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ON THE CONFORMAL MAPPING OF VARIABLE REGIONS

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We establish an estimate for the functional

$$I(f, g; \rho) = \int_{C_\rho} |f(t) - g(t)| \cdot |dt| ,$$

C_ρ **is the circumference** $|t| = \rho, 0 \leqq \rho < 1$. **Here** f **and** g **are normalized conformal mapping functions of** $|z| < 1$ **onto a pair of bounded, open, simply connected, origin containing domains in the** w **plane whose boundaries are near each other in some sense. In the second part of the paper we establish an estimate for the functional** $I(f', g'; \rho)$ **in case the boundaries are additionally assumed to be rectifiable.**

We are motivated by the fact that if one of the domains is a disc we get the case of "nearly circular" domains which has been much studied.

Aside from an absolute constant our estimates are geometric in nature, being expressed in terms of numbers which are derived from properties of the boundaries of the mapped domains. They are of interest to us because they hold uniformly for all $\rho, 0 \leqq \rho < 1$ and because they approach zero when one of the domains converges to the other as described in the paper.

1. DEFINITION 1. Let D_f and D_g denote a pair of open, bounded, simply connected sets in the w plane both of which contain the origin. Let Γ_f and Γ_g denote their respective boundaries. Let Δ denote the component of $D_f \cap D_g$ which contains the origin and let Γ denote the boundary of Δ. Let λ_f be the radius of the largest disk lying in the complement of Γ_f and having its center on Γ (if no such disk exists, write $\lambda_f = 0$). Let λ_g be analogously defined. The inner distance is defined by the formula

$$\varepsilon = \varepsilon(\Gamma_f, \Gamma_g) = \operatorname{Max}(\lambda_f, \lambda_g) .$$

The statement '$\phi(z)$ is a normalized mapping function' means that $\phi(z)$ is the conformal mapping function of one bounded, simply connected, origin containing domain onto another and that $\phi(0) = 0$, and $\phi'(0)$ is positive.

The symbol C_ρ will always be used to denote the locus $|t| = \rho$, $0 \leqq \rho < 1$.

Let R_1 and R_2 denote the radii of two circles with centers at $w = 0$

which are such that the boundaries Γ_f and Γ_g lie in the ring

$$0 < R_1 \leqq |w| \leqq R_2 .$$

THEOREM 1. *If $f(z)$ and $g(z)$ are the normalized mapping functions of $|z| < 1$ onto D_f and D_g respectively, if $0 < \varepsilon(\Gamma_f, \Gamma_g) < R_1$, then*

$$I(f, g; \rho) = \int_{C_\rho} |f(t) - g(t)| \cdot |dt| \leqq K_1 R_2 \left(\frac{\varepsilon}{R_1}\right)^{1/6} .$$

The number K_1 is an absolute constant, and the inequality holds uniformly for all ρ, $0 \leqq \rho < 1$.

Before proving Theorem 1 we state some results which are used in the proof.

LEMMA A. ([4], p. 349.) *Let D be a bounded, simply connected domain which contains the origin and let $z = \psi(w)$ be the normalized mapping function of D onto the disk $|z| < 1$ in the z plane. If w is a point of D at a distance δ from the boundary of D, then*

$$1 - |\psi(w)| \leqq 4\sqrt{\delta\psi'(0)} .$$

LEMMA B. ([3], p. 563.) *Let $w = \phi(z)$ be the normalized mapping function of $|z| < 1$ onto the domain whose boundary D lies in the ring $1 - \sigma \leqq |w| \leqq 1$, $0 < \sigma < 1$. Then*

$$\int_{C_\rho} |\phi(t) - t|^2 \cdot |dt| \leqq K_2 \sigma^2 .$$

The number K_2 is an absolute constant, and inequality holds uniformly for all ρ, $0 \leqq \rho < 1$.

