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REARRANGEMENT INEQUALITIES INVOLVING CONVEX FUNCTIONS

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Let $a = (a_1, \cdots, a_n)$ and $b = (b_1, \cdots, b_n)$ be n-tuples of non-negative numbers. Then

$$(1) \qquad \prod_{i=1}^{n} (a'_i+b'_i) \leqq \prod_{i=1}^{n} (a_i+b'_i) \leqq \prod_{i=1}^{n} (a^*_i+b'_i)$$

and

$$(2) \qquad \sum_{i=1}^{n} a^*_i b'_i \leqq \sum_{i=1}^{n} a_i b'_i \leqq \sum_{i=1}^{n} a'_i b'_i \, .$$

$a' = (a'_i, \cdots, a'_n)$ **and** $a^* = (a^*_1, \cdots, a^*_n)$ **are respectively the rearrangement of a in a nondecreasing or nonincreasing order. (1) was recently found by Minc and (2) is well known. In this note we show that these inequalities are special cases of rearrangement inequalities valid for functions having some convex properties.**

Let $x = (x_1, \cdots, x_n)$ be an n-tuple of real numbers. We denote by $x^* = (x^*_1, \cdots, x^*_n)$ the n-tuple x rearranged in a nonincreasing order $x^*_1 \geqq x^*_2 \geqq \cdots \geqq x^*_n$, and we denote by $x' = (x'_1, \cdots, x'_n)$ the same n-tuple rearranged in a nondecreasing order $x'_1 \leqq x'_2 \leqq \cdots \leqq x'_n$.

Recently Minc [2] proved that if $a = (a_1, \cdots, a_n)$ and $b = (b_1, \cdots, b_n)$ are real n-tuples such that $a_i, b_i \geqq 0$, $i = 1, \cdots, n$, then

$$(1) \qquad \prod_{i=1}^{n} (a'_i + b'_i) \leqq \prod_{i=1}^{n} (a_i + b'_i) \leqq \prod_{i=1}^{n} (a^*_i + b'_i) \, .$$

If $a_i > 0$ and $b_i \geqq 0$, $i = 1, \cdots, n$, then (1) is equivalent to

$$(1)' \qquad \sum_{i=1}^{n} \log\left(1 + \frac{b'_i}{a'_i}\right) \leqq \sum_{i=1}^{n} \log\left(1 + \frac{b'_i}{a_i}\right) \leqq \sum_{i=1}^{n} \log\left(1 + \frac{b'_i}{a^*_i}\right) .$$

(see also [4, Theorem 2] and [5]).

It is well known [1, Th. 368] that if $a = (a_1, \cdots, a_n)$ and $b = (b_1, \cdots, b_n)$ are real n-tuples, then

$$(2) \qquad \sum_{i=1}^{n} a^*_i b'_i \leqq \sum_{i=1}^{n} a_i b'_i \leqq \sum_{i=1}^{n} a'_i b'_i \, .$$

If $a_i > 0$ and $b_i \geqq 0$, $i = 1, \cdots, n$, then (2) is obviously equivalent to

$$(2)' \qquad \sum_{i=1}^{n} \left(\frac{b'_i}{a'_i}\right) \leqq \sum_{i=1}^{n} \left(\frac{b'_i}{a_i}\right) \leqq \sum_{i=1}^{n} \left(\frac{b'_i}{a^*_i}\right) .$$

In the present note we generalize (1)′ and (2)′ for more general functions. An inequality analogue to $(1)'$ is proved for functions $f(x)$ such that $f(e^x)$ is convex (Theorem 1), and an inequality analogue to $(2)'$ is proved for convex functions $f(x)$ (Theorem 2).

In our proof we use the following theorem of Mirsky [3]: Given two n-tuples $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$ such that $x_i \geqq 0$ and $y_i \geqq 0$, $i = 1, \cdots, n$. If

$$\sum_{i=1}^{k} y_i^* \leqq \sum_{i=1}^{k} x_i^* , \qquad k = 1, \cdots, n ,$$

then y lies in the convex hull of the set of vectors $(\delta_1 x_{\tau(1)}, \cdots, \delta_n x_{\tau(n)})$, where each δ_i takes the values 0 or 1 and τ ranges over all permutations of $(1, \cdots, n)$.

