. Let $\Lambda \subseteqq D(\mathscr{A}, U(\mathscr{A}))$ be a sub-$U(\mathscr{A})$ module. If $M = \sum_{i \in \Gamma} \bigoplus U(\mathscr{A})$, define $J(\mathscr{A}, M) = \sum_{i \in \Gamma} \bigoplus \Lambda_i$, where $\Lambda_i \simeq \Lambda$ for all i. If M is any unital right $U(\mathscr{A})$-module, let X_M be the free unital right $U(\mathscr{A})$-module on the set M. Let Ω_M be the composite $\sum_{m \in M} \bigoplus \Lambda_m = J(\mathscr{A}, X_M) \xrightarrow{i} \sum_{m \in M} \bigoplus D(\mathscr{A}, X_m) = D(\mathscr{A}, X_M) \xrightarrow{D(\mathscr{A}, \Pi)} D(\mathscr{A}, M)$, where Π is the canonical projection $\Pi: X_M \to M$. Define $J(\mathscr{A}, M) = \text{image } \Omega_M$.

It is easy to see that the two definitions of J on free bimodules agree.

Let $\nu\colon M_1 \to M_2$ be a map of $\mathscr{A}$-bimodules. ν induces $X_\nu\colon X_{M_1} \to X_{M_2}$ by applying ν to generators. Consider the diagram

$$\begin{tikzcd}
J(\mathscr{A}, X_{M_1}) \arrow[r, "J(\mathscr{A}, X_\nu)"] \arrow[d, "i"'] & J(\mathscr{A}, X_{M_2}) \arrow[d, "i"] \\
D(\mathscr{A}, X_{M_1}) \arrow[r, "D(\mathscr{A}, X_\nu)"] \arrow[d, "D(\mathscr{A}, \Pi)"'] & D(\mathscr{A}, X_{M_2}) \arrow[d, "D(\mathscr{A}, \Pi)"] \\
D(\mathscr{A}, M_1) \arrow[r, "D(\mathscr{A}, \nu)"] & D(\mathscr{A}, M_2)
\end{tikzcd}$$

where i is the inclusion. By restricting $D(\mathscr{A}, X_\nu)$ to Λ_m for each $m \in M_1$ we get $J(\mathscr{A}, X_\nu)$ making the entire diagram commutative.

Define

$$\begin{aligned} J(\mathscr{A}, \nu) &= D(\mathscr{A}, \nu)/\text{image } iD(\mathscr{A}, \Pi) \\ &= D(\mathscr{A}, \nu)/J(\mathscr{A}, M_1) . \end{aligned}$$

By commutativity, $J(\mathfrak{A}, \nu)$ takes on values in $J(\mathscr{A}, M_2)$ and is an epimorphism if ν is. Hence J is an inner derivation functor.

Finally, we show that θ and Ψ are inverses. Given $\Lambda \subseteqq D(\mathscr{A}, U(\mathscr{A}))$, $\theta\Psi(\Lambda) = \Psi(\Lambda)(\mathscr{A}, U(\mathscr{A})) = \Lambda$. Conversely, given an inner derivation functor J, $\theta(J) = J(\mathscr{A}, U(\mathscr{A}))$, $\Psi(\theta(J))(\mathscr{A}, U(\mathscr{A}) = J(\mathscr{A}, U(\mathscr{A}))$. Hence, by definition of Ψ and additivity of J, $\Psi(\theta(J)(\mathscr{A}, X_M) = J(\mathscr{A}, X_M)$ for any $\mathscr{A}$-bimodule M. Then, since both J, $\Psi\theta(J)$ are subfunctors of $D(\mathscr{A},\ )$ preserving epimorphsims, they must agree on all bimodules M.

DEFINITION. Let J be an inner derivation functor. $H_J^1(\mathscr{A}, M) = D(\mathscr{A}, M)/J(\mathscr{A}, M)$. If $\alpha\colon M_1 \to M_2$, $H_J^1(\mathscr{A}, \alpha)$ is the K-module homomorphism induced by $D(\mathscr{A}, \alpha)$. Clearly, this makes $H_J^1(\mathscr{A},\ )$ a functor from $\mathscr{A}$-bimodules to K-modules.

DEFINITION. Let $\{d_i\}_{i \in \Gamma} \subseteqq D(\mathscr{A}, U(\mathscr{A}))$. An inner derivation functor J is *generated by* $\{d_i\}_{i \in \Gamma}$ if J corresponds to the left $U(\mathscr{A})$-submodule of $D(\mathscr{A}, U(\mathscr{A})$ generated by $\{d_i\}_{i \in \Gamma}$. J is *finitely generated* if J is generated by some finite set $\{d_i\}_{i=1}^k \subseteqq D(\mathscr{A}, U(\mathscr{A}))$.

Let J be a finitely generated inner derivation functor, say by $\{d_i\}_1^k$. Let X_i be the free $\mathscr{A}$-bimodule on one generator x_i. Then there is a unique morphism of bimodules $\xi_i\colon U(\mathscr{A}) \to X_i$ such that $1_{U(\mathscr{A})}\xi_i = x_i$. We write $\bar{d}_i = d_i \circ \xi_i$, the composite. Note that $\bar{d}_i \in D(\mathscr{A}, X_i)$. Let Y be the $U(\mathscr{A})$-submodule of $\sum_1^k \oplus X_i$ generated by

$\{\mathscr{A}(\sum_1^k \bar{d}_i)\}$. Let $C_{\{d_i\}} = \sum_1^k X_i/Y$.

DEFINITION. $H^0_{J,\{d_i\}}(\mathscr{A}, M) = \mathrm{Hom}_{U(\mathscr{A})}(C_{\{d_i\}}, M)$. If $\alpha: M_1 \to M_2$, then $H^0_{J,\{d_i\}}(\mathscr{A}, \alpha)$ is the K-module morphism induced by $\mathrm{Hom}_{U(\mathscr{A})}(C_{\{d_i\}}, \alpha)$.

These definitions clearly make $H^0_{J,\{d_i\}}(\mathscr{A},\ \ )$ a functor from $\mathscr{A}$-bimodules to K-modules. For any short exact sequence of $\mathscr{A}$-bimodules $0 \to M' \to M \to M'' \to 0$, the sequence $0 \to H^0_{J,\{d_i\}}(\mathscr{A}, M') \to H^0_{J,\{d_i\}}(\mathscr{A}, M) \to H^0_{J,\{d_i\}}(\mathscr{A}, M'')$ is exact.

In the sequel, we use the notation $[x/x$ satisfies $P]$ to mean the submodule generated by the set of x satisfying P. If f and g are homomorphism, d a derivation, we write their composites as fg, $f\circ d$, $d\circ f$.

THEOREM 2. *Let M be an $\mathscr{A}$-bimodule, $f_m \in \mathrm{Hom}_{U(\mathscr{A})}(U(\mathscr{A}), M)$ such that $1_{U(\mathscr{A})} f_m = m \in M$. Then $H^0_{J,\{d_i\}}(\mathscr{A}, M)$ is isomorphic to the K-module of all k-tuples $(m_i)_1^k$ such that $\sum_1^k d_i \circ f_{m_i} = 0.]$*

Proof. This is immediate from the fact that $\sum_1^k d_i \circ f_{m_i} = \sum_1^k \bar{d}_i \circ \xi_i^{-1} f_{m_i} = (\sum_1^k \bar{d}_i)\circ f_{m_1,\cdots m_k}$, where $f_{m_1,\cdots m_k}\ \mathrm{Hom}_{U(\mathscr{A})}(\sum_1^k X_i, M)$ such that $x_i f_{m_1,\cdots m_k} = m_i$. But by the definition of $C_{\{d_i\}}$ as $\sum_1^k \oplus X_i/[\mathscr{A} \sum \bar{d}_i]$, $H^0_{J,\{d_i\}}(\mathscr{A}, M) = \mathrm{Hom}_{U(\mathscr{A})}(C_{\{d_i\}}, M) \simeq [f_{m_1,\cdots,m_k}/(\sum_1^k \bar{d}_i)\circ f_{m_1,\cdots,m_k} = 0.]$

LEMMA 1. *$D(\mathscr{A},\ \ )$ is a left exact functor from* $\mathscr{A}$-bimodules *to K-modules.*

Proof. Form the right $U(\mathscr{A})$-module $\mathscr{A} \otimes_k U(\mathscr{A})$. Let P be the submodule generated by $\{a_1 \otimes a_2^\rho - a_1 a_2 \otimes 1 + a_2 \otimes a_1^\lambda / a_1, a_2 \in \mathscr{A}\}$. Then it is easily seen that $D(\mathscr{A}, M) \simeq \mathrm{Hom}_{U(\mathscr{A})}(\mathscr{A} \otimes U(\mathscr{A})/P, M)$ for all M. But $\mathrm{Hom}_{U(\mathscr{A})}(\mathscr{A} \otimes U(\mathscr{A})/P,\ \ )$ is left exact.

Let $0 \to M' \xrightarrow{\chi} M \xrightarrow{\sigma} M'' \to 0$ be an exact sequence of $\mathscr{A}$-bimodules, J generated by $\{d_i\}_1^k$, $C_{\{d_i\}}$ defined as above. Let $f \in \mathrm{Hom}_{U(\mathscr{A})}(C_{\{d_i\}}, M'') = \mathrm{Hom}_{U(\mathscr{A})}(\sum_1^k \oplus X_i/Y, M'')$. Lift f uniquely to $f_1 \in \mathrm{Hom}_{U(\mathscr{A})}(\sum_1^k \oplus X_i, M'')$ and choose $f_2 \in \mathrm{Hom}_{U(\mathscr{A})}(\sum_1^k \oplus X_i, M)$ so that $f_2\sigma = f_1$.