LEMMA C. ([1], p. 165.) *If $F(z)$ and $\Theta(z)$ are regular in $|z| < 1$ if $\Theta(0) = 0$ and $|\Theta(z)| < 1$ in $|z| < 1$, then*

$$\int_{C_\rho} |F(\Theta(t))|^2 \cdot |dt| \leqq \int_{C_\rho} |F(t)|^2 \cdot |dt| ,$$

uniformly valid for all ρ, $0 \leqq \rho < 1$.

2. *Proof of Theorem* 1. (a) From Definition 1, each point of Γ will have distance at most ε from Γ_f. The inverse of $f(z)$ maps Δ onto a domain E which lies in $|z| < 1$. Let E_1 denote the boundary of E. From Lemma A, the set E_1 will lie in the ring

$$1 - 4\sqrt{\frac{\varepsilon}{f'(0)}} \leqq |z| \leqq 1 .$$

Since

$$f'(0) \geqq \inf_{|z|<1} \left| \frac{f(z)}{z} \right| \geqq R_1 ,$$

the set E_1 will lie in the ring

$$1 - 4\sqrt{\frac{\varepsilon}{R_1}} \leqq |z| \leqq 1 .$$

The above inequality fails to define a ring if $\varepsilon/R_1 \geqq 1/16$. We treat the two cases separately. Let $\omega(z)$ be the normalized mapping function of $|z| < 1$ onto E. If $\varepsilon/R_1 < 1/16$, we have from Lemma B,

$$J(\rho) = \int_{C_\rho} |\omega(t) - t|^2 \, |dt| \leqq 16K_2 \frac{\varepsilon}{R_1} .$$

For the case $1/16 \leqq \varepsilon/R_1 < 1$, we have trivially,

$$J(\rho) \leqq 4 \cdot 2\pi\rho \leqq 128\pi \cdot \frac{\varepsilon}{R_1} .$$

Thus, if $K_3 = \text{Max}\,[128\pi, 16K_2]$, then

$$J(\rho) \leqq K_3 \frac{\varepsilon}{R_1}, \; 0 < \varepsilon < R_1 . \tag{1}$$

(b) For $0 \leqq r \leqq 1$, $|z| < 1$ let

$$B_r(z) = f(z) - f(rz) .$$

Then

$$f(z) - f(\omega(z)) = B_r(z) - B_r(\omega(z)) + f(rz) - f(r\omega(z)) .$$

Hence

$$\begin{aligned} &\int_{C_\rho} |f(t) - f(\omega(t))| \cdot |dt| \\ &\quad \leqq \int_{C_\rho} |B_r(t)| \cdot |dt| + \int_{C_\rho} |B_r(\omega(t))| \cdot |dt| \\ &\qquad + \int_{C_\rho} |f(rt) - f(r\omega(t))| \cdot |dt| \equiv I_1 + I_2 + I_3 . \end{aligned} \tag{2}$$

If $f(z) = \sum_1^\infty a_k z^k$ then

$$\begin{aligned} I_1^2 &\leqq 2\pi\rho \cdot \int_{C_\rho} |B_r(t)|^2 \cdot |dt| = 2\pi\rho \Sigma \, |a_k|^2 \cdot \rho^{2k} \cdot (1 - r^k)^2 \cdot 2\pi\rho \\ &\leqq 4\pi^2 \Sigma \, |a_k|^2 (1 - r^k) \\ &= 4\pi^2 \Sigma \, |a_k|^2 (1 - r)(1 + r + r^2 + \cdots + r^{k-1}) \end{aligned}$$

$$\leqq 4\pi^2(1-r)\Sigma(|a_k|^2\cdot k) = 4\pi(1-r)\cdot(\text{area of } D_f)$$
$$\leqq 4\pi(1-r)\cdot\pi R_2^2 .$$

