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ON A SET OF POLYNOMIALS SUGGESTED BY LAGUERRE POLYNOMIALS

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Generating functions, integrals and recurrence relations are obtained for the polynomials $Z_n^\alpha(x; k)$ in x^k which form one set of the biorthogonal pair with respect to the weight function $e^{-x}x^\alpha$ over the interval $(0, \infty)$, the other set being that of polynomials in x.

A singular integral equation with $Z_n^\alpha(x;k)$ in the kernel is solved in terms of a generalized Mittag-Leffler's function and a unified formula for fractional integration and differentiation of the polynomials is derived.

It is known [7] that the polynomials $Z_n^\alpha(x; k)$ of degree n in x^k for positive integers k and Re $\alpha > -1$ are characterized up to a multiplicative constant by the above requirements. Konhauser [8] discussed the biorthogonality of the pair $\{Z_n^\alpha(x; k)\}$, $\{Y_n^\alpha(x; k)\}$ in the basic polynomials x^k and x, over the interval $(0, \infty)$ and with the admissible weight function $e^{-x}x^\alpha$ of the generalized Laguerre polynomial set $\{L_n^\alpha(x)\}$. Indeed the polynomials have several properties of interest and Konhauser [8] obtained among other things some recurrence relations and a differential equation for the polynomials $Z_n^\alpha(x; k)$ which are our primary concern in this paper. For $k = 2$, Preiser [11] obtained for these polynomials a generating function, a differential equation, integral representations and recurrence relations. Earlier Spencer and Fano [13] also used these polynomials for $k = 2$.

For $k = 1$, all the results proved in this paper reduce to those for $L_n^\alpha(x)$; in particular the integral equation (3.1) either reduces to or contains as still more special cases the integral equations solved by Widder [14], Buschman [1], Khandekar [6], Rusia [12] and Prabhakar ([10], $(7 \cdot 1)$). For $k = 2$, the results are essentially the same as those in [11] or [13].

2. Some properties of $Z_n^\alpha(x; k)$. We now obtain a generating function, a contour integral representation and a fractional integration formula for $Z_n^\alpha(x; k)$. In § 3, we need the Laplace transform and in § 4 derive a more general class of generating functions for the polynomials. Recurrence relations and a few other results will follow as natural consequences. We shall freely use the closed form ([8], (5))

$$(2.1) \qquad Z_n^\alpha(x; k) = \frac{\Gamma(kn + \alpha + 1)}{n!} \sum_{j=0}^{\infty} (-1)^j \binom{n}{j} \frac{x^{kj}}{\Gamma(kj + \alpha + 1)}$$

for Re $\alpha > -1$; naturally the results may be established from alternative characterizations of $Z_n^\alpha(x;k)$ but such a discussion does not seem to be of sufficient interest.

(i) *A generating function.* We obtain the generating function indicated in

$$e^t \phi(k, \alpha+1; -x^k t) = \sum_{n=0}^{\infty} \frac{Z_n^\alpha(x;k)\, t^n}{\Gamma(kn+\alpha+1)} \tag{2.2}$$

where $\phi(a, b; z)$ is the Bessel-Maitland function ([15], (1.3); [3], 18.1 (27)).

From (2.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{Z_n^\alpha(x;k)\, t^n}{\Gamma(kn+\alpha+1)} &= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{(-1)^m x^{km} t^n}{m!\,(n-m)!\,\Gamma(km+\alpha+1)} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m x^{km} t^{n+m}}{m!\, n!\, \Gamma(km+\alpha+1)} \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{m=0}^{\infty} \frac{(-x^k t)^m}{m!\, \Gamma(km+\alpha+1)} \\ &= e^t \phi(k, \alpha+1; -x^k t)\end{aligned}$$

and (2.2) is established.

Denoting $e^t \phi(k, \alpha+1; -x^k t)$ by $f(x,t)$ we at once find that $f(x,t)$ satisfies the partial differential equation

$$x \frac{\partial f}{\partial x} - \alpha t \frac{\partial f}{\partial t} + \alpha t f = 0 .$$

Substituting for $f(x,t)$ from (2.2) and equating the coefficients of t^n, we obtain the differential recurrence relation

$$x Z_n'^{\alpha}(x;k) = n k Z_n^\alpha(x;k) - k \frac{\Gamma(kn+\alpha+1)}{\Gamma(kn+\alpha-k+1)} Z_{n-1}^\alpha(x;k) ,$$

also obtained by Konhauser ([8], (6)) by direct calculations.

(ii) *Schläfli's Contour integral.* It is easy to show that

$$Z_n^\alpha(x;k) = \frac{\Gamma(kn+\alpha+1)}{n!\;\; 2\pi i} \int_{-\infty}^{(0+)} \frac{(t^k - x^k)^n e^t\, dt}{t^{kn+\alpha+1}} . \tag{2.3}$$

For $\displaystyle \frac{1}{2\pi i} \int_{-\infty}^{(0+)} \frac{(t^k - x^k)^n e^t\, dt}{t^{kn+\alpha+1}} = \sum_{j=0}^{n} (-1)^j \binom{n}{j} x^{kj} \frac{1}{2\pi i} \int_{-\infty}^{(0+)} t^{-(kj+\alpha+1)} e^t\, dt$

$$= \sum_{j=0}^{n} \frac{(-1)^j \binom{n}{j} x^{kj}}{\Gamma(kj+\alpha+1)}$$

using Hankel's formula ([3], 1.6(2))

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int_{-\infty}^{(0+)} e^t\, t^{-z}\, dt\,; \tag{2.4}$$

finally (2.3) follows from (2.1)