Since $\sum_1^k \bar{d}_i \in J(\mathscr{A}, \sum_1^k \oplus X_i)$, $(\sum_1^k \bar{d}_i)\circ f_2 \in J(\mathscr{A}, M) \subseteqq D(\mathscr{A}, M)$. Since $\mathscr{A} \sum_1^k \bar{d}_i \subseteqq Y$, $f_2\sigma = f_1$ and $f_1/Y = 0$, we have $(\sum_1^k \bar{d}_i)\circ f_2\sigma = 0$. Hence $\mathscr{A}(\sum_1^k \bar{d}_i)\circ f_2 \subseteqq M'\chi$ and, regarding M' as a submodule of M, $(\sum_1^k \bar{d}_i)\circ f_2$ can be considered as an element of $D(\mathscr{A}, M')$.

DEFINITION. $\delta^0_{\{d_i\}}: H^0_{J,\{d_i\}}(\mathscr{A}, M'') \to H^1_J(\mathscr{A}, M')$ is defined by $f\delta^0_{\{d_i\}} = (\sum_1^k \bar{d}_i)\circ f_2 + J(\mathscr{A}, M') \in D(\mathscr{A}, M')/J(\mathscr{A}, M')$.

LEMMA 2. *$\delta^0_{\{d_i\}}$ is well-defined and natural. Further, if $\{d'_i\}_1^{k'}$ is* *another finite generating set for J, there are K-module morphisms Φ, Ω, such that the square*

$$\begin{array}{ccc} H^0_{J,\{d_i\}}(\mathscr{A}, M'') & \xrightarrow{\delta^0_{\{d_i\}}} & H^1_J(\mathscr{A}, M') \\ \Omega\Big\downarrow \Big\uparrow\Phi & & =\Big\downarrow \Big\uparrow= \\ H^0_{J,\{d'_i\}}(\mathscr{A}, M'') & \xrightarrow{\delta^0_{\{d'_i\}}} & H^1_J(\mathscr{A}, M') \end{array}$$

commutes.

This is an easy exercise in diagram chasing.

By the last part of the preceeding lemma, we may drop the subscript on $\delta^0_{\{d_i\}} = \delta^0$. In order to begin the exactness proof, we need the following lemma.

LEMMA 3. *Let J be an inner derivation functor generated by $\{d_i\}_1^{k<\infty}$. Let $d \in J(\mathscr{A}, M)$. Then there exists an $f \in \mathrm{Hom}_{U(\mathscr{A})}(\sum_1^k \oplus X_i, M)$ such that $(\sum_1^k \bar{d}_i)\circ f = d$.*

Proof. There is a $\gamma \in \sum_{m\in M} J(\mathscr{A}, X_m)$ such that $\gamma J(\mathscr{A}, \Pi_M) = d$. Write $\gamma = \sum_m \beta_m$, $\beta_m \in J(\mathscr{A}, X_m)$ and $\beta_m \neq 0$ only finitely many times. Each $\beta_m = \sum_i u_{i,m} d_{i,m}$, $u_{i,m} \in U(\mathscr{A})$ where the second subscript indicates that d belongs to the mth direct summand. Then, we easily see that $d = \gamma J(\mathscr{A}, \Pi_M) = (\sum_i \bar{d}_i)\circ f$ where $x_i f = \sum_m m u_{i,m}$.

LEMMA 4. *If $0 \to M' \xrightarrow{\chi} M \xrightarrow{\sigma} M'' \to 0$ is an exact sequence of $\mathscr{A}$-bimodules, J an inner derivation functor generated by $\{d_i\}_1^k$, then the sequence*

$$\begin{aligned} 0 \longrightarrow H^0_{J,\{d_i\}}(\mathscr{A}, M') \longrightarrow H^0_{J,\{d_i\}}(\mathscr{A}, M) \longrightarrow H^0_{J,\{d_i\}}(\mathscr{A}, M'') \\ \longrightarrow H^1_J(\mathscr{A}, M') \longrightarrow H^1_J(\mathscr{A}, M) \longrightarrow H^1_J(\mathscr{A}, M'') \end{aligned}$$

is exact.

Proof. We have already seen exactness through $H^0_{J,\{d_i\}}(\mathscr{A}, M)$.

Exactness at $H^0_{J,\{d_i\}}(\mathscr{A}, M'')$.

Let $f \in H^0_{J,\{d_i\}}(\mathscr{A}, M) = \mathrm{Hom}_{U(\mathscr{A})}(C_{\{d_i\}}, M), fH^0_{J,\{d_i\}}(\mathscr{A}, \sigma) = f\sigma \in H^0_{J,\{d_i\}}(\mathscr{A}, M'')$. Then $(fH^0_{J,\{d_i\}}(\mathscr{A}, \sigma))\delta^0 = (\sum_1^k \bar{d}_i)\circ f + J(\mathscr{A}, M')$. But since $f \in \mathrm{Hom}_{U(\mathscr{A})}(C_{\{d_i\}}, M)$, $f/Y = 0$ and, therefore, $(\sum_1^k \bar{d}_i)\circ f = 0$. Then $H^0_{J,\{d_i\}}(\mathscr{A}, \sigma)\delta^0 = 0$.

Next, let $f \in \mathrm{Hom}_{U(\mathscr{A})}(C_{\{d_i\}}, M'')$ and $f\delta^0 = 0$. This means that if $\bar{f} \in \mathrm{Hom}_{U(\mathscr{A})}(\sum_1^k \oplus X_i, M)$ is any lifting of f, as before, then

$(\sum_1^k \bar{d}_i)\circ\bar{f} \in J(\mathscr{A}, M'\chi)$. Hence, there is $\tilde{f} \in \mathrm{Hom}_{U(\mathscr{A})}(\sum_1^k \oplus X_i, M')$ such that $(\sum_1^k \bar{d}_i)\circ\tilde{f}\chi = (\sum_1^k \bar{d}_i)\circ\bar{f}$ by the previous lemma. Consider $\bar{f} - \tilde{f}\chi \in \mathrm{Hom}_{U(\mathscr{A})}(\sum_1^k \oplus X_i, M)$. We have $(\sum_1^k \bar{d}_i)\circ(\bar{f} - \tilde{f}\chi) = 0$; hence $Y(\bar{f} - \tilde{f}\chi) = 0$, and $(\bar{f} - \tilde{f}\chi) \in \mathrm{Hom}_{U(\mathscr{A})}(C_{\{d_i\}}, M) = H^0_{J,\{d_i\}}(\mathscr{A}, M)$. Further $(\bar{f} - \tilde{f}\chi)H^0_{J,\{d_i\}}(\mathscr{A}, \sigma) = (\bar{f} - \tilde{f}\chi)\sigma = \bar{f}\sigma - \tilde{f}\chi\sigma = \bar{f}\sigma = f$. That is, $\bar{f} - \tilde{f}\chi$ is the required preimage.

Exactness at $H^1_J(\mathscr{A}, M')$.

Let $f \in H^0_{J,\{d_i\}}(\mathscr{A}, M'')$. Then $f\delta^0 \in D(\mathscr{A}, M')/J(\mathscr{A}, M')$ is gotten by restricting the image of some element of $J(\mathscr{A}, M)$ to M'. Hence $f\delta^0 H^1_J(\mathscr{A}, \chi) = 0$.

Let $d \in D(\mathscr{A}, M')$ be a representative of an element of $H^1_J(\mathscr{A}, M')$ with $(d + J(\mathscr{A}, M'))H^1_J(\mathscr{A}, \chi) = 0$. This means that $d\circ\chi \in J(\mathscr{A}, M)$. Hence, by the previous lemma, there exists $f \in \mathrm{Hom}_{U(\mathscr{A})}(\sum_1^k \oplus X_i, M)$ such that $(\sum_1^k \bar{d}_i)\circ f = d\circ\chi$. Consider $f\sigma \in \mathrm{Hom}_{U(\mathscr{A})}(\sum_1^k \oplus X_i, M'')$. $(\sum_1^k \bar{d}_i)\circ f\sigma = d\circ\chi\sigma = 0$. Hence $Yf\sigma = 0$ and $f\sigma \in \mathrm{Hom}_{U(\mathscr{A})}(C_{\{d_i\}}, M'') = H^0_{J,\{d_i\}}(\mathscr{A}, M'')$. Clearly $(f\sigma)\delta^0 = d + J(\mathscr{A}, M')$.

Exactness at $H^1_J(\mathscr{A}, M)$.