Thus, if $K_4^2 = 4\pi^2$,

$$I_1 \leqq K_4 R_2 \sqrt{1-r},\ 0 \leqq r \leqq 1 . \tag{3}$$

From Lemma C, the same bound is valid for I_2:

$$I_2 \leqq K_4 R_2 \sqrt{1-r},\ 0 \leqq r \leqq 1 . \tag{4}$$

(c) If $0 < r < \alpha < 1$, we have for the integrand of I_3:

$$\begin{aligned}
|f(rt) - f(r\omega(t))| &\leqq \frac{1}{2\pi}\int_{C_\alpha} |f(\gamma)|\cdot\left|\frac{1}{\gamma - rt} - \frac{1}{\gamma - r\omega}\right|\cdot|d\gamma| \\
&\leqq \frac{1}{2\pi}\int_{C_\alpha} |f(\gamma)|\cdot\left|\frac{r\omega - rt}{(\gamma - rt)(\gamma - r\omega)}\right|\cdot|d\gamma| \\
&\leqq \frac{\sup|f|\cdot r\,|\omega - t|}{2\pi}\int_{C_\alpha}\frac{|d\gamma|}{|\gamma - rt|\cdot|\gamma - r\omega|} \\
&\leqq \frac{R_2|\omega - t|}{2\pi}\left[\int_{C_\alpha}\frac{|d\gamma|}{|\gamma - rt|^2}\right]^{1/2}\cdot\left[\int_{C_\alpha}\frac{|d\gamma|}{|\gamma - r\omega|^2}\right]^{1/2} \\
&\leqq \frac{R_2|\omega - t|}{2\pi}\left[\frac{2\pi\alpha}{\alpha^2 - |rt|^2}\right]^{1/2}\cdot\left[\frac{2\pi\alpha}{\alpha^2 - |r\omega|^2}\right]^{1/2} .
\end{aligned}$$

Let $\alpha \to 1$ and we obtain

$$|f(rt) - f(r\omega(t))| \leqq \frac{R_2|\omega(t) - t|}{1 - r},\ 0 < r < 1 .$$

Hence, from (1)

$$\begin{aligned}
\int_{C_\rho} |f(rt) - f(r\omega(t))|\cdot|dt| &\leqq \frac{R_2}{1-r}\int_{C_\rho}|\omega(t) - t|\,|dt| \\
&\leqq \frac{R_2}{1-r}\left[\int_{C_\rho}|\omega - t|^2\,|dt|\right]^{1/2}\cdot\sqrt{2\pi\rho} \\
&\leqq \frac{R_2}{1-r}\left[2\pi K_3\left(\frac{\varepsilon}{R_1}\right)\right]^{1/2} .
\end{aligned} \tag{5}$$

If we combine (2), (3), (4) and (5), we obtain the estimate

$$\begin{aligned}
&\int_{C_\rho} |f(t) - f(\omega(t))|\cdot|dt| \\
&\quad\leqq 2K_4 R_2\sqrt{1-r} + \frac{R_2}{1-r}\left[2\pi K_3\left(\frac{\varepsilon}{R_1}\right)\right]^{1/2},\ 0 < r < 1 .
\end{aligned} \tag{6}$$

(d) The whole argument can be repeated with $g(z)$ in place of $f(z)$. In this case we shall have an estimate analogous to (6):

$$\begin{aligned}(6')\qquad &\int_{C_\rho} |g(t) - g(\omega_1(t))|\cdot|dt| \\ &\leqq 2K_4R_2\sqrt{1-r} + \frac{R_2}{1-r}\left[2\pi K_3\left(\frac{\varepsilon}{R_1}\right)\right]^{1/2},\ 0 < r < 1 .\end{aligned}$$

The function $\omega_1(z)$ is the normalized mapping function of $|z| < 1$ onto the image of Δ under the inverse of $g(z)$. Since $f(\omega(z))$ and $g(\omega_1(z))$ are both normalized mapping functions from $|z| < 1$ onto Δ it follows from the uniqueness that

$$(7)\qquad f(\omega(z)) \equiv g(\omega_1(z)),\ |z| < 1 .$$

If we combine (6), (6′), (7) and choose r so that $1 - r = (\varepsilon/R_1)^{1/3}$, the conclusion of the theorem is established.

