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A DENSITY WHICH COUNTS MULTIPLICITY

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P. Erdös, using analytic theorems, has proven the following results: Let $f(x)$ be the number of integers m such that $\phi(m) \leqq x$, where ϕ is the Euler function, and let $g(x)$ be the number of integers n such that $\sigma(n) \leqq x$, where σ is the usual sum of divisors function. Then there are positive (but undetermined) constants c_1 and c_2 such that $f(x) = c_1x + o(x)$ and $g(x) = c_2(x) + o(x)$. The constants c_1 and c_2 can be calculated using complex analysis including the Wiener-Ikehara Theorem. A major purpose of this paper is to give an elementary proof that $\lim_{x\to\infty} f(x)/x$ exists and, in the process, calculate the value of the limit. These considerations of multiplicity motivate a generalization of natural density which counts multiplicity. This paper contains an investigation of this generalization.

Let $A = \{a_i\}_{i=1}^{\infty}$ be a sequence of positive real numbers $\geqq 1$. For a positive integer j, define $\#(A, j)$ to be the number of integers i such that $a_i \leqq j$ (that is, the number of elements of A *counting multiplicity* which are $\leqq j$). If $\liminf_{j\to\infty} \#(A, j)/j = \alpha$ (we allow $\alpha = \infty$) we say A has Δ-asymptotic density α and we define $\underline{\Delta}(A) = \alpha$. We also define $\bar{\Delta}(A) = \limsup_{j\to\infty} \#(A, j)/j$. If $\underline{\Delta}(A) = \bar{\Delta}(A)$ we say A has Δ-natural density α and we define $\Delta(A) = \alpha$. It is clear that a reordering of A does not affect $\underline{\Delta}(A)$ or $\bar{\Delta}(A)$. It is also clear that $\underline{\Delta}(A) = \underline{\Delta}(\{[a_i]\}_{i=1}^{\infty})$ and $\bar{\Delta}(A) = \bar{\Delta}(\{[a_i]\}_{i=1}^{\infty})$ where $[a_i]$ is the greatest integer which does not exceed a_i. Unless otherwise specified all sequences in this paper will be of positive real numbers.

Throughout this paper d will denote natural density, i.e., the classical analog of Δ where multiplicity is not counted; Z^+ will denote the set of positive integers; Q^+ will denote the positive rational numbers; R^+ will denote the set of positive real numbers; p will always be a prime; and $P = \{p_i\}_{i=1}^{\infty}$ will be the sequence, in the natural order, of primes.

If γ: $Z^+ \to R^+$ then to γ there corresponds the unique sequence $\gamma(1), \gamma(2), \cdots$. We will write γ in place of this sequence. Thus, for example, in the notation of this paper $\Delta(\phi)$ and $\Delta(\sigma)$ exist and are positive [5]. If for instance $\gamma = \tau$, where $\tau(n) =$ the number of positive integer divisors of the positive integer n, then it is clear that $\Delta(\tau) = \infty$.

If $A = \{a_i\}_{i=1}^{\infty}$ and $B = \{b_j\}_{j=1}^{\infty}$ are sequences then define $A + B$ to be the sequence, in the natural order, of positive real numbers x such that there exist i and $j \in Z^+$ with $a_i + b_j = x$, and x appears in this sequence the precise number of distinct ways we can write $x = a_{i_1} + b_{j_1}$. Note that it is possible to have $x = a_{i_1} + b_{j_1}$ and yet for x not to be a member of $A + B$. This happens precisely when some positive number $y < x$ is representable infinitely often in the form $y = a_i + b_j$. Finally if A and B are sets of positive reals then define $A \backslash B$ to be the complement of B in A.

1. Number theoretic functions. In this section we investigate the densities of certain sequences related to the ϕ function and other functions.

We first prove some lemmas which we will use to calculate $\Delta(\phi)$.

DEFINITION 1.1. For each $n \in Z^+$ and $k \in Z^+$ define

$$\phi_k(n) = n \prod_{\substack{p \mid n \\ p \leq p_k}} \frac{p-1}{p} \; ;$$

cf. [8, p. 56].