Clearly $H^1_J(\mathscr{A}, \chi)H^1_J(\mathscr{A}, \sigma) = 0$. Suppose $d \in D(\mathscr{A}, M)$ is a representative of an element of $H^1_J(\mathscr{A}, M)$ and $(d + J(\mathscr{A}, M'')H^1_J(\mathscr{A}, \sigma) = 0$. This means $d\circ\sigma \in J(\mathscr{A}, M'')$. Then there exists $f \in \mathrm{Hom}_{U(\mathscr{A})}(\sum_1^k \oplus X_i, M'')$ such that $(\sum_1^k \bar{d}_i)\circ f = d\sigma$ and there exists $\bar{f} \in \mathrm{Hom}_{U(\mathscr{A})}(\sum_1^k \oplus X_i, M)$ such that $\bar{f}\sigma = f$. Consider $d - (\sum_1^k \bar{d}_i)\circ\bar{f} \in D(\mathscr{A}, M)$. $(d - (\sum_1^k \bar{d}_i)\circ\bar{f})D(\mathscr{A}, \sigma) = d\circ\sigma - (\sum_1^k \bar{d}_i)\circ\bar{f}\sigma = d\sigma - (\sum_1^k \bar{d}_i)\circ f = 0$. Hence $d - (\sum_1^k \bar{d}_i)\circ\bar{f}$ can be considered as an element of $D(\mathscr{A}, M')$ and, as such, $(d - \sum_1^k \bar{d}_i\circ\bar{f})D(\mathscr{A}, \chi) \in D(\mathscr{A}, M)$. But $(\sum_1^k \bar{d}_i)\circ\bar{f} \in J(\mathscr{A}, M)$ and so $(d - \sum_1^k \bar{d}_i\circ\bar{f})D(\mathscr{A}, \chi) = d(J(\mathscr{A}, M))$. That is, $(d - \sum_1^k \bar{d}_i\circ\bar{f}) + J(\mathscr{A}, M') \in H^1_J(\mathscr{A}, M')$ is the required preimage.

2. Exactness of the long sequence.

DEFINITION. For $n \geqq 2$, $\mathscr{A}$ a T-algebra, M a T-bimodule for $\mathscr{A}$, $H^n(\mathscr{A}, M)$ is the K-module of equivalence classes of singular extensions of length n of M by $\mathscr{A}$. Let

$$E = 0 \longrightarrow M \xrightarrow{\chi} M_{n-2} \longrightarrow M_{n-3} \longrightarrow \cdots \longrightarrow \mathscr{B} \longrightarrow \mathscr{A} \longrightarrow 0$$

be a representative of an element of $H^n(\mathscr{A}, M)$ and $\alpha \in \mathrm{Hom}_{U(\mathscr{A})}(M, N)$. Then $EH^n(\mathscr{A}, \alpha) \in H^n(\mathscr{A}, N)$ is the equivalence class of the sequence

$$0 \longrightarrow N \longrightarrow N_{n-2} \longrightarrow M_{n-3} \longrightarrow \cdots \longrightarrow \mathscr{B} \longrightarrow \mathscr{A} \longrightarrow 0$$

where $N_{n-2} = R_1/R_2$; $R_1 = N \oplus M_{n-2}$, R_2 is the submodule of R_1 generated by $\{(-m\alpha, m\chi)/m \in M\}$. Under these definitions $H^n(\mathscr{A},\ )$ is a functor form $\mathscr{A}$-bimodules to K-modules. For further details see Gerstenhaber or Maclane.

Let $0 \to M' \to M \to M'' \to 0$ be exact. We now adapt a method of Barr [1] to define a connecting homomorphism $\delta^n: H^n(\mathscr{A}, M'') \to H^{n+1}(\mathscr{A}, M')$, $n \geqq 2$, and $\delta^1: D(\mathscr{A}, M'') \to H^2(\mathscr{A}, M')$ and to show that the long sequence $0 \to D(\mathscr{A}, M') \to D(\mathscr{A}, M) \to D(\mathscr{A}, M'') \to H^2(\mathscr{A}, M') \to \cdots \to H^n(\mathscr{A}, M) \to H^n(\mathscr{A}, M'') \to H^{n+1}(\mathscr{A}, M') \to \cdots$ is exact. Note that we have dropped the subscript J from H^n because, for $n \geqq 2$, $H^n(\mathscr{A}, M)$ is independent of the inner derivation functor chosen.

DEFINITION. A long T-singular extension is called *generic* if it admits a morphism to any long T-singular extension.

LEMMA 5. *Generic extensions exist.*

Proof. See Barr [1] or Gerstenhaber [5].

Briefly the construction of a T-generic extension for $\mathscr{A}$ is as follows. Let $\overline{\mathscr{F}}$ be the free T-algebra on the set $\mathscr{A}$, $\bar{N}$ the kernel of the canonical projection $\overline{\mathscr{F}} \to \mathscr{A}$. Letting $\mathscr{F} = \overline{\mathscr{F}}/\bar{N}^2$, the sequence $0 \to N \to \mathscr{F} \to \mathscr{A} \to 0$ is universal (or generic) for short singular extensions of $\mathscr{A}$. Let $X_i \to N$ be an $\mathscr{A}$-projective resolution of N. Then $X_i \to \mathscr{F} \xrightarrow{\tau} \mathscr{A} \to 0$ is a generic extension of $\mathscr{A}$.

DEFINITION. If M is an $\mathscr{A}$-bimodule, $E(\mathscr{A}, M)$ is the *split null extension* of M *by* $\mathscr{A}$. It is the algebra on the K module $\mathscr{A} \oplus M$ with multiplication $(a_1, m_1)(a_2, m_2) = (a_1a_2, a_1m_2 - m_1a_2)$. The equivalence class of the sequence $0 \to M \to E(\mathscr{A}, M) \to \mathscr{A} \to 0$ is the 0 element of $H^2(\mathscr{A}, M)$.

A morphism $\alpha \in \mathrm{Hom}_{U(\mathscr{A})}(M, N)$ induces $E(\mathscr{A}, \alpha) \in \mathrm{Hom}_T(E(\mathscr{A}, M), E(\mathscr{A}, N))$, the algebra homorphisms, in the obvious fashion.

LEMMA 6. *If $\mathscr{F}$ is generic for the algebra $\mathscr{A}$, then $D(\mathscr{F}, \ )$ is exact on $\mathscr{A}$-bimodules (regarded as $\mathscr{F}$-bimodules by pullback along $\tau: \mathscr{F} \to \mathscr{A}$).*

Proof. We need only show that if $M \xrightarrow{\sigma} M'' \to 0$ is exact then $D(\mathscr{F}, M) \to D(\mathscr{F}, M'') \to 0$ is exact. Let $\pi: \overline{\mathscr{F}} \to \mathscr{F}$ be the canonical projection, $d'' \in D(\mathscr{F}, M'')$.

We write $\mathrm{Hom}_T(\ ,\ )$ to mean algebra homomorphisms. d'' induces $\tilde{d}'' \in \mathrm{Hom}_T(\mathscr{F}, E(\mathscr{A}, M''))$ defined by $f\tilde{d}'' = (f\tau, fd'')$ for $f \in \mathscr{F}$; and $\tilde{d}''$ induces $\bar{d}'' \in \mathrm{Hom}_T(\overline{\mathscr{F}}, E(\mathscr{A}, M'')$ defined by $\bar{d}'' = \pi\tilde{d}''$.

We have

$$\begin{tikzcd} & & \overline{\mathscr{F}} \arrow[dll, dashed, "\bar{d}"'] \arrow[d, "\bar{d}''"] & \\ E(\mathscr{A}, M) \arrow[rr, "E(\mathscr{A},\, \sigma)"'] & & E(\mathscr{A}, M'') \arrow[r] & 0 \end{tikzcd}$$

where $\bar{d} \in \mathrm{Hom}_T(\overline{\mathscr{F}}, E(\mathscr{A}, M))$ exists by freeness of $\overline{\mathscr{F}}$. Since $(a, m)E(\mathscr{A}, \sigma) = (a, m\sigma)$ we must have $\bar{d}$ of the form $\bar{f}\bar{d} = (\bar{f}\pi\tau, m)$ for some $m \in M$. This implies that $\bar{d}$ is induced by a derivation $\tilde{d}: \overline{\mathscr{F}} \to M$, where M is regarded as an $\overline{\mathscr{F}}$-bimodule by pullback along $\pi\tau$. Since $(\bar{n}_1\bar{n}_2)\tilde{d} = (\bar{n}_1\pi\tau)\bar{n}_2 + \bar{n}_1(\bar{n}_2\pi\tau) = 0\bar{n}_2 + \bar{n}_1 0 = 0$, $\bar{N}^2 d = 0$. Hence $\tilde{d}$ induces $d \in D(\mathscr{F}, M)$ which is clearly the required preimage.

Suppose we have an $\mathscr{A}$-bimodule M with the sequence $X \xrightarrow{\varepsilon} \mathscr{F} \xrightarrow{\tau} \mathscr{A} \to 0$ exact and $d \in D(\mathscr{F}, M)$. It is easy to verify that $\varepsilon \circ d \in \mathrm{Hom}_{U(\mathscr{A})}(X, M)$.

LEMMA 7. *If* $0 \to N \xrightarrow{\beta} \mathscr{F} \xrightarrow{\tau} \mathscr{A} \to 0$ *is generic for short singular extensions of* $\mathscr{A}$, *then for any* $\mathscr{A}$-*bimodule* M, $H^2(\mathscr{A}, M) \simeq \mathrm{Hom}_{U(\mathscr{A})}(N, M)/D(\mathscr{F}, M)D(\beta, M)$.

Proof. The preceeding remark shows that $D(\mathscr{F}, M)D(\beta, M) \subseteq \mathrm{Hom}_{U(\mathscr{A})}(N, M)$. Let $f_2 \in \mathrm{Hom}_{U(\mathscr{A})}(N, M)$. Let $\mathscr{B}$ be the T-algebra $E(\mathscr{F}, M)/G$, where M is an $\mathscr{F}$-bimodule by pullback along τ, G the ideal consisting of the elements $\{(-n\beta, nf_2)/n \in N\}$. It is easy to see that the diagram

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "\beta"] \arrow[d, "f_2"] & \mathscr{F} \arrow[r, "\tau"] \arrow[d, "f_1"] & \mathscr{A} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & M \arrow[r, "\chi"'] & \mathscr{C} \arrow[r, "\sigma"'] & \mathscr{A} \arrow[r] & 0 \end{tikzcd}$$

is exact and commutative, where for $g \in \mathscr{F}$, $gf_1 = (g, 0) + G$; for $m \in M$, $m\chi = (0, m) + G$; for $(g, m) + G \in \mathscr{B}$, $((g, m) + G)\sigma = g\tau$.