2. Two rearrangement inequalities.

THEOREM 1. *Let $a = (a_1, \cdots, a_n)$ and $b = (b_1, \cdots, b_n)$ be n-tuples satisfying $a_i > 0$ and $b_i \geqq 0$, $i = 1, \cdots, n$. Let $f(x)$ be a real valued function defined for $x \geqq 1$ such that $F(x) = f(e^x)$ is convex for $x \geqq 0$ and $f(1) \leqq f(x)$ for $x \geqq 1$. Then*

$$\sum_{i=1}^{n} f\left(1 + \frac{b_i'}{a_i'}\right) \leqq \sum_{i=1}^{n} f\left(1 + \frac{b_i'}{a_i}\right) \leqq \sum_{i=1}^{n} f\left(1 + \frac{b_i'}{a_i^*}\right) . \tag{3}$$

If $F(x)$ is strictly convex, then equality in the right inequality of (3) *holds if and only if $b'/a^* = (b_1'/a_1^*, \cdots, b_n'/a_n^*)$ is a rearrangement of $b'/a = (b_1'/a_1, \cdots, b_n'/b_n)$, and equality in the left inequality of* (3) *holds if and only if $b'/a' = (b_1'/a_1', \cdots, b_n'/a_n')$ is a rearrangement of b'/a.*

Proof. We first prove the theorem for $n = 2$. In this case the theorem becomes: Let $0 < a_1 \leqq a_2$ and $0 \leqq b_1 \leqq b_2$. Then

$$f\left(1 + \frac{b_1}{a_1}\right) + f\left(1 + \frac{b_2}{a_2}\right) \leqq f\left(1 + \frac{b_1}{a_2}\right) + f\left(1 + \frac{b_2}{a_1}\right) . \tag{4}$$

If $F(x)$ is strictly convex, then equality in (4) holds if and only if $a_1 = a_2$ or $b_1 = b_2$.

Denote

$$1 + \frac{b_1}{a_1} = u_1, \ \ 1 + \frac{b_2}{a_2} = u_2, 1 + \frac{b_2}{a_1} = v_1, \ 1 + \frac{b_1}{a_2} = v_2 .$$

We have,

$$1 \leqq u_1 \leqq v_1, \ \ 1 \leqq u_2 \leqq v_1 . \tag{5}$$

By (1) for $n = 2$, or directly, we obtain

$$\begin{aligned} u_1u_2 &= \left(1+\frac{b_1}{a_1}\right)\left(1+\frac{b_2}{a_2}\right) = \frac{(a_1+b_1)(a_2+b_2)}{a_1a_2} \\ &\leqq \frac{(a_1+b_2)(a_2+b_1)}{a_1a_2} = \left(1+\frac{b_2}{a_1}\right)\left(1+\frac{b_1}{a_2}\right) = v_1v_2 \,. \end{aligned} \tag{6}$$

Denote

$$\log u_i = \tilde{u}_i,\ \log v_i = \tilde{v}_i \,, \qquad i = 1, 2 \,. \tag{7}$$

From (5), (6) and (7) it follows that

$$\begin{cases} \tilde{u}_1 \leqq \tilde{v}_1, & \tilde{u}_2 \leqq \tilde{v}_1 \,, \\ \tilde{u}_1 + \tilde{u}_2 \leqq \tilde{v}_1 + \tilde{v}_2 \,. \end{cases} \tag{8}$$

By the theorem of Mirsky stated above, it follows from (8) that $\tilde{u} = (\tilde{u}_1, \tilde{u}_2)$ lies in the convex hull of the set of vectors $(\delta_1\tilde{v}_{\tau(1)}, \delta_2\tilde{v}_{\tau(2)})$, where δ_1 and δ_2 take the values 0 or 1 and τ is a permutation of $(1, 2)$. As $F(x) = f(e^x)$ is convex for $x \geqq 0$, $F(x_1) + F(x_2)$ is convex in the quadrant $x_1 \geqq 0$, $x_2 \geqq 0$ and thus obtains its maximum in the above convex hull on one of its vertices. Hence,

