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PRIMARY RINGS AND DOUBLE CENTRALIZERS

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This note is devoted to proving the theorem that every right quasi-projective module over a semi-primary ring R has the double centralizer property if and only if R is a direct sum of primary rings, and to discussing some of its consequences. In particular, this theorem places a strong necessary condition on a large class of the balanced rings of Camillo which are both a specialization of Thrall's $QF-1$ rings and a generalization of the uniserial rings of Köthe.

Let R be an associative ring with identity. An R-module M is *quasi-projective* [19] (*quasi-injective* [10]) in case $\mathrm{Hom}_R(M,\ )$ ($\mathrm{Hom}_R(\ ,M)$) preserves the exactness of all short exact sequences with middle term M. If the natural ring homomorphism of R into the double centralizer, $\mathrm{Hom}_C(M, M)$, where $C = \mathrm{Hom}_R(M, M)$, is onto, the R-module M is said to have the *double centralizer property.*

Let N denote the Jacobson radical of R. Then R is semi-primary in case N is nilpotent and R/N is semi-simple. If, in addition, R/N is a simple ring, R is said to be *primary.* A semi-primary ring R is a direct sum of primary rings if and only if for each pair of primitive idempotents e and f in R $fNe \neq 0$ implies $fR \cong eR$. In other words, direct sums of primary rings are just those semi-primary rings in which the composition factors in each primitive one-sided ideal are pairwise isomorphic. It follows that if N is nilpotent and R/N^2 is a direct sum of primary rings, then so is R.

We shall need the following notation. The *socle*, $S(M)$, of an R-module M is its largest semi-simple submodule. The *top*, $T(M)$, of M is the largest semi-simple factor module of M. The former is equal to the annihilator in M of N, the latter is M/NM (or M/MN). In particular, if e is a primitive idempotent in R then $T(Re)$ is the unique simple factor module of Re.

With the above, we are ready to prove the main result.

THEOREM. *Every right* (*equivalently, left*) *quasi-projective module over a semi-primary ring R has the double centralizer property if and only if R is a direct sum of primary rings.*

Proof. ($\Rightarrow$) Let R be semi-primary. In view of the above comments and the fact that quasi-projective modules over factor rings of R are quasi-projective as R-modules, we may assume that $N^2 = 0$. Suppose that, for some primitive idempotents e and f in R,

$fNe \neq 0$ and $fR \not\cong eR$. We prove this implication by constructing a factor ring $\hat{R}$ of R over which some faithful projective right module does not have the double centralizer property. To this end, let $A = \{r \in R \mid fRr = 0\}$, the right annihilator in R of fR. Write $\bar{R} = R/A$, $\bar{N} = (N + A)/A$, $\bar{e} = e + A$ and $\bar{f} = f + A$. Here $\bar{N}$ is the radical of $\bar{R}$ and, because $fR \cap A = 0$, $\bar{e}$ and $\bar{f}$ are primitive idempotents in $\bar{R}$ with $\bar{f}\bar{N}\bar{e} \neq 0$. Now repeat the process on the left to construct from $\bar{R}$ and $\bar{B} = \{\bar{r} \in \bar{R} \mid \bar{r}\bar{R}\bar{e} = 0\}$ a factor ring $\hat{R}$ possessing radical $\hat{N}$ and primitive idempotents $\hat{f}$ and $\hat{e}$ with $\hat{f}\hat{N}\hat{e} \neq 0$. Since the modules $\bar{f}\bar{R}_{\bar{R}}$ and ${}_{\hat{R}}\hat{R}\hat{e}$ are faithful it follows, as in the discussion preceding [8, Th. 3], that every minimal left ideal in $\bar{R}$ is isomorphic to $T(\bar{R}\bar{f})$ and every minimal right ideal in $\hat{R}$ is isomorphic to $T(\hat{e}\hat{R})$. Thus, noting that $\bar{N} \subseteqq S({}_{\bar{R}}\bar{R})$ and $\hat{N} \subseteqq S(\hat{R}_{\hat{R}})$, we glean the additional information that $\hat{e}\hat{N} = 0$ and $\hat{N}\hat{f} = 0$. According to the implication (b) $\Rightarrow$ (d) of [8, Th. 4] (whose proof is valid for semiprimary rings), the present proof will be complete once we show that $T(\hat{R}\hat{e})$ is not isomorphic to a minimal left ideal in $\hat{R}$ and that $\mathrm{Ext}^1_{\hat{R}}(T(\hat{R}\hat{e}), \hat{R}) \neq 0$. Because $\hat{N}^2 = 0$ we have $S({}_{\hat{R}}\hat{R}) = \hat{N} + \hat{T}$ where $\hat{T}$ is the sum of the simple primitive left ideals in $\hat{R}$. But $\hat{e}\hat{N} = 0$ and $\hat{N}\hat{e} \neq 0$ so $T(\hat{R}\hat{e})$ cannot be embedded in either summand. On the other hand, $\hat{N}\hat{f} = 0$ implies that $T(\hat{R}\hat{f})$ is a direct summand of ${}_{\hat{R}}\hat{R}$, so an essential extension ${}_{\hat{R}}M$ of $T(\hat{R}\hat{f})$ with $M/T(\hat{R}\hat{f}) \cong T(\hat{R}\hat{e})$ will show that $\mathrm{Ext}^1_R(T(\hat{R}\hat{e}), R) \neq 0$. To obtain such an extension, let $E = E(T(\hat{R}\hat{f}))$, the injective hull [4] of $T(\hat{R}\hat{f})$ over $\hat{R}$. Then since $\hat{f}\hat{R}\hat{e} \neq 0$ and $\hat{e}\hat{R}\hat{f} = 0$ we have, by [7, Lemma (1.1), (c)], that $\hat{e}E/T(\hat{R}\hat{f}) \neq 0$. This and the fact that $E/T(\hat{R}\hat{f})$ is semi-simple yield an $\hat{R}$-module M with $T(\hat{R}\hat{f}) \leqq M \leqq E$ and $M/T(\hat{R}\hat{f}) \cong T(\hat{R}\hat{e})$.