Conversely, for any short singular extension $0 \to M \xrightarrow{\chi} \mathscr{B} \xrightarrow{\sigma} \mathscr{A} \to 0$, since $0 \to N \xrightarrow{\beta} \mathscr{F} \xrightarrow{\tau} \mathscr{A} \to 0$ is generic, there is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] \arrow[d, "f_2"] & \mathscr{F} \arrow[r] \arrow[d, "f_1"] & \mathscr{A} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & M \arrow[r, "\chi"'] & \mathscr{B} \arrow[r, "\sigma"'] & \mathscr{A} \arrow[r] & 0 \end{tikzcd}$$

where f_1 is an algebra morphism, f_2 is an $\mathscr{A}$-bimodule morphism.

Suppose $f_1': \mathscr{F} \to \mathscr{B}$, $f_2': N \to M$ also yield a commutative diagram. Let $f = f_1 - f_1'$. Since $f_1\sigma = f_1'\sigma = \tau$, $f\sigma = 0$ and f is a K-linear map into M. Let $x_1, x_2 \in \mathscr{F}$. Then

$$\begin{aligned}(x_1x_2)f &= (x_1f_1)(x_2f_1) - (x_1f_1')(x_2f_1')\\ &= (x_1f_1)(x_2f_1) - (x_1f_1)(x_2f_1') + (x_1f_1)(x_2f_1') - (x_1f_1')(x_2f_1')\\ &= (x_1f_1)(x_2f) + (x_1f)(x_2f_1')\\ &= x_1(x_2f) + (x_1f)x_2\end{aligned}$$

regarding M as an $\mathscr{F}$-bimodule by pullback along τ. Hence $f = f_1 - f_1' \in D(\mathscr{F}, M)$ and so

$$H^2(\mathscr{A}, M) \simeq \mathrm{Hom}_{U(\mathscr{A})}(N, M)/D(\mathscr{F}, M)D(\beta, M) .$$

LEMMA 8. *If* $X \xrightarrow{\varepsilon} \mathscr{F} \xrightarrow{\tau} \mathscr{A} \to 0$ *is exact, then* $\ker(D(\varepsilon, M)\colon D(\mathscr{F}, M) \to \mathrm{Hom}_{U(\mathscr{A})}(X, M)) = D(\mathscr{A}, M)$.

Proof. We have $X \xrightarrow{\varepsilon} \underset{\downarrow d\atop M}{\mathscr{F}} \xrightarrow{\tau} \mathscr{A} \to 0$ with $d \in \ker(D\mathscr{F}, M) \to \mathrm{Hom}_{U(\mathscr{A})}(X, M))$. Hence $\mathrm{Hom}_{U(\mathscr{A})}(X, M)$ is 0. Then says (image ε) $d = 0$. By exactness $\ker(\tau)d = 0$. Then for $g \in \mathscr{F}$, $(g + \ker\tau)\,\bar{d} = gd$ is a well-defined derivation from $\mathscr{A}$ to M and is the required one.

Let $X_i \xrightarrow{\varepsilon} \mathscr{F} \xrightarrow{\tau} \mathscr{A} \to 0$ be a generic resolution of $\mathscr{A}$. Define $\bar{H}^i(\mathscr{A}, M)$ to be the i-th cohomology module of the complex $0 \to D(\mathscr{A}, M) \to \mathrm{Hom}_{U(\mathscr{A})}(X_1, M) \to \cdots \to \mathrm{Hom}_{U(\mathscr{A})}(X_k, M) \to$.

LEMMA 9. $\bar{H}^0(\mathscr{A}, M) \simeq D(\mathscr{A}, M)$; $\bar{H}^n(\mathscr{A}, M) \simeq H^{n+1}(\mathscr{A}, M)$, $n \geqq 1$.

Proof. $\bar{H}^0(\mathscr{A}, M) = \ker(D(\mathscr{F}, M) \to \mathrm{Hom}_{U(\mathscr{A})}(X_1, M)) \simeq D(\mathscr{A}, M)$ by Lemma 8. $\bar{H}^1(\mathscr{A}, M) = \ker(\mathrm{Hom}_{U(\mathscr{A})}(X_1, M) \to \mathrm{Hom}_{U(\mathscr{A})}(X_2, M))/D(\mathscr{F}, M)D(\varepsilon, M) \simeq \mathrm{Hom}_{U(\mathscr{A})}(N, M)/D(\mathscr{F}, M)D(\beta, M)$, since $X_2 \to X_1 \to N \to 0$ is exact and $\mathrm{Hom}_{U(\mathscr{A})}(\ \ , M)$ is left exact, $\simeq H^2(\mathscr{A}, M)$ by Lemma 7.

For $n \geqq 2$, let $0 \to M \to P_{n-1} \to \cdots \to P_1 \to \mathscr{B} \to \mathscr{A} \to 0$ be a singular extension of length $n+1$ and let $C = \ker(\mathscr{B} \to \mathscr{A})$. Since $0 \to N \to \mathscr{F} \to \mathscr{A} \to 0$ is generic, we can fill in

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & \mathscr{F} & \longrightarrow & \mathscr{A} & \longrightarrow & 0\\ & & \downarrow{\scriptstyle \bar{f}_2} & & \downarrow{\scriptstyle \bar{f}_1} & & \downarrow{\scriptstyle =} & & \\ 0 & \longrightarrow & C & \longrightarrow & \mathscr{B} & \longrightarrow & \mathscr{A} & \longrightarrow & 0\end{array}$$

to a commutative diagram with $\bar{f}_1$ a morphism of algebras, $\bar{f}_2$ of $\mathscr{A}$-bimodules; and, since $X_2 \to N \to 0$ is a projective resolution, we can fill in

$$
\begin{tikzcd}
X_{n+1} \arrow[r, "\partial"] & X_n \arrow[r] \arrow[d, "f_n"] & \cdots \arrow[r] & X_1 \arrow[r, "\varepsilon"] \arrow[d, "f_1"] & N \arrow[r] \arrow[d, "f_2"] & 0 \\
0 \arrow[r] & M \arrow[r] & \cdots \arrow[r] & P_1 \arrow[r] & C \arrow[r] & 0
\end{tikzcd}
$$

to a commutative diagram with $0 = \partial f_n \colon X_{n+1} \to M$. Then f_n is a cocycle and the coset of f_n is in $H^n(\mathscr{A}, M)$. A straighforward application of the Chain Comparison Theorem shows that f_n is unique up to cohomology class.

LEMMA 10. *Let $0 \to M' \to M \to M'' \to 0$ be exact. Then there are natural homomorphisms, δ^n, so that the long sequence*

$$
\begin{aligned}
0 &\longrightarrow D(\mathscr{A}, M') \longrightarrow D(\mathscr{A}, M) \longrightarrow D(\mathscr{A}, M'') \xrightarrow{\delta^1} H^2(\mathscr{A}, M') \\
&\longrightarrow H^2(\mathscr{A}, M) \longrightarrow H^2(\mathscr{A}, M'') \xrightarrow{\delta^2} H^3(\mathscr{A}, M') \longrightarrow \cdots \\
&\longrightarrow H^n(\mathscr{A}, M'') \xrightarrow{\delta^n} H^{n+1}(\mathscr{A}, M') \longrightarrow \cdots
\end{aligned}
$$

is exact.

Proof. Taking a generic resolution $X_i \to \mathscr{F} \to \mathscr{A} \to 0$, we get a commutative diagram

$$
\begin{tikzcd}
0 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \\
0 \arrow[r] & D(\mathscr{F}, M') \arrow[r] \arrow[d, dash] & D(\mathscr{F}, M) \arrow[r] \arrow[d, dash] & D(\mathscr{F}, M'') \arrow[r] \arrow[d, dash] & 0 \\
0 \arrow[r] & \mathrm{Hom}_{U(\mathscr{A})}(X_1, M') \arrow[r] \arrow[d, dashed] & \mathrm{Hom}_{U(\mathscr{A})}(X_1, M) \arrow[r] \arrow[d, dashed] & \mathrm{Hom}_{U(\mathscr{A})}(X_1, M'') \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & \mathrm{Hom}_{U(\mathscr{A})}(X_n, M') \arrow[r] \arrow[d, dashed, dash] & \mathrm{Hom}_{U(\mathscr{A})}(X_n, M) \arrow[r] \arrow[d, dashed, dash] & \mathrm{Hom}_{U(\mathscr{A})}(X_n, M'') \arrow[r] \arrow[d, dashed, dash] & 0 \\
 & {} & {} & {} &
\end{tikzcd}
$$

where the second row is exact by Lemma 6, the others since the X_i are projective. By Lemma 9, the long exact sequence corresponding to this is as asserted.

THEOREM 3. *Let $0 \to M' \to M \to M'' \to 0$ be exact, J an inner derivation functor generated by $\{d_i\}_1^{k<\infty}$. Then the long sequence*

$$
\begin{aligned}
0 &\longrightarrow H^0_{J,\{d_i\}}(\mathscr{A}, M') \longrightarrow H^0_{J,\{d_i\}}(\mathscr{A}, M) \longrightarrow H^0_{J,\{d_i\}}(\mathscr{A}, M'') \\
&\xrightarrow{\delta^0} H^1_J(\mathscr{A}, M') \longrightarrow H^0_J(\mathscr{A}, M) \longrightarrow H^1_J(\mathscr{A}, M'') \longrightarrow H^2(\mathscr{A}, M') \\
&\longrightarrow \cdots \longrightarrow H^n(\mathscr{A}, M'') \xrightarrow{\delta^n} H^{n+1}(\mathscr{A}, M') \longrightarrow \cdots \longrightarrow
\end{aligned}
$$

is exact.

