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ON UNIFORM CONVERGENCE FOR WALSH-FOURIER SERIES

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In 1940 R. Salem formulated a sufficient condition for a continuous and periodic function to have a trigonometric Fourier series which converges uniformly to the function. In this paper we will formulate a similar condition, which implies that the Walsh-Fourier series of such a function has this property. Furthermore we show that our result is stronger than certain classical results, and that it also implies the uniform convergence of the Walsh-Fourier series of certain classes of continuous functions of generalized bounded variation. The latter is analogous to results obtained by L. C. Young and R. Salem for trigonometric Fourier series.

Let $\{\varphi_n(x)\}$ be the sequence of Rademacher functions, i.e.,

$$\varphi_0(x) = +1 \left(0 \leqq x < \frac{1}{2}\right), \qquad \varphi_0(x) = -1 \left(\frac{1}{2} \leqq x < 1\right),$$
$$\varphi_0(x+1) = \varphi_0(x) .$$

$\varphi_n(x) = \varphi_0(2^n x)$, $(n = 1, 2, 3, \cdots)$. In [3] R. E. A. C. Paley gave the following definition for the Walsh functions $\{\psi_n(x)\}$: $\psi_0(x) \equiv 1$, and, if $n = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_r}$, with $n_1 > n_2 > \cdots > n_r$, then $\psi_n(x) = \varphi_{n_1}(x)\varphi_{n_2}(x) \cdots \varphi_{n_r}(x)$. J. L. Walsh [6] proved that the system $\{\psi_n(x)\}$ is a complete orthonormal system. For every Lebesgue-integrable function $f(x)$ of period 1 there is a corresponding Walsh-Fourier series (WFS):

$$f(x) \sim \sum_{k=0}^{\infty} c_k \psi_k(x), \quad \text{with} \quad c_k = \int_0^1 f(t)\psi_k(t)dt .$$

As in the case of trigonometric Fourier series (TFS), we can find a simple expression for the partial sums of a WFS,

$$S_n(f, x) = \sum_{k=0}^{n-1} c_k \psi_k(x) = \int_0^1 f(x \dotplus t) D_n(t)dt ,$$

where $D_n(t) = \sum_{k=0}^{n-1} \psi_k(t)$. For the meaning of $\dotplus$ and for further notations, definitions and properties of the WFS we refer to [2].

2. In [4], Chapter VI, R. Salem proved the following theorem: Let $f(x)$ be a continuous function of period 2π. For odd n, let

$$T_n(x) = \sum_{p=0}^{(n-1)/2} (p+1)^{-1}[f(x + 2p\pi/n) - f(x + (2p+1)\pi/n)]$$

and let $Q_n(x)$ be obtained from $T_n(x)$ by changing π into $-\pi$. Then, if $\lim_{n\to\infty} T_n(x) = \lim_{n\to\infty} Q_n(x) = 0$ uniformly in x, the TFS of $f(x)$ converges uniformly to $f(x)$. R. Salem also showed that this theorem implies both the Dini-Lipschitz test for continuous functions with modulus of continuity $\omega(f, \delta) = o(\log \delta^{-1})^{-1}$ as $\delta \to 0$, and Jordan's theorem on continuous functions of bounded variation. Finally, he extended this last theorem to certain classes of continuous functions of generalized bounded variation. For a proof of Salem's results, see also [1], Chapter IV, § 5.

3. Our main result about WFS can be stated as follows:

THEOREM. *Let $f(x)$ be a continuous function of period* 1. *Let*

$$U_n(x) = \sum_{p=1}^{2^n-1} p^{-1} \, | f(x + 2p/2^{n+1}) - f(x + (2p+1)/2^{n+1}) | \, .$$

Then, $\lim_{n\to\infty} U_n(x) = 0$ *uniformly in x implies that* $\lim_{k\to\infty} S_k(f, x) = f(x)$ *uniformly in x.*

Proof. For each natural number k we have

$$S_k(f, x) - f(x) = \int_0^1 D_k(t)[f(x+t) - f(x)]dt \, .$$

Let $k = 2^n + k'$, with $0 \leqq k' < 2^n$, then, according to [2], p. 386, we have $D_k(t) = D_{2^n}(t) + \psi_{2^n}(t) \cdot D_{k'}(t)$, where

$$D_{2^n}(t) = \begin{cases} 2^n \text{ on } [0, 2^{-n}) \\ \\ 0 \text{ on } [2^{-n}, 1) \end{cases} , \quad D_{k'}(t) = k' \text{ on } [0, 2^{-n}) \, ,$$

and

$$\psi_{2^n}(t) = \begin{cases} +1 \text{ on } [2p/2^{n+1}, (2p+1)/2^{n+1}) \\ \\ -1 \text{ on } [(2p+1)/2^{n+1}, (2p+2)/2^{n+1}) \end{cases} \qquad \text{for } p = 0, 1, \cdots, 2^n - 1$$