Throughout the remainder of the paper we shall assume the situation of Theorem 1 with the added hypothesis that Γ_f and Γ_g are rectifiable Jordan curves of lengths L_f and L_g. In this case it is well-known that $\bar{D}_f$ is the continuous image of $|z| \leqq 1$ and that if $f'(z)$ is defined at the boundary by

$$f'(e^{i\theta}) = \operatorname*{Lim}_{z\to e^{i\theta}} \frac{f(z) - f(e^{i\theta})}{z - e^{i\theta}},\ |z| \leqq 1 ,$$

then $f'(e^{i\theta})$ exists almost everywhere, is Lebesgue summable, and

$$L_f = \int_0^{2\pi} |f'(e^{i\theta})|\, d\theta .$$

3. The following definition ([4], p. 337) and lemma ([4], p. 337) are useful.

DEFINITION α. Let c denote a crosscut of D_f which does not pass through $w = 0$. Let T denote that subregion of D_f determined by c which does not contain $w = 0$. Let λ denote the diameter of c and let Λ denote the diameter of T. For any $\delta > 0$ consider all possible crosscuts c for which $\lambda \leqq \delta$. The crosscut modulus is defined is defined to be

$$\eta_f(\delta) = \sup_{\lambda \leqq \delta} \Lambda .$$

The crosscut modulus is monotonic and has the property:

$$\eta_f(\delta) \to 0 \quad \text{as} \quad \delta \to 0 .$$

LEMMA D. *Let A_f denote the area of D_f. Let z_0 be any point on $|z| = 1$ and k_s the part of the circle $|z - z_0| = s$ which lies in*

$|z| < 1$. *Then for every* $s, 0 < s < 1$, *there exists a* σ, $s \leqq \sigma \leqq s^{1/2}$ *such that the image of* k_σ *is a crosscut of length*

$$l_\sigma \leqq \left(\frac{2\pi A_f}{\log \frac{1}{s}} \right)^{1/2} .$$

We introduce the abbreviation:

$$\nu_f(\delta) = \eta_f\left(\left(\frac{2\pi A_f}{\log \frac{1}{\delta}} \right)^{1/2} \right), 0 < \delta < 1 . \tag{8}$$

An immediate consequence of Lemma D is

LEMMA 1.

$$h_f(r) = \operatorname*{Sup}_{|z|=1} |f(z) - f(rz)| \leqq \nu_f(1 - r), 0 < r < 1 .$$

4. DEFINITION 2. For $m \geqq 2$, let $\{w_1, w_2, w_3, \cdots, w_m\}$ be any set of m distinct points taken in cyclic order on Γ_f and so distributed that Γ_f is partitioned into m subarcs of equal length, each subarc having length $L_{f/m}$. Let l_λ be the length of the perimeter of the cyclically determined polygon, and let λ, the norm of the partition, be defined by

$$\lambda = \operatorname{Max} [|w_1 - w_m|, |w_2 - w_1|, |w_3 - w_2|, \cdots, |w_m - w_{m-1}|] .$$

The number l_λ can be written as

$$l_\lambda = |w_1 - w_m| + \sum_{k=1}^{m-1} |w_{k+1} - w_k| .$$

For any $\delta > 0$ consider all partitions for which $\lambda \leqq \delta$. Let

$$U_f(\delta) = \operatorname*{Inf}_{\lambda \leqq \delta} l_\lambda .$$

It is easily shown that $\operatorname{Sup} U_f(\delta) = L_f$. We define the modulus of rectifiability to be

$$\zeta_f(\delta) = L_f - U_f(\delta) .$$

The modulus $\zeta_f(\delta)$ is monotonic and has the property: $\zeta_f(\delta) \to 0$ as $\delta \to 0$.