Proof. We have already seen the exactness of $0 \to H^0_{J,\{d_i\}}(\mathscr{A}, M') \to \cdots \to H^1_J(\mathscr{A}, M'')$. Note that the maps $H^1_J(\mathscr{A}, M') = D(\mathscr{A}, M')/J(\mathscr{A}, M') \to D(\mathscr{A}, M)/J(\mathscr{A}, M) = H^1_J(\mathscr{A}, M)$, and $H^1_J(\mathscr{A}, M) \to H^1_J(\mathscr{A}, M'')$ are induced by $D(\mathscr{A}, M') \to D(\mathscr{A}, M)$, $D(\mathscr{A}, M) \to D(\mathscr{A}, M'')$ respectively.

Since $J(\mathscr{A},\ )$ is epimorphism preserving, $J(\mathscr{A}, M'')$ is in image $(D(\mathscr{A}, M) \to D(\mathscr{A}, M''))$, and since $D(\mathscr{A}, M) \to D(\mathscr{A}, M'') \xrightarrow{\delta^1} H^2(\mathscr{A}, M)$ is exact, δ^1 induces δ^1: $H^1_J(\mathscr{A}, M'') = D(\mathscr{A}, M'')/J(\mathscr{A}, M'') \to H^2(\mathscr{A}, M)$, the kernel of which is image $(D(\mathscr{A}, M)/J(\mathscr{A}, M) \to D(\mathscr{A}, M'')/J(\mathscr{A}, M''))$. Combining, $0 \to \cdots \to H^1_J(A, M'')$ has been shown exact, $H^1_J(\mathscr{A}, M) \to H^1_J(\mathscr{A}, M'') \xrightarrow{\delta^1} H^2(\mathscr{A}, M')$ is exact by the previous remarks, and $H^1_J(\mathscr{A}, M'') \xrightarrow{\delta^1} H^2(\mathscr{A}, M') \to H^2(\mathscr{A}, M) \to \cdots$ is exact by Lemma 10. This proves the theorem.

3. **Extensions.** We briefly indicate extensions of previous theory to other cases of interest. First the relative (K-split) theory. The zeroth and first cohomology modules are as before. $H^n(\mathscr{A}, M)$, $n \geqq 2$, is defined as the K-module of equivalence classes of K-split extensions of length n. Once we note that a split generic resolution always exists, the previous theorems are easily seen to hold with this new definition of the cohomology modules. For a T-algebra, let $\overline{\mathscr{F}_K}$ be a free T-algebra on the module $\mathscr{A}$ (rather than on the set $\mathscr{A}$), $\bar{N}_K$ the kernel of $\overline{\mathscr{F}_K} \to \mathscr{A} \to 0$, the canonical projection. Then, with $N_K = \bar{N}_K/\bar{N}_K^2$, $\mathscr{F}_k = \overline{\mathscr{F}_K}/\bar{N}_K^2$, $0 \to N_K \to \mathscr{F}_K \to \mathscr{A} \to 0$ is generic for short singular K-split extensions of $\mathscr{A}$.

We next consider unital cohomology. Let $\mathscr{A}$ be a T-algebra with unit $1_{\mathscr{A}}$. The algebra $U_1(\mathscr{A}) = U(\mathscr{A})/[1^\lambda_{\mathscr{A}} - 1_{U(\mathscr{A})}, 1^\rho_{\mathscr{A}} - 1_{U(\mathscr{A})}]$ is the unital universal T-multiplication envelope for $\mathscr{A}$. It has the property that any unital T-bimodule for $\mathscr{A}$, M, is a unital right $U_1(\mathscr{A})$ module and conversely. Then instead of working in the category of $\mathscr{A}$-bimodules, we may work in the category of unital $\mathscr{A}$-bimodules. After showing a correspondance between inner derivation functors in this category and left $U_1(\mathscr{A})$-submodules of $D(\mathscr{A}, U_1(\mathscr{A}))$, all of the previous constructions and results go through without change.

The following discussion of cohomology of algebras with involution will find application in Glassman [7], in the cohomology of Jordan algebras. If $(\mathscr{A}, \sigma)$ is a T-algebra with involution (automorphism of period 2), then (M, σ) is an $(\mathscr{A}, \sigma)$ bimodule if $E(\mathscr{A}, M)$ is an algebra with involution (automorphism of period 2) under the map $(a, 0)\sigma = (a\sigma, 0)$, $(0, m)\sigma = (0, m\sigma)$. Morphisms of $\mathscr{A}$-bimodules with involution are just morphisms of $\mathscr{A}$-bimodules which, in addition, commute with the involution.

The universal envelope with involution (automorphism of period

2) for $(\mathscr{A}, \sigma)$ is the associative algebra $U(\mathscr{A}) \oplus U(\mathscr{A})\bar{\sigma}$ with multiplication $\bar{\sigma}^2 = 1$, $\bar{\sigma}a^\lambda = (a\sigma)^\rho\bar{\sigma}$, $\bar{\sigma}a^\rho = (a\sigma)^\lambda\bar{\sigma}$ ($\bar{\sigma}a^\lambda = (a\sigma)^\lambda\bar{\sigma}$, $\bar{\sigma}a^\rho = (\bar{\sigma}a)^\rho\bar{\sigma}$). $U(\mathscr{A}) \oplus U(\mathscr{A})\bar{\sigma} = (U(\mathscr{A}), \bar{\sigma})$ has the property that any $\mathscr{A}$-bimodule with involution (automorphism of period 2), (M, σ), is a right unital $(U(\mathscr{A}), \bar{\sigma})$-module and conversely; and $(U(\mathscr{A}), \bar{\sigma})$ is the free $(\mathscr{A}, \sigma)$-bimodule with involution (automorphism of period 2) on one generator. We define $D((\mathscr{A}, \sigma), (M, \sigma)) = [d \in D(\mathscr{A}, M)/\sigma\circ d = d\circ\sigma]$. We define an inner derivation functor as an epimorphism preserving subfunctor of $D((\mathscr{A}, \sigma),\ )$ and, again, show correspondance between inner derivation functors and right $U(\mathscr{A}, \bar{\sigma})$ submodules of $D((\mathscr{A}, \sigma), (U(\mathscr{A}), \bar{\sigma}))$.

The previous constructions and theorems follow without change, now working in the category of modules with involution (automorphism of period 2). However, the involution (automorphism of period 2) allows a refinement in the choice of H^0 which we will now describe.

Write $(X(x), \bar{\sigma})$, the free bimodule with involution on one generator. By X we will mean $(X, \bar{\sigma})$ considered without its involution. X is free on two generators, x and $x\bar{\sigma}$. Suppose that J is an inner derivation functor with the property $[\mathscr{A}J((\mathscr{A}, \sigma), (X, \bar{\sigma}))] \subseteqq F \subseteqq X$. Here J is generated by $\{d_i\}_1^k$, $[\mathscr{A}J((\mathscr{A}, \sigma), (X, \bar{\sigma})]$ is the submodule generated by the image of $\mathscr{A}$ under all inner derivations, F is a free $U(\mathscr{A})$ submodule of X on one generator which is closed under $\bar{\sigma}/F$. Then letting $[\mathscr{A} \sum_1^k \bar{d}_i, \bar{\sigma}]$ be the submodule with involution (automorphism of period 2) generated by $\mathscr{A} \sum_1^k \bar{d}_i$, we define $C^F_{J,\{d_i\}} = \sum_1^k \oplus (F, \bar{\sigma}/F)/[\mathscr{A} \sum_1^k \bar{d}_i, \bar{\sigma}]$ and get a long exact sequence as before.

Of particular interest are the cases where F is generated by $x - x\bar{\sigma}$, or $x + x\bar{\sigma}$. Consider the former. $\mathrm{Hom}_{(U(\mathscr{A}),\bar{\sigma})}((C^F_{J,\{d_i\}}, \bar{\sigma}), (M, \sigma)) = \mathrm{Hom}_{(U(\mathscr{A}),\bar{\sigma})}(\sum_1^k \oplus (F, \bar{\sigma}/F)/[(\mathscr{A} \sum_1^k \bar{d}_i, \bar{\sigma}), (M, \sigma)) \simeq \{(m_1, \cdots, m_k)/m_i \in M$, m_i skew and $\sum_1^k \bar{d}_i\circ\tilde{f}_{m_i} = 0\}$, where $(x - x\bar{\sigma})\tilde{f}_{m_i} = m_i$, $\simeq \{m_1 - m_1\sigma, \cdots, m_k - m_k\sigma)/m_i \in M, \sum_1^k \bar{d}_i\circ\tilde{f}_{m_i - m_i\sigma} = 0\}$. On the other hand $\mathrm{Hom}_{(U(\mathscr{A}),\bar{\sigma})}(C_{J,\{d_i\}}, (M, \sigma)) \simeq \mathrm{Hom}_{(U(\mathscr{A}),\bar{\sigma})}(\sum_1^k \oplus (X, \bar{\sigma})/[(\mathscr{A} \sum_1^k \bar{d}_i, \bar{\sigma}), (M, \sigma)) \simeq \{m_1, \cdots, m_k)/\sum_1^k \bar{d}_i\circ f_{m_i} = 0\}$, where $x_i f_{m_i} = m_i$, $\simeq \{(m_1, \cdots, m_k)/\sum_1^k \bar{d}_i\circ\tilde{f}_{m_i - m_i}\sigma = 0\}$.