($\Leftarrow$) If R is a direct sum of primary rings then every faithful projective R-module must be a generator and every factor ring of R has the same property. According to [9, Th. 3.3] or [11, Th. 1.10] a quasi-projective R-module M is faithful and projective over a factor ring of R. Thus M is a generator over that factor ring and has the double centralizer property by [5, Th. 1]. This completes the proof.

Camillo [2] calls a ring R *balanced* in case each of its right modules has the double centralizer property (equivalently, each factor ring of R is QF-1 in the sense of Thrall [18]). He proved that every balanced ring is semi-perfect (for properties of semi-perfect and perfect rings, see [1]) with nil radical N and observed that a direct sum of rings is balanced if and only if so is each of the direct summands. In the case where N is nilpotent, our theorem together with a recent theorem of Morita and Tachikawa reduces the study of balanced rings to that of local rings. (R is *local* if R/N is a division ring.)

COROLLARY 1. *Every indecomposable semi-primary balanced ring is Morita equivalent to a local balanced ring.*

Proof. According to Theorem 2 of the appendix of [14], if R and S are Morita equivalent rings (i. e., if their categories of modules are isomorphic in the sense of Morita [13]) then R is QF-1 when S is. Moreover, from the results of [13] it follows that if R and S are Morita equivalent then each factor ring of R bears the same relationship to a factor ring of S. Thus QF-1 and balanced are both categorical concepts. But every primary ring R is Morita equivalent to a local ring (in fact, isomorphic to a full matrix ring over a local ring eRe, e a primitive idempotent in R); and according to our theorem indecomposable balanced semi-primary rings are primary.

A ring R has *dominant dimension*, dom. dim. (R), at least n in case there is an exact sequence

$$0 \to R \to E_1 \to \cdots \to E_n$$

with E_i an injective projective left R-module, $i = 1, \cdots, n$ (see, for example, [16]). A left artinian ring is QF-3 in case it has dominant dimension at least 1. Thus, according to [7, Th. 3.6], a left artinian ring is (right artinian and) generalized uniserial if and only if each of its factor rings has dominant dimension at least 1. Every QF ring has infinite dominant dimension (Nakayama conjectured the converse, at least for finite dimensional algebras (see [15])) and, according to the proof of Lemma 2 in Nakayama's [17], a ring is *uniserial* [12] (= a direct sum of primary generalized uniserial rings) if and only if each of its factor rings is QF. Now, because every faithful projective (equivalently, every faithful injective) module over a QF-3 ring R has the double centralizer property precisely when dom. dim. $(R) \geqq 2$ (see [8] and [16]), we see that this condition on the factor rings is much stronger than necessary.

COROLLARY 2. *A left artinian ring R is uniserial if* (*and only if*) *each factor ring of R has dominant dimension at least* 2.

REMARKS. (a) The sufficiency part of the theorem is valid for right perfect rings. The necessity part holds for any semi-perfect ring in which central idempotents can be lifted modulo N^2, as they can in a semi-primary ring.

(b) It is not difficult to prove that a left module over a right perfect ring is rationally complete (i. e., has no proper rational extension (see [6, p. 58])) if and only if its only essential extensions by simple modules are by simple submodules of itself. Thus the argument of the theorem shows that a semi-primary ring R is a direct sum of primary rings if and only if the left regular representation of each factor ring of R is rationally complete (cf., [3]).

(c) A left artinian ring is a direct sum of primary rings if and only if each of its left quasi-injective modules has the double centralizer property. This follows from [8, Th. 5], [9, Corollary 1.3], and our present theorem.

(d) Since every factor ring of a uniserial ring is QF, uniserial rings are balanced. In [2] Camillo proved that balanced commutative rings are uniserial. By Corollary 2, balanced generalized uniserial rings are also uniserial. In fact, all the balanced rings that we know of are uniserial rings.[1]

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