Therefore,

$$| S_k(f, x) - f(x) | \leqq \left| \int_0^1 D_{2^n}(t) \, [f(x+t) - f(x)]dt \right|$$
$$+ \left| \int_0^1 \psi_{2^n}(t) D_{k'}(t)[f(x+t) - f(x)]dt \right| = A + B \, .$$

For the first term of this sum we have

$$A \leqq 2^n \int_0^{2^{-n}} |f(x \dotplus t) - f(x)|\, dt \leqq \omega(f, 2^{-n}) .$$

For the second term we have

$$\begin{aligned} B &= \Bigg| \sum_{p=0}^{2^n-1} \Bigg(\int_{2p/2^{n+1}}^{(2p+1)/2^{n+1}} D_{k'}(t)[f(x \dotplus t) - f(x)]dt \\ &\quad - \int_{(2p+1)/2^{n+1}}^{(2p+2)/2^{n+1}} D_{k'}(t)[f(x \dotplus t) - f(x)]dt \Bigg) \Bigg| \\ &= \Bigg| \sum_{p=0}^{2^n-1} \int_{2p/2^{n+1}}^{(2p+1)/2^{n+1}} (D_{k'}(t)[f(x \dotplus t) - f(x)] \\ &\quad - D_{k'}(t + 2^{-n-1})[f(x \dotplus (t + 2^{-n-1})) - f(x)])dt \Bigg| . \end{aligned}$$

Now we observe that, since $k' < 2^n$, $D_{k'}(t)$ is a sum of functions $\psi_i(t)$ with $i < 2^n$. Each of these functions is constant on the intervals $[k/2^n, (k+1)/2^n)$, $(k = 0, 1, \cdots, 2^n - 1)$. Therefore, if $t \in [2p/2^{n+1}, (2p+1)/2^{n+1})$, then $D_{k'}(t) = D_{k'}(t + 2^{-n-1}) = D_{k'}(2p/2^{n+1})$. Thus we have

$$\begin{aligned} B &= \Bigg| \sum_{p=0}^{2^n-1} \int_{2p/2^{n+1}}^{(2p+1)/2^{n+1}} D_{k'}(p/2^n)[f(x \dotplus t) - f(x \dotplus (t + 2^{-n-1}))]dt \Bigg| \\ &= \Bigg| \sum_{p=0}^{2^n-1} \int_0^{2^{-n-1}} D_{k'}(p/2^n)[f(x \dotplus (t + 2p/2^{n+1})) \\ &\quad - f(x \dotplus (t + (2p+1)/2^{n+1}))]dt \Bigg| \\ &= \Bigg| \sum_{p=0}^{2^n-1} 2^{-n-1} \int_0^1 D_{k'}(p/2^n)[f(x \dotplus (t + 2p)/2^{n+1}) \\ &\quad - f(x \dotplus (t + 2p + 1)/2^{n+1})]dt \Bigg| \\ &\leqq \Bigg| 2^{-n-1} \int_0^1 D_{k'}(0)[f(x \dotplus t/2^{n+1}) - f(x \dotplus (t + 1)/2^{n+1})]dt \Bigg| \\ &\quad + \Bigg| \sum_{p=1}^{2^n-1} 2^{-n-1} \int_0^1 \cdots dt \Bigg| \\ &\leqq 2^{-n-1} \cdot k' \cdot \omega(f, 2^{-n-1}) + \Bigg| \int_0^1 \Bigg(2^{-n-1} \sum_{p=1}^{2^n-1} \cdots \Bigg) dt \Bigg| = B_1 + B_2 . \end{aligned}$$

Using the fact that for $u \in (0, 1)$, $|D_k(u)| < 2u^{-1}$, [2], Lemma 1, we obtain the following inequality for the integrand, I, of B_2:

$$\begin{aligned} |I| \leqq \sum_{p=1}^{2^n-1} 2^{-n-1} \cdot 2^{n+1} \cdot p^{-1} \, |f(x \dotplus (t + 2p)/2^{n+1}) \\ - f(x \dotplus (t + 2p + 1)/2^{n+1})| . \end{aligned}$$

Now we observe that for every $t \in [0, 1)$ there is an $\tilde{x} \in [0, 1)$, $\tilde{x} = \tilde{x}(t)$, such that $x \dotplus (t + q)/2^{n+1} = \tilde{x} \dotplus q/2^{n+1}$ for all $q = 1, 2, \cdots, 2^{n+1} - 1$. Therefore

$$|I| \leqq \sum_{p=1}^{2^n-1} p^{-1} |f(\tilde{x} \dotplus 2p/2^{n+1}) - f(\tilde{x} \dotplus (2p+1)/2^{n+1})| = U_n(\tilde{x}) .$$

Under the hypothesis of our theorem $U_n(\tilde{x}) \to 0$ uniformly in $\tilde{x}$ as $n \to \infty$. This implies that $B_2 \to 0$ uniformly in x as $n \to \infty$, and so, $\lim_{k\to\infty} (S_k(f, x) - f(x)) = 0$ uniformly in x.

4. In this section we will show that our main theorem implies two classical results for WFS. The first is the Dini-Lipschitz test for WFS, which was first proved in [2], Th. XIII. A generalization of it can be found in [5], § (3.5).