Thus, by using $C^{[x - x\bar{\sigma}]}$ we have limited consideration to the skew elements of M. In the general case, F will be generated by an element y such that $y\bar{\sigma} = yu$, $u \in U(\mathscr{A})$ invertible. So, by using $C^{[y]}$, we will limit consideration to k-tuples (m_i) where $m_i\sigma = m_iu$.

4. Comparison with known theories.

Maximal and minimal inner derivation functor. Let J be the inner derivation functor corresponding to the 0 submodule of $D(\mathscr{A}, U(\mathscr{A}))$. It is clear that $J(\mathscr{A}, M) = 0$ for all $\mathscr{A}$-bimodules M. Since ϕ, the empty set, generates J, we have $C_\phi = 0$ and $H^0_{J,\phi}(\mathscr{A}, M) = \mathrm{Hom}_{U(\mathscr{A})}(C_\phi, M) = 0$. Also $H^1_J(\mathscr{A}, M) = D(\mathscr{A}, M)/J(\mathscr{A}, M) = D(\mathscr{A}, M)$. Then, given an exact sequence $0 \to M' \to M \to M'' \to 0$, the sequence of cohomology modules is $0 \to D(\mathscr{A}, M') \to D(\mathscr{A}, M) \to D(\mathscr{A}, M'') \to H^2(\mathscr{A}, M') \to \cdots \to$. This is the minimal inner derivation functor and has been discussed, for the commutative associative case, by Barr [1].

If J corresponds to the submodule $D(\mathscr{A}, U(\mathscr{A}))$ of $D(\mathscr{A}, U(\mathscr{A}))$, we call J the maximal inner derivation functor.

The classical inner derivation functor.

DEFINITION. If $\mathscr{A}$ is a T-algebra, the *Lie transformation algebra of* $\mathscr{A}$ is the Lie algebra generated by $\{a_R, a_L/a \in \mathscr{A}\}$, the collection of right and left multiplications of $\mathscr{A}$ by elements of $\mathscr{A}$. We denote this $\mathscr{L}(\mathscr{A})$.

Write $X(x) = U(\mathscr{A})$, the free right $U(\mathscr{A})$ module on one generator. Then, as elements of $E(\mathscr{A}, X)$, the product of two elements of X is 0. Thus, we see that a non-zero element of $\mathscr{L}(E(\mathscr{A}, X))$ mapping $\mathscr{A} \to X$ must have the form $\sum_i p_i$ where p_i is of the form $[a_{1_{s_1}}[\cdots[a_{r_{s_r}}(xu)_s]\cdots]$. Here $a_j \in \mathscr{A}$, $u \in U(\mathscr{A})$, s_j, $s = L$ or R. If $f \in \operatorname{Hom}_{U(\mathscr{A})}(X, X)$ $[a_{1_{s_1}}[\cdots[a_{r_{s_r}}(xu)_s] \circ f = [a_{1_{s_1}}[\cdots[a_{r_{s_r}}(xfu)_s]\cdots]$. Hence $D(\mathscr{A}, U(\mathscr{A})) \cap \mathscr{L}(E(\mathscr{A}, U(\mathscr{A})))$ is a left sub-$U(\mathscr{A})$-module of $D(\mathscr{A}, U(\mathscr{A}))$.

DEFINITION. The *classical inner derivation functor* I is the inner derivation functor corresponding to $D(\mathscr{A}, U(\mathscr{A})) \cap \mathscr{L}(E(\mathscr{A}, U(\mathscr{A})))$.

a. *Classical unital associative cohomology.* Let $\mathscr{A}$ be associative with unit, $U_1(\mathscr{A}) = \mathscr{A} \otimes \mathscr{A}^0$, the unital universal enveloping algebra. Schafer has shown that a derivation $d: \mathscr{A} \to \mathscr{A}$ is in $\mathscr{L}(\mathscr{A})$ if and only if it has the form $a_R - a_L$, $a \in \mathscr{A}$. From this it is clear that if M is an $\mathscr{A}$-bimodule, a derivation from $\mathscr{A}$ to M is in $\mathscr{L}(E(\mathscr{A}, M))$ if and only if it has the form $m_R - m_L$, $m \in M$.

Writing $X(x) = U_1(\mathscr{A})$, the free unital $\mathscr{A}$-bimodule on one generator, $d \in I(\mathscr{A}, X)$ if and only if $d = (xu)_R - (xu)_L$, $u \in U_1(\mathscr{A})$. But then $d = (x_R - x_L) \circ f_u$, where $f_u \in \operatorname{Hom}_{U_1(\mathscr{A})}(X, X)$ takes $x \to xu$. Thus, the set $\{x_R - x_L\}$ generates I. If Y is the $U_1(\mathscr{A})$-submodule of X generated by $\mathscr{A}(x_R - x_L) = \{ax - xa/a \in \mathscr{A}\}$, then $C_{\{x_R - x_L\}} = X/Y = X/[ax - xa] \simeq \mathscr{A}$ (as $\mathscr{A}$-bimodules) under the map $axb \to ab$. So we have $H^0_{I,\{x_R - x_L\}}(\mathscr{A}, M) = \operatorname{Hom}_{\mathscr{A} \otimes \mathscr{A}^0}(\mathscr{A}, M)$ and $H^0_{I,\{x_R - x_L\}}(\mathscr{A}, M) = [m \in M/am - ma = 0 \text{ for all } a \in \mathscr{A}\}]$.

The Hochschild relative cohomology groups for an associative algebra with 1 are defined by $\tilde{H}^n(\mathscr{A}, M) = \operatorname{Ext}^n_{(\mathscr{A} \otimes \mathscr{A}^0, K)}(\mathscr{A}, M)$. It is well-known that $\tilde{H}^0(\mathscr{A}, M) \cong [m \in M/am - ma = 0 \text{ for all } a \in \mathscr{A}] = H^0_{I,\{x_R - x_L\}}(\mathscr{A}, M)$; $\tilde{H}^1(\mathscr{A}, M) = D(\mathscr{A}, M)/I(\mathscr{A}, M) = H^1(\mathscr{A}, M)$; $H^2(\mathscr{A}, M) =$ the K module of equivalence classes of split short singular extensions of M by $\mathscr{A} = H_K^2(\mathscr{A}, M)$. Since $\tilde{H}^n$ and H^n both vanish on relative injectives for $n \geqq 2$, we have

THEOREM 4. *If $\mathscr{A}$ is associative with* 1, *Hochschild cohomology agrees with unital classical split cohomology.*

b. *Classical unital associative cohomology with involution.* Let $(\mathscr{A}, \sigma)$ be an associative algebra with unit and involution over a commutative ring K with unit and 2^{-1}, $(U_1(\mathscr{A}), \bar{\sigma})$ the universal unital enveloping algebra with involution for $(\mathscr{A}, \sigma)$, $(X(x), \bar{\sigma}) \simeq (U_1(\mathscr{A}), \bar{\sigma})$ the free unital $\mathscr{A}$-bimodule with involution on one generator.

Let (M, σ) be a bimodule with involution for $(\mathscr{A}, \sigma)$. We have defined $D((\mathscr{A}, \sigma), (M, \sigma)) = \{d \in D(\mathscr{A}, M)/\sigma \circ d = d \circ \sigma\}$ and have noted that $d \in I(\mathscr{A}, M) = D(\mathscr{A}, M) \cap \mathscr{L}(E(\mathscr{A}, M))$ if and only if $d = m_R - m_L$, $m \in M$.

LEMMA 11. *$d \in I(\mathscr{A}, M)$ satisfies $\sigma \circ d = d \circ \sigma$ if and* only if $d = m_R - m_L$ with m skew in M.

Proof. Suppose $m \in M$, $m\sigma = -m$. Let $a \in \mathscr{A}$. Then $(am - ma)\sigma = m\sigma(a\sigma) - a\sigma(m\sigma) = -m(a\sigma) + (a\sigma)m = (a\sigma)m - m(a\sigma)$. Conversely, suppose $m \in M$, and $m_R - m_L$ commutes with σ. This is equivalent to the operator identity $\sigma m_R - \sigma m_L = \sigma(m\sigma)_L - \sigma(m\sigma)_R$. Since σ is onto, we may rewrite this $(m_R + m\sigma_R) = (m_L + m\sigma_L)$ or $(m + m\sigma)_R = (m + m\sigma)_L$. Writing $m = \frac{1}{2}(m + m\sigma) + \frac{1}{2}(m - m\sigma)$, we have

$$\begin{aligned} m_R - m_L &= \tfrac{1}{2}(m + m\sigma)_R - \tfrac{1}{2}(m + m\sigma)_L + \tfrac{1}{2}(m - m\sigma)_R - \tfrac{1}{2}(m - m\sigma)_L \\ &= \tfrac{1}{2}(m - m\sigma)_R - \tfrac{1}{2}(m - m\sigma)_L \,. \end{aligned}$$

But $m - m\sigma$ is skew.