LEMMA 1.1.1. *$\Delta(\phi_k) = \prod_{p \leq p_k} (1 + (1/p(p-1)))$ for each $k \in Z^+$.*

Proof. Pick $k \in Z^+$ and define $P^k = \{p_1, p_2, \cdots, p_k\}$. To each subset P_j^k $(j = 1, 2, \cdots, 2^k)$ of P^k there corresponds the sequence of positive integers which are divisible by each member of P_j^k and by no member of $P^k \backslash P_j^k$. These sequences are pairwise disjoint and their union is Z^+.

For a subset P_j^k of P^k say $\{n_{j,i}\}_{i=1}^{\infty}$ is the corresponding sequence. It is clear that

$$(*) \qquad \#(\phi_k, n) = \sum_{j=1}^{2^k} \#(\{\phi_k(n_{j,i})\}_{i=1}^{\infty}, n) \quad \text{for each} \quad n \in Z^+.$$

Now for a fixed P_j^k the density of $\{n_{j,i}\}_{i=1}^{\infty}$ is clearly

$$\prod_{p \in P_j^k} \frac{1}{p} \prod_{p \in P^k \backslash P_j^k} \frac{p-1}{p} \, .$$

Also for each integer m in this sequence we have

$$\phi_k(m) = m \prod_{p \in P_j^k} \frac{p-1}{p} \, .$$

Therefore

$$\Delta(\{\phi_k(m)\}_{m \text{ in the sequence defined by } P_j^k})$$
$$= \Big(\prod_{p \in P_j^k} \frac{p}{p-1}\Big)\Big(\prod_{p \in P_j^k} \frac{1}{p}\Big)\Big(\prod_{p \in P^k \backslash P_j^k} \frac{p-1}{p}\Big) = \prod_{p \in P_j^k} \frac{1}{p-1} \prod_{p \in P^k \backslash P_j^k} \frac{p-1}{p} \, .$$

So by (*) we have

$$\begin{aligned}\Delta(\phi_k) &= \sum_{j=1}^{2^k}\Bigl(\prod_{p\in P_j^k}\frac{1}{p-1}\prod_{p\in P^k\setminus P_j^k}\frac{p-1}{p}\Bigr)\\ &= \sum_{j=1}^{2^k}\frac{\prod_{p\in P^k\setminus P_j^k}\frac{(p-1)^2}{p}}{\prod_{p\in P^k}(p-1)} = \frac{\prod_{p\in P^k}\Bigl(1+\frac{(p-1)^2}{p}\Bigr)}{\prod_{p\in P^k}(p-1)} = \prod_{p\in P^k}\Bigl(1+\frac{1}{p(p-1)}\Bigr)\end{aligned}$$

and the lemma is proved.

Note. $\lim_{k\to\infty}\Delta(\phi_k) = \prod_{p\in P}\Bigl(1+\frac{1}{p(p-1)}\Bigr) < \infty$.

LEMMA 1.1.2. *Choose* $n\in Z^+$, $n>1$, *and say* $r\in Z^+$ *satisfies* $p_1p_2\cdots p_r \leqq n$. *Then* $\#(\phi_r, n) \leqq n(\Delta(\phi_r)+1)$. *In fact if*

$$n = tp_1p_2\cdots p_r,\ t\geqq 1,\ t\in Q^+ ,$$

then $\#(\phi_r, n) \leqq n(\Delta(\phi_r) + 1/t)$.

Proof. Say $n = tp_1\cdots p_r$ $(t\geqq 1)$. Then if

$$P_j^r = \{q_1, \cdots, q_s\} \subset \{p_1, \cdots, p_r\}$$

we have $R_{j,r} \overset{\mathrm{def}}{=}$ the number of integers m such that $\phi_r(m)\leqq n$ and $q_1\cdots q_s \mid m$ and none of the members of $P^r\setminus P_j^r$ divide m = the number of integers $m\leqq n(q_1/q_1-1)\cdots(q_s/q_s-1)$ which are divisible by $q_1\cdots q_s$ and divisible by no member of P^r/P_j^r. Say $T_{j,r}$ is the smallest integer $\geqq t(q_1/q_1-1)\cdots(q_s/q_s-1)$. Then clearly $R_{j,r}\leqq$ the number of integers m which do not exceed $p_1\cdots p_rT_{j,r}$ and which are divisible by $q_1\cdots q_s$ and divisible by no member of $P^r\setminus P_j^r$. But since $T_{j,r}$ is an integer we have

$$\begin{aligned}R_{j,r} &\leqq (p_1\cdots p_rT_{j,r})\frac{1}{q_1\cdots q_s}\prod_{p\in P^r\setminus P_j^r}\frac{p-1}{p}\\ &\leqq p_1\cdots p_r\Bigl(t\frac{q_1}{q_1-1}\cdots\frac{q_s}{q_s-1}+1\Bigr)\frac{1}{q_1\cdots q_s}\prod_{p\in P^r\setminus P_j^r}\frac{p-1}{p} .\end{aligned}$$