COROLLARY 1. *Let $f(x)$ be a continuous function of period* 1 *and let $\omega(f, \delta) = o(\log \delta^{-1})^{-1}$ as $\delta \to 0$. Then the* WFS *of $f(x)$ converges uniformly to $f(x)$.*

Proof. We see immediately that

$$| U_n(x) | \leqq \sum_{p=1}^{2^n-1} p^{-1}\omega(f, 2^{-n-1}) \leqq \omega(f, 2^{-n-1})C \log 2^n$$

for some constant C. Thus $\lim_{n\to\infty} U_n(x) = 0$ uniformly in x.

The next corollary is Jordan's test for WFS, which was first proved in [6], Th. IV.

COROLLARY 2. *Let $f(x)$ be a continuous function of period* 1. *If $f(x)$ is of bounded variation on* [0, 1], *then its* WFS *converges uniformly to $f(x)$.*

Proof. We can find a nondecreasing sequence of natural numbers $\{m(n)\}$ such that (a) $m(n) < 2^n - 1$ for all n, (b) $m(n) \to \infty$ as $n \to \infty$, (c) $\omega(f, 2^{-n-1}) \log m(n) \to 0$ as $n \to \infty$. Then,

$$\begin{aligned} U_n(x) | &\leqq \omega(f, 2^{-n-1})\left[1 + \frac{1}{2} + \cdots + \frac{1}{m(n)}\right] \\ &\quad + \sum_{p=m(n)+1}^{2^n-1} p^{-1} |f(x \dotplus 2p/2^{n+1}) - f(x \dotplus (2p+1)/2^{n+1})| \\ &\leqq C\omega(f, 2^{-n-1}) \log m(n) + (m(n)+1)^{-1} \operatorname{Var}(f) . \end{aligned}$$

Thus $\lim_{n\to\infty} U_n(x) = 0$ uniformly in x.

Finally we will prove a theorem for WFS analogous to certain results of L. C. Young [7] and R. Salem [4] for TFS, and which is an extension of Jordan's theorem. First we will give a definition of bounded Φ-variation.

Let $\varphi(u)$ be a continuous, strictly increasing function defined for $u \geqq 0$, such that $\varphi(0) = 0$ and $\lim_{u\to\infty} \varphi(u) = \infty$. Let ψ be the inverse of φ. Next, let $\Phi(u) = \int_0^u \varphi(t)dt$ and $\Psi(u) = \int_0^u \psi(t)dt$. Functions so obtained, are called complementary in the sense of W. H. Young, and they satisfy the following inequality, due to W. H. Young: if $a, b \geqq 0$, then $ab \leqq \Phi(a) + \Psi(b)$, see [8], *p.* 16.

DEFINITION. A function $f(x)$ on $[0, 1)$ is said to be of bounded Φ-variation if there is an $M < \infty$ such that for each finite partition $0 \leqq x_1 < x_2 \cdots < x_n \leqq 1$ we have $\sum_{i=1}^{n-1} \Phi(|f(x_{i+1}) - f(x_i)|) < M$.

We can prove the following

COROLLARY 3. *Let $\Phi(x)$ and $\Psi(x)$ be functions complementary in the sense of W. H. Young and let $\sum_{k=1}^{\infty} \Psi(k^{-1}) < \infty$. Let $f(x)$ be a continuous function of period* 1 *and of bounded Φ-variation. Then $\lim_{n\to\infty} S_n(f, x) = f(x)$ uniformly in x.*

Proof. Since $\sum_{k=1}^{\infty} \Psi(k^{-1}) < \infty$, we can find a sequence $\{\varepsilon(k)\}$ of positive numbers, decreasing to 0 as $k \to \infty$, and for which

$$\sum_{k=1}^{\infty} \Psi(k\varepsilon(k))^{-1}) < \infty .$$

Let

$$|f(x + 2p/2^{n+1}) - f(x + (2p + 1)/2^{n+1})| = \Delta_p .$$

Then, according to Young's inequality, we have

$$\Delta_p \cdot (p\varepsilon(p))^{-1} \leqq \Phi(\Delta_p) + \Psi((p\varepsilon(p))^{-1}) .$$

From our hypothesis it follows that there is a constant $N < \infty$ such that for each m

$$\sum_{p=m}^{2^n-1} \Delta_p(p\varepsilon(p))^{-1} \leqq \sum_{p=m}^{2^n-1} \Phi(\Delta_p) + \sum_{p=m}^{2^n-1} \Psi((p\varepsilon(p))^{-1}) < N .$$

Therefore,

$$\sum_{p=m}^{2^n-1} \Delta_p p^{-1} < N\varepsilon(m) .$$

Choosing $\{m(n)\}$ as in the proof of Corollary 2, we have

$$|U_n(x)| \leqq \omega(f, 2^{-n-1})\left[1 + \frac{1}{2} + \cdots + \frac{1}{m(n)}\right] + N\varepsilon(m(n) + 1) ,$$

i.e., $U_n(x) \to 0$ uniformly in x as $n \to \infty$.

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