With $(X(x), \bar{\sigma}) \simeq (U_1(\mathscr{A}), \bar{\sigma})$, the free unital bimodule with involution on one generator, we define the classical inner derivation functor $I((\mathscr{A}, \sigma),\ )$ to be the one generated by $D((\mathscr{A}, \sigma), (X, \bar{\sigma})) \cap \mathscr{L}(E(\mathscr{A}, X))$. From the previous lemma we see that $d \in I((\mathscr{A}, \sigma), (X, \bar{\sigma}))$ if and only if $d = (xu - (xu)\bar{\sigma})_R - (xu - (xu)\bar{\sigma})_L$, $u \in (U_1(\mathscr{A}), \bar{\sigma})$. But then $d = ((x - x\bar{\sigma})_R - (x - x\bar{\sigma}_L) \circ f_u$, where $f_u \in \mathrm{Hom}_{(U_1(\mathscr{A}), \bar{\sigma})}((X, \bar{\sigma}), X, \bar{\sigma}))$ takes $x \to xu$.

Writing $x = x - \bar{\sigma}$, I is generated by $\tilde{x}_R - \tilde{x}_L$. Noting that $\tilde{x}$ generates a free submodule F of X and recalling the previous discussion of cohomology of algebras with involution, we define $(C^F_{I, \{x_R - x_L\}}, \bar{\sigma}) = (F, \bar{\sigma}/F)/[\mathscr{A}(x_R - x_L), \bar{\sigma}]$ and find $\mathrm{Hom}_{(U_1(\mathscr{A}), \bar{\sigma})}((C^F_{I, \{x_R - x_L\}}, \bar{\sigma}), (M, \sigma)) = [m \in M/m$ skew and $am - ma = 0$ for all $a \in \mathscr{A}]$.

We note that $(\mathscr{A}, -\sigma)$ is also a bimodule (but not an algebra) with involution. The map taking $\tilde{x} - 1_{\mathscr{A}}$ defines an isomorphism

$(C^F_{I,\{x_R - x_L\}}, \bar{\sigma}) \simeq (\mathscr{A}, \text{-}\sigma)$. Harris [8] has constructed an explicit $(U_1(\mathscr{A}), \bar{\sigma})$ K-split projective resolution of $(\mathscr{A}, -\sigma)$, $X_n \to (\mathscr{A}, -\sigma)$. He has shown that $\mathrm{Hom}_{(U_1(\mathscr{A}), \bar{\sigma})}((X_n, M, \sigma))$ is isomorphic to the space of n-linear functions $g: \mathscr{A} \otimes \cdots \otimes \mathscr{A} \to M$ such that $(a_1, \cdots, a_n)g\sigma = \omega_n(a_n\sigma, \cdots, a_1\sigma)g$, $\omega_n = (-1)^{1/2}(n-1)(n-1)(n-2)$. We have already seen that $\mathrm{Hom}_{(U_1(\mathscr{A}), \bar{\sigma})}((\mathscr{A}, -\sigma), (M, \sigma)) \cong [m \in M/am - ma = 0$ for all $a \in \mathscr{A}$, m skew]. We will now show correspondances between certain linear maps and cocycles and coboundaries. Following standard notation, we write these on the left. Harris shows that 1-cocycles are linear functions $g: \mathscr{A} \to M$ such that $g(ab) = ag(b) + g(a)b$ and $g(a\sigma) = g(a)\sigma$ for all a, b in $\mathscr{A}$; i.e., these are derivations commuting with involution. 1-coboundaries are functions $g: a \to am - ma$ such that $g \circ \sigma = \sigma \circ g$. By Lemma 11, these are just $\{m_R - m_L/m$ skew in $M\}$. Hence $\mathrm{Ext}^1_{(U_1\mathscr{A}), \bar{\sigma})}((\mathscr{A}, -\sigma), (M, \sigma)) = D((\mathscr{A}, \sigma), (M, \sigma))/I((\mathscr{A}, \sigma), (M, \sigma)) = H^1_I((\mathscr{A}, \sigma), (M, \sigma))$.

2-cocycles are bilinear functions $g: \mathscr{A} \otimes \mathscr{A} \to M$ with $a_1g(a_2, a_3) - g(a_1a_2, a_3) + g(a_1, a_2a_3) - g(a_1, a_2)a_3 = 0$ for all $a_i \in \mathscr{A}$, and $g(a_1, a_2)\sigma = g(a_2\sigma, a_1\sigma)$.

Now let K be a field characteristic $\neq 2$,

$$0 \longrightarrow (M, \sigma) \longrightarrow (\mathscr{B}, \sigma) \xrightarrow{\tau} (\mathscr{A}, \sigma) \longrightarrow 0$$

be a short singular extension of associative algebras with involution. We can choose a linear splitting δ for $(\mathscr{B}, \sigma) \xrightarrow{\tau} (\mathscr{A}, \sigma)$ that respects involution. For this, choose a basis for $\mathscr{A}$, say $\{a_1, \cdots, a_n\}$. Choose $b_1 \in \mathscr{B}$ such that $b_1\tau = a_1$. Define

$$a_1\delta = \begin{cases} b_1 \text{ if } a_1 \notin Ka_1 \\ \left(\dfrac{1}{k+1}\right)(b_1 + kb_1\sigma) \text{ if } a_1\sigma = ka_1, \text{ and } -1 \neq k \in K \\ \frac{1}{2}(b_1 - b_1\sigma) \text{ if } a_1\sigma = -a_1 \,. \end{cases}$$

Since $k^2 = 1$, we can define $a_1\sigma\delta = a_1\delta\sigma$.

Suppose $a_1\delta, \cdots, a_r\delta, a_1\sigma\delta, \cdots, a_r\sigma\delta$ have been defined so that δ commutes with involution on $[a_1, \cdots, a_r, a_1\sigma, \cdots, a_r\sigma]$. Suppose a_{r+1} is the first $a_i \notin [a_1, \cdots, a_r\sigma]$. Then we can choose as above and continue inductively.

Let δ be so chosen and write $h(a, b) = a\delta b\delta - (ab)\delta \in M$. Then

$$\begin{aligned} h(a, b)\sigma &= ((a\delta b\delta) - (ab)\delta)\sigma \\ &= b\delta\sigma a\delta\sigma - (ab)\delta\sigma = b\sigma\delta a\sigma\delta - (ab)\sigma\delta \\ &= b\sigma\delta a\sigma\delta - (b\sigma a\sigma)\delta = h(h\sigma, a\sigma) \,. \end{aligned}$$

Hence we can associate a 2-cocycle to each singular extension of M by $\mathscr{A}$. Suppose we have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (M, \sigma) & \longrightarrow & (\mathscr{B}, \sigma) & \longrightarrow & (\mathscr{A}, \sigma) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \longrightarrow & (M, \sigma) & \longrightarrow & (\mathscr{B}, \sigma) & \longrightarrow & (\mathscr{A}, \sigma) & \longrightarrow & 0 . \end{array}$$

Then $(m, a)\alpha = (m + h(a), a)$ where h is a 2-coboundary. But since α commutes with involution $(m, a)\alpha\sigma = (m+h(a), a)\sigma = (m\sigma + h(a)\sigma, a\sigma)$. Also $(m, a)\alpha\sigma = (m\sigma, a\sigma)\alpha = (m\sigma+h(a\sigma), a\sigma)$. Hence $h(a)\sigma = h(a\sigma)$. Since Harris's cohomology modules clearly vanish on relative injectives for $n \geqq 2$ as do the classical ones we have

THEOREM 5. *If $(\mathscr{A}, \sigma)$ is associative with unit over a commutative ring with 2^{-1}, then Harris's 0-th and 1-st cohomology modules are classical; if K is a field of characteristic $\neq 2$, $(\mathscr{A}, \sigma)$ an algebra over K, Harris's modules are classical for all $n \geqq 0$.*

c. *Classical Lie cohomology.* Let $\mathscr{A}$ be a Lie algebra over a commutative ring with unit K, M a Lie bimodule for $\mathscr{A}$. We denote multiplication in $\mathscr{A}$ by brackets and multiplication of M by $\mathscr{A}$ by juxtaposition. Schafer has shown that a derivation from $\mathscr{A} \to \mathscr{A}$ is in $\mathscr{L}(\mathscr{A})$ if and only if it is of the form a_L, $a \in \mathscr{A}$. From this it is clear that a derivation from $\mathscr{A}$ to M is in $\mathscr{L}(E(\mathscr{A}, M))$ if and only if it has the form m_L, $m \in M$.

Writing $X(x) \simeq U(\mathscr{A})$, the free $\mathscr{A}$-bimodule on one generator, $d \in I(\mathscr{A}, X)$ if and only if $d = (xu)_L$, $u \in U(\mathscr{A})$. But then $d = x_L \circ f_u$, where $f_u \in \mathrm{Hom}_{U(\mathscr{A})}(X, X)$ takes $x \to xu$. Thus the set $\{x_L\}$ generates I. If Y is the $U(\mathscr{A})$ submodule of X generated by $\mathscr{A}x_L$, then $C_{I,\{x_L\}} = X/Y$. Even over a ring, the Poincare-Birkhoff-Witt theorem shows that $U(\mathscr{A})$ is linearly generated by monomials in the generators for $\mathscr{A}$ and $1_{U(\mathscr{A})}$, and that there is an augmentation $U(\mathscr{A})\, \varepsilon\, K1_{U(\mathscr{A})}$. Then $X/Y \simeq K$, K regarded as an $\mathscr{A}$-bimodule by pullback along ε.

To compute the modules $\mathrm{Ext}^n_{(U(\mathscr{A}),K)}(K, M)$, the Koszul resolution may be used, and as was the case for associative algebras, we have

THEOREM 6. *If $\mathscr{A}$ is Lie, $H^n_K(\mathscr{A}, M) \simeq \mathrm{Ext}^n_{(U(\mathscr{A}),K)}(K, M)$ for all $n \geqq 0$.*

d. *Classical Lie cohomology with automorphism of period* 2. In a later paper, this case will be used to discuss cohomology of Jordan algebras.