Now $\#(\phi_r, n) = \sum_{j=1}^{2^r} R_{j,r}$. So

$$\begin{aligned}\#(\phi_r, n) &\leqq \sum_{j=1}^{2^r}\Bigl(p_1\cdots p_r\Bigl(t\prod_{p\in P_j^r}\frac{p}{p-1}+1\Bigr)\prod_{p\in P_j^r}\frac{1}{p}\prod_{p\in P^r\setminus P_j^r}\frac{p-1}{p}\Bigr)\\ &= tp_1\cdots p_r\sum_{j=1}^{2^r}\Bigl(\prod_{p\in P_j^r}\frac{1}{p-1}\prod_{p\in P^r\setminus P_j^r}\frac{p-1}{p}\Bigr)\\ &\quad + p_1\cdots p_r\sum_{j=1}^{2^r}\Bigl(\prod_{p\in P_j^r}\frac{1}{p}\prod_{p\in P^r\setminus P_j^r}\frac{p-1}{p}\Bigr) = n\Bigl(\Delta(\phi_r)+\frac{1}{t}\Bigr)\end{aligned}$$

and the lemma is proved.

LEMMA 1.1.3. *Choose* $n \in Z^+$, $n > 1$, *and say* $r \in Z^+$ *is defined by* $p_1 \cdots p_r \leqq n < p_1 \cdots p_{r+1}$. *Then we have*

$$\phi(m) \leqq n \Rightarrow \phi_r(m) \leqq \frac{p_{r+1}}{p_{r+1}-1} \cdots \frac{p_{2r+1}}{p_{2r+1}-1} n .$$

Thus

$$\#(\phi, n) \leqq \#\left(\phi_r, \left[\frac{p_{r+1}}{p_{r+1}-1} \cdots \frac{p_{2r+1}}{p_{2r+1}-1} n\right]\right) .$$

Proof. Suppose m has more than $r+1$ distinct prime divisors. Then $\phi(m) \geqq (p_{r+2}-1)(p_{r+1}-1) \cdots (p_1-1) \geqq p_1 \cdots p_{r+1} > n$, a contradiction. So m has at most $r+1$ distinct prime divisors.

Now

$$\phi_r(m) = \phi(m) \prod_{\substack{p \mid m \\ p > p_r}} \frac{p}{p-1} \leqq n \prod_{\substack{p \mid m \\ p > p_r}} \frac{p}{p-1} \leqq n \frac{p_{r+1}}{p_{r+1}-1} \cdots \frac{p_{2r+1}}{p_{2r+1}-1}$$

since m has at most $r+1$ distinct prime divisors and the lemma is proved.

THEOREM 1.1.

$$\Delta(\phi) = \prod_{p \in P}\left(1 + \frac{1}{p(p-1)}\right) = \frac{\zeta(2) \cdot \zeta(3)}{\zeta(6)} ,$$

where ζ *denotes the Riemann Zeta function.*

Proof. It is well known [7, p. 246] that $\zeta(s) = \prod_{p \in P} (1/1 - p^{-s})$ for $s > 1$. Thus it follows that $\prod_{p \in P} (1 + (1/p(p-1))) = (\zeta(2) \cdot \zeta(3)/\zeta(6))$. So it only remains to show that $\Delta(\phi) = \prod_{p \in P} (1 + (1/p(p-1)))$.

For $r \in Z^+$ let $g_r = (p_{r+1}/p_{r+1}-1) \cdots (p_{2r+1}/p_{2r+1}-1)$. It follows from Mertens' Theorem and Tchebychef's Theorem [7, pp. 351 and 9] that $\lim_{r \to \infty} g_r = 1$. Choose $n \in Z^+$, $n > 1$, and say $r \in Z^+$ is defined by $p_1 \cdots p_r \leqq n = tp_1 \cdots p_r < p_1 p_2 \cdots p_{r+1}$, where $t \geqq 1$.