$$\begin{aligned} &f\left(1+\frac{b_1}{a_1}\right) + f\left(1+\frac{b_2}{a_2}\right) = f(u_1) + f(u_2) = F(\tilde{u}_1) + F(\tilde{u}_2) \\ &\leqq \max\{F(\delta_1\tilde{v}_{\tau(1)}) + F(\delta_2\tilde{v}_{\tau(2)})\} \leqq F(\tilde{v}_1) + F(\tilde{v}_2) \\ &= f(v_1) + f(v_2) = f\left(1+\frac{b_1}{a_2}\right) + f\left(1+\frac{b_2}{a_1}\right) \,. \end{aligned}$$

Here we used the fact that $F(0) \leqq F(x)$ for $x \geqq 0$. (4) is thus proved.

It is obvious that if $a_1 = a_2$ or $b_1 = b_2$ then equality holds in (4). We have to show that if $F(x)$ is strictly convex and if

$$0 < a_1 < a_2 \quad \text{and} \quad 0 \leqq b_1 < b_2 \tag{9}$$

then the inequality in (4) is strict. As $F(x)$ is strictly convex, it is enough to show that if (9) holds then $\tilde{u}$ does not coincide with one of the vertices $(\delta_1\tilde{v}_{\tau(1)}, \delta_2\tilde{v}_{\tau(2)})$. From (9) follows $\tilde{u}_1 < \tilde{v}_1$ and $\tilde{u}_2 < \tilde{v}_1$. Therefore if $\tilde{u} = (\tilde{u}_1, \tilde{u}_2)$ is a vertex, then $\tilde{u}_1 = 0$ or $\tilde{u}_2 = 0$. But $b_2 > 0$. Hence, $\tilde{u}_1 = 0$ and $(\tilde{u}_1, \tilde{u}_2)$ coincides with the vertex $(0, \tilde{v}_2)$. But from $\tilde{u}_2 = \tilde{v}_2$ it follows that $b_1 = b_2$, which contradicts (9).

The theorem for $n \geqq 3$ follows now by induction on n as in [2].

We prove the right inequality of (3) together with its equality statement.

If $a_1 = a_1^*$ then the result, including the equality statement, follows by the induction.

Assume now that $a_1 = a_k^*$ and $a_l = a_1^*$, where $k, l \neq 1$. Using the proved result for $n = 2$ and the induction hypothesis for $n - 1$, we obtain

$$\begin{aligned}
\sum_{i=1}^{n} f\Big(1 + \frac{b'_i}{a_i}\Big) &= f\Big(1 + \frac{b'_1}{a^*_k}\Big) + f\Big(1 + \frac{b'_l}{a^*_1}\Big) + \sum_{\substack{i=2 \\ i \neq l}}^{n} f\Big(1 + \frac{b'_i}{a_i}\Big) \\
&\leqq f\Big(1 + \frac{b'_1}{a^*_1}\Big) + \Big\{f\Big(1 + \frac{b'_l}{a^*_k}\Big) + \sum_{\substack{i=2 \\ i \neq l}}^{n} f\Big(1 + \frac{b'_i}{a_i}\Big)\Big\} \\
&\leqq f\Big(1 + \frac{b'_1}{a^*_1}\Big) + \sum_{i=2}^{n} f\Big(1 + \frac{b'_i}{a^*_i}\Big) = \sum_{i=1}^{n} f\Big(1 + \frac{b'_i}{a^*_i}\Big) ,
\end{aligned} \tag{10}$$

and the right inequality of (3) is proved.

If equality holds in the right inequality of (3), then equality holds in all the inequalities of (10). Hence, using the proved equality statement for $n = 2$ and the induction hypothesis for $n - 1$, it follows that

$$a^*_1 = a^*_k = a_1 = a_l \tag{11}$$

or

$$b'_1 = b'_l \tag{12}$$

holds, and

$$\begin{aligned}
&\Big(\frac{b'_2}{a^*_2}, \cdots, \frac{b'_n}{a^*_n}\Big) \text{ is a rearrangement of} \\
&\qquad\qquad \Big(\frac{b'_2}{a_2}, \cdots, \frac{b'_{l-1}}{a_{l-1}}, \frac{b'_l}{a_1}, \frac{b'_{l+1}}{a_{l+1}}, \cdots, \frac{b'_n}{a_n}\Big) .
\end{aligned} \tag{13}$$

Combining (11) or (12) with (13), it follows that b'/a^* is a rearrangement of b'/a, and the proof of the right inequality is completed.