LEMMA 2. *If* $L_f(r)$ *is the length of the level curve in* D_f *which is the image of* $|z| = r$, *then*

$$L_f - L_f(r) \leqq \zeta_f(\sqrt{\nu(1 - r)}) + 2L_f\sqrt{\nu_f(1 - r)} + 4\nu_f(1 - r), 0 < r < 1 .$$

Proof. Let the positive integer m be defined by

$$m = \left[\frac{L_f}{\sqrt{\nu_f}(1-r)}\right] + 2 . \tag{9}$$

Let $w_1, w_2, \cdots, w_m$ be a set of points in cyclic order Γ_f, so arranged that Γ_f is partitioned into m equal subarcs, each subarc having length L_f/m. Clearly the norm of the partition does not exceed L_f/m and if l_m is the length of the perimeter of the polygon, then

$$L_f - l_m \leqq \zeta_f\left(\frac{L_f}{m}\right) . \tag{10}$$

We define the points z_k, $\tilde{w}_k$ by $w_k = f(z_k)$, $\tilde{w}_k = f(rz_k)$. The set $\tilde{w}_k$ determines a polygon inscribed in the level curve in D_f which is the image of $|z| = r$. Comparing corresponding sides of the polygons, we have from Lemma 1,

$$\begin{aligned} |w_{k+1} - w_k| &\leqq |w_{k+1} - \tilde{w}_{k+1}| + |\tilde{w}_{k+1} - \tilde{w}_k| + |\tilde{w}_k - w_k| \\ &\leqq h_f(r) + |\tilde{w}_{k+1} - \tilde{w}_k| + h_f(r) \\ &\leqq 2\nu_f(1-r) + |\tilde{w}_{k+1} - \tilde{w}_k| . \end{aligned}$$

Similarly,

$$|\tilde{w}_{k+1} - \tilde{w}_k| \leqq 2\nu_f(1-r) + |w_{k+1} - w_k| .$$

Thus, if l'_m is the length of the perimeter of the level curve polygon,

$$|l'_m - l_m| \leqq 2m\nu_f(1-r) . \tag{11}$$

Noting that $l'_m \leqq L_f(r)$, we have from (10) and (11)

$$\begin{aligned} L_f - L_f(r) &\leqq L_f - l'_m \leqq L_f - l_m + |l_m - l'_m| . \\ &\leqq \zeta_f\left(\frac{L_f}{m}\right) + 2m\nu_f(1-r) . \end{aligned} \tag{12}$$

From (9)

$$\frac{L_f}{\sqrt{\nu_f}(1-r)} \leqq m \leqq \frac{L_f}{\sqrt{\nu_f}(1-r)} + 2 .$$

The conclusion follows from (10), (11) and (12).

In the estimate of Lemma 2, it would appear that the first term should dominate the others and this will be so if ζ_f is sufficiently weak. However, it is possible (e.g., if D_f is a disk) for the term $2L_f\sqrt{\nu_f}$ to be dominant. For purpose of final estimate we introduce the boundary functional

$$\beta_f(\delta) = \zeta_f(\sqrt{\nu_f(\delta)}) + 2L_f\sqrt{\nu_f(\delta)} + 4\nu_f(\delta),\ 0 < \delta < 1 . \tag{13}$$

LEMMA 3.

$$\int_{C_\rho} |f'(t) - f'(rt)| \cdot |dt| \leqq 2\sqrt{L_f \beta_f(1-r)}\,,$$
$$0 < r < 1\,, \quad \text{for all} \quad \rho, 0 \leqq \rho < 1\,.$$

Proof. The function $\sqrt{f'(z)}$ (i.e., the branch which is positive at the origin) is regular in $|z| < 1$. If $\sqrt{f'(z)} = \sum_0^\infty c_k z^k$, it is well known that $\Sigma |c_k|^2$ is convergent and

$$L_f = \int_0^{2\pi} \sqrt{f'(e^{i\theta})}\overline{\sqrt{f'(e^{i\theta})}} d\Theta = 2\pi \Sigma |c_k|^2\,,$$
$$L_f(r) = \int_0^{2\pi} \sqrt{f'(re^{i\theta})}\overline{\sqrt{f'(re^{i\theta})}} r d\Theta = 2\pi r \cdot \Sigma |c_k|^2 r^{2k},\ 0 < r < 1\,.$$