Now, $\#(\phi_r, n) = \#(\phi_{r-1}, n) + (\#(\phi_r, n) - \#(\phi_{r-1}, n))$. But

$$\#(\phi_r, n) - \#(\phi_{r-1}, n)$$

is the number of integers m such that $p_r \mid m$ and

$$n < \phi_{r-1}(m) \leqq \frac{p_r}{p_{r-1}} n .$$

This number is the sum (over $j = 1, 2, \cdots, 2^{r-1}$) of the number of integers less than or equal to

$$\Big(\prod_{p\in P_j^{r-1}}\frac{p}{p-1}\Big)\frac{np_r}{p_r-1}$$

and greater than

$$\Big(\prod_{p\in P_j^{r-1}}\frac{p}{p-1}\Big)n$$

which are divisible by p_r and each $p \in P_j^{r-1}$ and not divisible by any $p \in P^{r-1}\backslash P_j^{r-1}$. It then follows that

$$\begin{aligned}&\#(\phi_r, n) - \#(\phi_{r-1}, n)\\ &\leqq \sum_{j=1}^{2^{r-1}}\Big\{\Big(\frac{2np_r}{p_r(p_r-1)}\Big(\prod_{p\in P_j^{r-1}}\frac{p}{p-1}\Big) - \frac{n}{2p_r}\Big(\prod_{p\in P_j^{r-1}}\frac{p}{p-1}\Big)\Big)\\ &\quad\times \prod_{p\in P_j^{r-1}}\frac{1}{p}\prod_{p\in P^{r-1}\backslash P_j^{r-1}}\frac{p-1}{p}\Big\} = n\Delta(\phi_{r-1})\Big(\frac{2}{p_r-1}-\frac{1}{2p_r}\Big) = o(n)\,.\end{aligned}$$

So $\#(\phi_r, n) = \#(\phi_{r-1}, n) + o(n)$.

By Lemma 1.1.2 we have

$$\#(\phi_{r-1}, n) \leqq n\Big(\Delta(\phi_{r-1}) + \frac{1}{p_r}\Big) = n\Delta(\phi_r) + o(n)\,.$$

So $\#(\phi_r, n) \leqq n\Delta(\phi_r) + o(n)$. By Lemma 1.1.3 we have $\#(\phi, n) \leqq \#(\phi_r[g_r n])$. So $\#(\phi, n) \leqq [g_r n]\Delta(\phi_r) + o([g_r n]) = n\Delta(\phi_r) + o(n)$. Divide by n and let $n \rightarrow \infty$ to get $\overline{\lim}_{n\rightarrow\infty} \#(\phi, n)/n \leqq \lim_{r\rightarrow\infty} \Delta(\phi_r)$.

Finally $\underline{\Delta}(\phi) \geqq \lim_{k\rightarrow\infty} \Delta(\phi_k)$ because if we choose $k \in Z^+$ then for n large we have $\#(\phi, n) \geqq \#(\phi_k, n) \geqq n(\Delta(\phi_k) - 1/k)$ and so

$$\liminf_{n\rightarrow\infty} \#(\phi, n)/n \geqq \Delta(\phi_k) - 1/k$$

for each $k \in Z^+$. Thus $\Delta(\phi) = \lim_{r\rightarrow\infty} \Delta(\phi_r) = \prod_{p\in P}(1 + (1/p(p-1))$ and the theorem is proved.

A related result due to P. Erdös may be found in [4, pp. 211–213].

DEFINITION 1.2. For $t \geqq 1$, t a real number, a positive integer n is said to be t-abundant if $\sigma(n) \geqq tn$.

H. Davenport [3] has shown that for t as above, the sequence of t-abundant positive integers has a natural density.

THEOREM 1.2. *For each $k \in Z^+$ let d_k = the natural density of the k-abundant integers. Then* $\sum_{k=1}^{\infty} d_k \leqq \Delta(\phi) = (\zeta(2)\cdot\zeta(3)/\zeta(6))$.