Let $(\mathscr{A}, \sigma)$ be a Lie algebra with automorphism of period 2 over a commutative ring K with unit and 2^{-1}, $(U(\mathscr{A}), \bar{\sigma})$ the universal enveloping algebra with automorphism of period 2 for $(\mathscr{A}, \sigma)$, $(X(x), \bar{\sigma}) \simeq (U(\mathscr{A}), \bar{\sigma})$ the free $\mathscr{A}$-bimodule with automorphism of period 2 on one generator x. Let (M, σ) be a bimodule with automorphism of period 2 for $(\mathscr{A}, \sigma)$. We have defined $D((\mathscr{A}, \sigma), (M, \sigma)) = [d \in D(\mathscr{A}, M)/\sigma \circ d = d \circ \sigma]$ and have noted that $d \in I(\mathscr{A}, M) = D(\mathscr{A}, M) \cap \mathscr{L}(E(\mathscr{A}, M))$ if and only if $d = m_L$, $m \in M$.

LEMMA 12. *$d \in I(\mathscr{A}, M)$ satisfies $\sigma \circ d = d \circ \sigma$ if and only if $d = m_L$ with m symmetric in M.*

Proof. Suppose $m \in M$, $m\sigma = m$. Let $a \in \mathscr{A}$. Then $(ma)\sigma = m\sigma a\sigma = m(a\sigma)$. Conversely, suppose $m \in M$ is such that m_L commutes with σ. This is equivalent to the operator identity $\sigma(m\sigma)_L = \sigma(m_L)$. Since σ is onto, we may write this $(m\sigma)_L = m_L$. Writing $m = \frac{1}{2}(m + m\sigma) + \frac{1}{2}(m - m\sigma)$, $m_L = \frac{1}{2}(m+m\sigma)_L + \frac{1}{2}(m - m\sigma)_L = \frac{1}{2}(m + m\sigma)_L$. But $\frac{1}{2}(m + m\sigma)_L$ is symmetric.

This shows that $d \in I((\mathscr{A}, \sigma), (X, \bar{\sigma}))$ if and only if $d = (xu + (xu)\bar{\sigma})_L$, $u \in (U(\mathscr{A}), \bar{\sigma})$. But then $d = (x + x\bar{\sigma})_L \circ f_u$ where $f_u \in \mathrm{Hom}_{(U(\mathscr{A}), \bar{\sigma})}((X, \bar{\sigma}), (X, \bar{\sigma}))$ takes $x \to xu$. Thus, with $\tilde{x} = x + x\bar{\sigma}$, I is generated by $\{\tilde{x}_L\}$. Noting that x generates a free submodule F of X, F closed under $\bar{\sigma}$, we define $(C^F_{I,\{\tilde{x}_L\}}, \bar{\sigma}) = (F, \bar{\sigma}/F)/[\mathscr{A}(\tilde{x}_L), \bar{\sigma}]$ and find that $\mathrm{Hom}_{(U(\mathscr{A}), \bar{\sigma})}((C^F_{I,\{\tilde{x}_L\}}, \bar{\sigma}), (M, \sigma)) = [m \in M/m$ symmetric and $ma = 0$ for all $a \in \mathscr{A}\}$. It is easy to see, as was done for $X/Y \simeq K$, that $C^F_{I,\{\tilde{x}_L\}}$ is isomorphic to $(K, 1)$, 1 denoting the identity automorphism, under the map $\tilde{x} \to 1$.

For K a field of characteristic $\neq 2$, Harris [9] has constructed a projective $(U(\mathscr{A}), \bar{\sigma})$ resolution of $(K, 1)$. Defining $\tilde{H}^n((\mathscr{A}, \sigma), (M, \sigma))$ as the n-th cohomology of this complex. Harris has shown that $H^0((\mathscr{A}, \sigma), (M, \sigma)) \cong [m \in M/m$ symmetric and $ma = 0$ for all $a \in \mathscr{A}] \simeq H^0_{I,\{\tilde{x}_L\}}((\mathscr{A}, \sigma), (M, \sigma))$; $H^1((\mathscr{A}, \sigma), (M, \sigma)) \cong$ the K-module generated by those derivations f from $\mathscr{A}$ to M such that $f(x\sigma) = f(x)\sigma$ modulo inner derivations of the form $f(a) = ma$ with m symmetric $\simeq H^1_I((\mathscr{A}, \sigma), M, \sigma))$; $H^2((\mathscr{A}, \sigma), (M, \sigma)) \cong$ the K-module generated by those Lie 2-cocycles g such that $g(a\sigma, b\sigma) = g(a, b)\sigma$ for all a, b in $\mathscr{A}$ modulo those 2-coboundaries given by linear maps commuting with the automorphism σ, $\simeq H^2((\mathscr{A}, \sigma), (M, \sigma))$.

THEOREM 7. *If $\mathscr{A}$ is a Lie algebra over a field of characteristic $\neq 2$, $\mathscr{A}$ with automorphism of period 2, then its cohomology modules as defined by Harris are classical.*

e. *Classical unital commutative associative cohomology.* If $\mathscr{A}$ is commutative associative with 1, $U_1(\mathscr{A}) \simeq \mathscr{A}$ with $\lambda = \rho = 1: \mathscr{A} \to U_1(\mathscr{A})$. If M is a unital commutative associative bimodule for the associative algebra $\mathscr{A}$, $I(\mathscr{A}, M) = [m_R - m_L/m \in M]$. But since M is commutative $am = ma$ for all $a \in \mathscr{A}$, and $I(\mathscr{A}, M) = 0$. Thus, in this case, classical cohomology is minimal.

If K is a field, F a field extension of K regarded as a commutative associative algebra over K, then Gerstenhaber has shown that $H^2(F, F) = 0$ if and only if F is separable extension. But since F is certainly an injective F-bimodule, the case F not separable provides as example for which $H^2(F, \ )$ does not vanish on injectives.

THEOREM 5. *If $\mathscr{A}$ is a commutative associative algebra with* 1, *classical unital cohomology is minimal. If $F \supsetneqq K$ is a nonseparable field extension, there is no inner derivation functor J, no module C_J for which the right derived functors of* $\mathrm{Hom}_F(C_J, \ )$ *are* $\{H_J^n(F, \ )\}$.

BIBLIOGRAPHY

1. M. Barr, *A cohomology theory for commutative algebras*, I. Proc. Amer. Math. Soc. **16** (1965), 1379-1384, II. Proc. Amer. Math. Soc. **16** (1965), 1385-1391.
2. M. Barr and G. S. Rinehart, *Cohomology as the derived functor of derivations* (to appear)
3. H. Cartan and S. Eilenberg, *Homological algebra*, Princeton University Press, 1956.
4. J. Dixmier, *Homologie des anneaux de Lie*, Ann. Sci. Ecole Norm Sup. **74** (1957), 25-83.
5. M. Gerstenhaber, *A Uniform cohomology theory for algebras*, Proc. Nat. Acad. Sci. **52** (1964), 626-629.
6. ———, *On the deformations of rings and algebras*, Ann. of Math. **79** (1964), 59-103.
7. N. Glassman, *Cohomology of non-associative algebras II* (to appear, J. of Algebra).
8. B. Harris, *Derivations of Jordan algebras*, Pacific J. Math. **9** (1959), 495-512.
9. ———, *Cohomology of Lie triple systems and Lie algebras with involution*, Trans. Amer. Math. Soc. **98** (1961), 148-162.
10. G. Hochschild, *Cohomology of restricted Lie algebras*, Amer. J. Math. **76** (1954), 555-580.
11. ———, *On the cohomology groups of an associative algebra*, Amer. J. Math. **46** (1945), 58-67.
12. ———, *On the cohomology theory for associative algebras*, Ann. of Math. **47** (1946), 568-579.
13. ———, *Cohomology and representations of associative algebras*, Duke Math. J. **14** (1947), 921-948.
14. ———, *Semi-simple algebras and generalized derivations*, Amer. J. Math. **64** (1942), 677-694.
15. N. Jacobson, *Lie algebras*, Interscience, 1962.
16. ———, *Jordan algebras* (to appear)
17. ———, *Derivation algebras and multiplication algebras of semisimple Jordan algebras*, Ann. of Math. **50**. (1949), 866-874.
18. A. Knopfmacher, *Universal envelopes for non-associative algebras*, Quart. J. Math., Oxford Ser. (12) **13** (1962), 264-282.
19. M. Koecher, *Embeddings of Jordan algebras in Lie algebras*, Yale Notes, 1966.
20. S. Maclane, Homology, Academic Press, 1963.
21. J. P. May, *The cohomology of restricted Lie algebras and of Hopf algebras*, J. of Algebra. **3** (1966), 123-146.
22. B. Mitchell, *The theory of categories*, Academic Press, 1965.

23. R. Schafer, *An introduction to non-associative algebras*, Academic Press, 1966.
24. ———, *Inner derivations of non-associative algebras*, Bull. Amer, Math, Soc. **55** (1949), 769-776.
25. U. Shukla, *Cohomologie des algebras associatives*, Ann. Sci. Ecole Norm. Sup. **78** (1961), 163-209.
26. C. Watts, *A characterization of additive functors*, Proc. Amer. Math. Soc. **11** (1960), 5-11.

Received March 17, 1969.

YALE UNIVERSITY