We write

$$\left[\int_{C_\rho} |f'(t) - f'(rt)| \cdot |dt|\right]^2$$
$$\leqq \int_{C_\rho} |\sqrt{f'(t)} - \sqrt{f'(rt)}|^2 \cdot |dt| \cdot \int_{C_\rho} |\sqrt{f'(t)} + \sqrt{f'(rt)}|^2 \cdot |dt|$$
$$= I_1 \cdot I_2\,,$$
$$I_1 = 2\pi\rho \cdot \Sigma |c_k|^2 \rho^{2k}(1 - 2r^k + r^{2k}) \leqq 2\pi\Sigma |c_k|^2 (1 - r^{2k})$$
$$= L_f - L_f(r) \cdot \frac{1}{r} \leqq L_f - L_f(r)\,,$$
$$I_2 = 2\pi\rho\, \Sigma |c_k|^2 \rho^{2k}(1 + 2r^k + r^{2k}) \leqq 2\pi\Sigma(|c_k|^2 \cdot 4) = 4L_f\,.$$

From these inequalities and Lemma 2, the conclusion is apparent.

5. Final estimates.

We assert:

THEOREM 2. *If Γ_f and Γ_g are rectifiable Jordan curves of lengths L_f and L_g, if $0 < \varepsilon/R_1 < 1$, then*

$$I(f', g'; \rho) \leqq 2[\sqrt{L_f} + \sqrt{L_g} + M/R_1^{1/2}]\sqrt{\beta_f(\sigma)} + 2\sqrt{L_g \mu}\,,$$
$$\textit{uniformly for all} \quad \rho, 0 \leqq \rho < 1\,,$$

where $\sigma = (\varepsilon/R_1)^{1/24}$, $\mu = |L_f - L_g|$, $M = \text{Max}\,[K_1R_2, 2\sqrt{L_g K_1 R_2}]$.

Proof. Write

$$I(f', g'; \rho) \leqq \int_{C_\rho} |f'(t) - f'(rt)| \cdot |dt| + \int_{C_\rho} |f'(rt) - g'(rt)| \cdot |dt|$$
$$+ \int_{C_\rho} |g'(rt) - g'(t)| \cdot |dt| = I_1 + I_2 + I_3\,.$$

Choose $1 - r = \sigma$, from Lemma 3,

$$I_1 \leqq 2\sqrt{L_f \beta_f(\sigma)} .$$

Let $0 < \rho < \alpha < 1$, then, from Theorem 1

$$I_2 \leqq \int_{C_\rho}\left[\frac{1}{2\pi}\int_{C_\alpha}\left|\frac{f(\gamma)-g(\gamma)}{(\gamma - rt)^2}\right|\cdot|d\gamma|\right]\cdot|dt| \leqq \frac{K_1 R_2(\varepsilon/R_1)^{1/6}}{(1-r)^2}$$
$$= K_1 R_2 \sigma^2 \leqq K_1 R_2 \sigma .$$

From the proof of Lemma 3 (with g in place of f)

$$I_3 \leqq 2\sqrt{L_g}(L_g - L_g(r))^{1/2} ,$$

and

$$L_g - L_g(r) \leqq |L_g - L_f| + L_f - L_f(r) + |L_f(r) - L_g(r)|$$
$$= \mu + A + B .$$

From Lemma 2, $A \leqq \beta_f(\sigma)$, and

$$B = \left|\int_{C_\rho}[|f'(rt)| - |g'(rt)|]\,|dt| \leqq \int_{C_\rho}|f'(rt) - g'(rt)|\cdot|dt|\right|$$
$$= I_2 \leqq K_1 R_2 \sigma^2 .$$