Proof. It is known that $\phi(n)\sigma(n)/n^2 < 1$ for each integer $n > 1$

[7, p. 267]. So if $n \in](k-1)N, kN]$ and $\sigma(n) \geqq kn$ then $\phi(n) \leqq N$. Thus for $k \in Z^+$ and for N large, depending on k, we have

$$\begin{aligned} \#(\phi, N) &\geqq N + d_2(2N - N) + d_3(3N - 2N) + \cdots \\ &\quad + d_k(kN - (k-1)N) - \frac{N}{k} \\ &= N(1 + d_2 + d_3 + \cdots + d_k - 1/k) \\ &= N(d_1 + d_2 + \cdots + d_k - 1/k) . \end{aligned}$$

Now divide by N and let $N \to \infty$. We then have

$$\Delta(\phi) \geqq \lim_{k\to\infty} (d_1 + d_2 + \cdots + d_k - 1/k) = \sum_{k=1}^{\infty} d_k$$

and the theorem is proved.

2. General theorems. We begin this section by stating some results whose proofs are not difficult.

1. If $A = \{a_i\}_{i=1}^{\infty}$ is a sequence such that $\Delta(A) = \infty$ then there exists a sequence $\{i_j\}_{j=1}^{\infty}$ of positive integers with $\sum_{j=1}^{\infty} a_{i_j}/i_j < \infty$.
2. If $A = \{a_i\}_{i=1}^{\infty}$ is a sequence such that $\Delta(A) = 0$ then $\sum_{a_i \leq r} a_i = o(r^2)$ and $\sum_{a_i \leq r} 1/a_i = o(\log r)$.
3. If $A = \{a_i\}_{i=1}^{\infty}$ is a sequence such that $\infty > \Delta(A) > 0$ then $\sum_{a_i \leq r} a_i \sim \Delta(A) r^2/2$ and $\sum_{a_i \leq r} 1/a_i \sim \Delta(A) \log r$.

THEOREM 2.1. *Let $A = \{a_i\}_{i=1}^{\infty}$ be a sequence such that $\Delta(A) = \infty$. Then there exists a strictly increasing sequence $\{i_j\}_{j=1}^{\infty}$ of positive integers with $d(\{i_j\}_{j=1}^{\infty}) = 0$ and $\Delta(\{a_{i_j}\}_{j=1}^{\infty}) = \infty$.*

Proof. It suffices to assume $\lim_{i\to\infty} a_i = \infty$ because otherwise the proof is immediate.

Case I. $a_1 \leqq a_2 \leqq a_3 \leqq \cdots$.
First, there is no loss of generality in supposing $a_1 < a_2 < a_3 < \cdots$ because if $a_i = a_{i+1} = \cdots = a_{i+r-1} < a_{i+r}$ for some i then define

$$\varepsilon = \min \left(a_{i+r} - a_i, \begin{array}{c} \text{the distance from } a_i \text{ to the} \\ \text{smallest integer greater than } a_i \end{array} \right)$$

and replace a_{i+t} by $a_i + t\varepsilon/r$ for $t = 0, 1, \cdots, r-1$.

We now define a subsequence B of A by induction. Let $a_1 \in B$. If each of $a_1, a_2, \cdots, a_{k-1}$ has already been either included in B or excluded from B, place a_k in B if

$$\frac{\#(B, a_{k-1}) + 1}{a_k} \leqq \sqrt{\frac{\#(A, a_k)}{a_k}}$$

and exclude a_k from B if the inequality fails. It then follows that $\#(B, a_k)/a_k \sim \sqrt{\#(A, a_k)/a_k}$ and so $\Delta(B) = \infty$. Also if we write $B = \{a_{i_j}\}_{j=1}^\infty$ then we have $d(\{i_j\}_{j=1}^\infty) = 0$ because

$$\frac{n}{i_n} = \frac{\#(\{i_j\}_{j=1}^\infty, i_n)}{i_n} = \frac{\#(B, a_{i_n})}{\#(A, a_{i_n})} = \frac{a_{i_n}}{\#(A, a_{i_n})}\frac{\#(B, a_{i_n})}{a_{i_n}}$$
$$\sim \sqrt{\frac{a_{i_n}}{\#(A, a_{i_n})}}\left(\sqrt{\frac{a_{i_n}}{\#(A, a_{i_n})}}\frac{\#(B, a_{i_n})}{a_{i_n}}\right)$$

which tends to $0.1 = 0$ as $n \to \infty$.