The proof of the left inequality is similar.

For $f(x) = \log x$, (3) reduces to (1)$'$. We note that although $F(x) = x$ is not strictly convex, the statement of equality appearing in (3) holds true for this case too. This follows from the fact that the general equality statement for $n \geqq 3$ was derived only from its validity for $n = 2$, and for $f(x) = \log x$ it is easy to check directly that it holds for $n = 2$.

THEOREM 2. *Let $a = (a_1, \cdots, a_n)$ and $b = (b_1, \cdots, b_n)$ be n-tuples satisfying $a_i > 0$ and $b_i \geqq 0$, $i = 1, \cdots, n$. Let $f(x)$ be a real valued function defined and convex for $x \geqq 0$ and satisfying $f(0) \leqq f(x)$ for $x \geqq 0$. Then*

$$\sum_{i=1}^{n} f\Big(\frac{b'_i}{a'_i}\Big) \leqq \sum_{i=1}^{n} f\Big(\frac{b'_i}{a_i}\Big) \leqq \sum_{i=1}^{n} f\Big(\frac{b'_i}{a^*_i}\Big) . \tag{14}$$

If $f(x)$ is strictly convex, then the same equality statement as in Theorem 1 holds.

Proof. For $n = 2$, (14) becomes: Let $0 < a_1 \leqq a_2$ and $0 \leqq b_1 \leqq b_2$. Then

$$f\left(\frac{b_1}{a_1}\right) + f\left(\frac{b_2}{a_2}\right) \leqq f\left(\frac{b_1}{a_2}\right) + f\left(\frac{b_2}{a_1}\right). \tag{15}$$

As before, we first prove the theorem for $n = 2$. Denote

$$\frac{b_1}{a_1} = x_1, \quad \frac{b_2}{a_2} = x_2, \quad \frac{b_2}{a_1} = y_1, \quad \frac{b_1}{a_2} = y_2 .$$

Using (2) for $n = 2$, we obtain

$$\begin{cases} x_1 \leqq y_1, \quad x_2 \leqq y_1 , \\ x_1 + x_2 \leqq y_1 + y_2 . \end{cases} \tag{16}$$

From (16) it follows that $x = (x_1, x_2)$ lies in the convex hull of the set of vectors $(\delta_1 y_{\tau(1)}, \delta_2 y_{\tau(2)})$.

From here on the proof proceeds very similar to the proof of Theorem 1, and we omit the details.

For $f(x) = x$, (14) reduces to (2)′. The equality statement of Theorem 1 holds, as before, also in this case, although $f(x)$ is not strictly convex.

We bring an additional example. The function $f(x) = x \log (x + 1)$ is strictly convex for $x \geqq 0$ and satisfies $f(0) \leqq f(x)$. Hence, applying Theorem 2, we obtain

$$\begin{aligned} \sum_{i=1}^{n} \left(\frac{b_i'}{a_i'}\right) \log\left(1 + \frac{b_i'}{a_i'}\right) &\leqq \sum_{i=1}^{n} \left(\frac{b_i'}{a_i}\right) \log\left(\frac{b_i'}{a_i} + 1\right) \\ &\leqq \sum_{i=1}^{n} \left(\frac{b_i'}{a_i^*}\right) \log\left(\frac{b_i'}{a_i^*} + 1\right) \end{aligned} \tag{17}$$

or

$$\prod_{i=1}^{n} \left(\frac{b_i'}{a_i'} + 1\right)^{b_i'/a_i'} \leqq \prod_{i=1}^{n} \left(\frac{b_i'}{a_i} + 1\right)^{b_i'/a_i} \leqq \prod_{i=1}^{n} \left(\frac{b_i'}{a_i^*} + 1\right)^{b_i'/a_i^*} . \tag{17$'$}$$

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