Thus,

$$I_3 \leqq 2\sqrt{L_g}(\mu + A + B)^{1/2} \leqq 2\sqrt{L_g}(\mu^{1/2} + A^{1/2} + B^{1/2}) .$$

Combining estimates we have

$$I(f', g'; \rho) \leqq 2(\sqrt{L_f} + \sqrt{L_g})\sqrt{\beta_f(\sigma)} + 2\sqrt{L_g \mu} + (2\sqrt{L_g K_1 R_2} + K_1 R_2)\sigma . \tag{14}$$

From (8) and (13) and the definition of η_f,

$$\sqrt{\beta_f(\sigma)} \geqq 2(\nu_f(\sigma))^{1/2} \geqq \eta_f\left(\left(\frac{2\pi A_f}{\log\frac{1}{\sigma}}\right)^{1/2}\right)^{1/2} \geqq \left(\frac{2\pi A_f}{\log\frac{1}{\sigma}}\right)^{1/4}$$
$$\geqq ((2\pi^2 R_1^2)^{1/4}\cdot\sigma) \geqq R_1^{1/2}\sigma .$$

Hence

$$(2\sqrt{L_g K_1 R_2} + K_1 R_2)\sigma \leqq \frac{2M\sqrt{\beta_f(\sigma)}}{R_1^{1/2}} \tag{15}$$

the conclusion follows from (14) and (15).

LEMMA 4. *If* $\mu = |L_f - L_g|$ *and if*

$$I^* = \operatorname*{Sup}_\rho I(f', g'; \rho),\ 0 \leqq \rho < 1 , \quad \textit{then} \quad \mu \leqq I^* .$$

Proof. We have

$$|L_f(\rho) - L_g(\rho)| = \left| \int_{C_\rho} |f'(t)|\cdot|dt| - \int_{C_\rho} |g'(t)|\cdot|dt| \right|$$
$$\leqq \left| \int_{C_\rho} |f'(t) - g'(t)|\cdot|dt| = I(f', g'; \rho) \leqq I^* .\right.$$

Let $\rho \to 1$ on the left and the lemma is proved.

LEMMA 5.

$$|f(e^{i\theta}) - g(e^{i\theta})| \leqq I^* .$$

Proof. The Fejer-Riesz inequality asserts that

$$A = \int_{-1}^{1} |H(x)|^p \, dx \leqq \frac{1}{2}\int_0^{2\pi} |H(e^{i\alpha})|^p \, d\alpha = B ,$$
$$p > 0 \text{ and } x \text{ is real} .$$

Here $H(z)$ which is regular in $|z| < 1$ belongs to the Hardy class H^p in $|z| \leqq 1$. Let $p = 1$ and we make the choice $H(z) = \rho e^{i\theta}(f'(z\rho e^{i\theta}) - g'(z\rho e^{i\theta}))$. Noting that $A \geqq \left| \int_0^1 H(x)dx \right|$, that $2B = I(f', g'; \rho) \leqq I^*$, we let $\rho \to 1$ and we get the conclusion of the lemma.

We are now able to state our convergence theorem as

THEOREM 3. *If the f boundary is held fixed and the g boundary is allowed to vary, a necessary and sufficient condition that $I(f', g'; \rho) \to 0$ uniformly for all ρ, $0 \leqq \rho < 1$, is that $\mu + \sigma \to 0$.*

Proof. We get the sufficiency from Theorem 2. From Lemma 4 we see that $I^* \to 0$ implies that $\mu \to 0$ which is one part of the necessity. From Lemma 5, we see that if I^* is arbitrarily small the boundary point $f(e^{i\theta})$ will be arbitrarily close to the g boundary and vice versa. So we have $I^* \to 0$ implies $\varepsilon \to 0$ implies that $\inf R_1 > 0$ so that $I^* \to 0$ implies that $\sigma \to 0$. This completes the proof of Theorem 3.

Without estimate, S. E. Warschawski [2] established a result that is similar to Theorem 3.

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