Case II. We make no assumptions about the monotonicity of A. However, without loss of generality, we may still assume $a_i = a_j \Rightarrow i = j$, for we can always order A by size, deal with A as in Case I, and then apply the inverse of the permutation used to order A to the new sequence which is derived from A by use of the ε's.

Now order A by size and call this sequence $A^* = \{a_i^*\}_{i=1}^\infty$. We have $a_i^* < a_{i+1}^*$ for all $i \in Z^+$. It follows immediately that if any $n - 1$ elements are deleted from A the minimum of the remaining elements is $\leqq a_n^*$. It is also clear that if $A_1^* = \{a_{2i-1}\}_{i=1}^\infty$ then $\Delta(A_1^*) = \infty$.

Apply Case I to A^* to get a subsequence $B^* = \{a_{i_j}^*\}_{j=1}^\infty$ of A^* such that $\Delta(B^*) = \infty$ and $d(\{i_j\}_{j=1}^\infty) = 0$. Now define t_1 by $a_{t_1} = \min(\{a_{i_1}, a_{i_1+1}, a_{i_1+2}, \cdots\})$. It follows that $t_1 \geqq i_1$ and $a_{t_1} \leqq a_{i_1}^*$. Define t_2 by $a_{t_2} = \min(\{a_{i_2}, a_{i_2+1}, a_{i_2+2}, \cdots\}\backslash\{a_{t_1}\})$. It follows that $t_2 \geqq i_2$ and $a_{t_2} \leqq a_{i_2+1}^*$. In general define t_j by

$$a_{t_j} = \min(\{a_{i_j}, a_{i_j+1}, a_{i_j+2}, \cdots\}\backslash\{a_{t_1}, a_{t_2}, \cdots, a_{t_{j-1}}\}) .$$

It follows that $t_j \geqq i_j$ and $a_{t_j} \leqq a_{i_j+j-1}^*$.

Since $t_j \geqq i_j$ for all $j \in Z^+$, it follows that $d(\{t_j\}_{j=1}^\infty) = 0$. Also $\Delta(\{a_{ij}^*\}_{j=1}^\infty) = \infty$ so $\Delta(\{a_{i_{2j-1}}^*\}_{j=1}^\infty) = \infty$ so $\Delta(\{a_{i_j+j-1}^*\}_{j=1}^\infty) = \infty$. It then follows that $\Delta(\{a_{t_j}\}_{j=1}^\infty) = \infty$ and the theorem is proved.

To emphasize that care must be taken in the choice of $\{i_j\}_{j=1}^\infty$ in the above theorem we note the following result.

THEOREM 2.2. *Suppose* $\{i_j\}_{j=1}^\infty$ *is a sequence of positive integers such that* $d(\{i_j\}_{j=1}^\infty) = 0$. *Then there exists a strictly increasing sequence* $A = \{a_i\}_{i=1}^\infty$ *such that* $\lim_{i\to\infty} a_i = \infty$, $\Delta(A) = \infty$, *and* $\Delta(\{a_{i_j}\}_{j=1}^\infty) = 0$.

THEOREM 2.3. *For each number* α *such that* $0 \leqq \alpha \leqq \infty$ *there exist two sequences* A *and* B *such that* $\Delta(A) = \Delta(B) = 0$ *and* $\Delta(A + B) = \alpha$.

Proof. If $\alpha = 0$ choose $A = B$ to be the sequence of factorials.

If $\alpha = \infty$ choose $A = B = P$. Then by the Prime Number Theorem $\Delta(A + B) = \infty$.

Suppose $0 < \alpha < \infty$. Choose β and $\gamma \in R^+$ so that $(1/4)\pi\beta\gamma = \alpha$. Let $A = \{n^2/\beta^2\}_{n=1}^{\infty}$ and $B = \{n^2/\gamma^2\}_{n=1}^{\infty}$. Clearly $\Delta(A) = 0 = \Delta(B)$. Also, the number of elements in $A + B$ which are $\leqq n$ is the number of lattice points (k, m) in the positive quadrant of the ellipse

$$k^2/\beta^2 + m^2/\gamma^2 \leqq n .$$

This number is $(1/4)\pi\beta\gamma n + O(\sqrt{n})$. Thus $\Delta(A + B) = (1/4)\pi\beta\gamma = \alpha$ and the theorem is proved.

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