For $k = 1$, (2.3) reduces to the known result ([2], p. 269)

$$L_n^\alpha(x) = \frac{\Gamma(n+\alpha+1)}{n!\; 2\pi i}\int_{-\infty}^{(0+)} \left(1 - \frac{x}{t}\right)^n e^t\, \frac{dt}{t^{\alpha+1}}\,.$$

If α is also a positive integer than the integrand in (2.3) is a single-valued analytic function of t with the only singularity $t = 0$. Hence we can deform the contour into $|t| = b\,|x|$ and the substitution $t = x\,u$ then leads to

$$x^\alpha\, Z_n^\alpha(x\,; k) = \frac{\Gamma(kn+\alpha+1)}{n!\; 2\pi i}\int_C (u^k\text{-}1)^n\, e^{xu}\, u^{-(kn+\alpha+1)}\, du \tag{2.5}$$

where C denotes the circle $|u| = b$. Indeed C may be replaced by any simple closed contour surrounding the point $u = 0$. For $k = 2$, (2.5) reduces to the integral representation by Preiser ([11], (5.22)).

Using (2.5), it follows that

$$\frac{\partial^k}{\partial x^k}\left[\frac{n!\, x^{\alpha+k}\, Z_n^{\alpha+k}(x\,;k)}{\Gamma(kn+k+\alpha+1)}\right] = \frac{n!\; x^\alpha\, Z_n^\alpha(x\,;k)}{\Gamma(kn+\alpha+1)}$$

and
$$\left(\frac{\partial^k}{\partial x^k} - 1\right)\left[\frac{n!\, x^{\alpha+k}\, Z_n^{\alpha+k}(x\,;k)}{\Gamma(kn+k+\alpha+1)}\right] = \frac{(n+1)!\; x^\alpha\, Z_{n+1}^\alpha(x\,;k)}{\Gamma(kn+k+\alpha+1)}$$

which leads to the pure recurrence relation

$$x^k\, Z_n^{\alpha+k}(x\,;k) = (kn+\alpha+1)_k\, Z_n^\alpha(x\,;k) - (n+1)\, Z_{n+1}^\alpha(x\,;k)\,. \tag{2.6}$$

For $k = 2$, (2.6) reduces to ([11], (5.39)).

(iii) *Fractional integrals and derivatives.* We show that

$$I^\mu\,[x^\alpha\, Z_n^\alpha(x\,;k)] = \frac{\Gamma(kn+\alpha+1)}{\Gamma(kn+\alpha+\mu+1)}\, x^{\alpha+\mu}\, Z_n^{\alpha+\mu}(x\,;k) \tag{2.7}$$

for $\operatorname{Re}\alpha > -1$ and $\operatorname{Re}\mu > -\operatorname{Re}(1+\alpha)$ where for suitable f and complex μ, $I^\mu f(x)$ denotes the μth order fractional integral (or fractional derivative) of $f(x)$ (see [10], § 2).

When $\operatorname{Re}\mu > 0$, we write [10]

$$\begin{aligned} I^\mu\,[x^\alpha Z_n^\alpha(x\,;k)] &= \int_0^x \frac{(x-t)^{\mu-1}}{\Gamma(\mu)}\, t^\alpha\, Z_n^\alpha(t\,;k)\, dt \\ &= \frac{\Gamma(kn+\alpha+1)}{n!\;\Gamma(\mu)}\sum_{j=0}^{n}\frac{(-n)_j}{\Gamma(kj+\alpha+1)}\int_0^x t^{kj+\alpha}(x-t)^{\mu-1}\, dt \end{aligned}$$

$$= \frac{\Gamma(kn+\alpha+1)}{n\,!} \sum_{j=0}^{n} (-n)_j \frac{x^{kj+\alpha+\mu}}{\Gamma(kj+\alpha+\mu+1)} \, ;$$

hence for $\mathrm{Re}\,\mu > 0$ and $\mathrm{Re}\,\alpha > -1$, we obtain

$$(2.8) \qquad I^{\mu}[x^{\alpha} Z_n^{\alpha}(x\,;\mathrm{k})] = \frac{\Gamma(kn+\alpha+1)}{\Gamma(kn+\alpha+\mu+1)} x^{\alpha+\mu} Z_n^{\alpha+\mu}(x\,;k) \, .$$

But (2.8) may be written as

$$(2.9) \qquad x^{\alpha} Z_n^{\alpha}(x\,;k) = \frac{\Gamma(kn+\alpha+1)}{\Gamma(kn+\alpha+\mu+1)} I^{-\mu}[x^{\alpha+\mu} Z_n^{\alpha+\mu}(x;k)] \, ,$$

the inversion being valid for $\mathrm{Re}\,\mu > 0$ and the assumptions made.

Putting $\mu' = -\mu$, $\alpha' = \alpha + \mu$, we obtain for $\mathrm{Re}\,\mu' < 0$

$$x^{\alpha'+\mu'} Z_n^{\alpha'+\mu'}(x\,;k) = \frac{\Gamma(kn+\alpha'+\mu'+1)}{\Gamma(kn+\alpha'+1)} I^{\mu'}[x^{\alpha'} Z_n^{\alpha'}(x;k)]$$

which is (4.1) with the letters α, μ accented. Ignoring the accents we can write

$$(2.10) \qquad I^{\mu}[x^{\alpha} Z_n^{\alpha}(x\,;k)] = \frac{\Gamma(kn+\alpha+1)}{\Gamma(kn+\alpha+\mu+1)} x^{\alpha+\mu} Z_n^{\alpha+\mu}(x\,;k)$$

for $\mathrm{Re}\,\mu < 0$, $\mathrm{Re}\,\alpha > -1$ and $\mathrm{Re}\,(\alpha+\mu) > -1$.

When $\mathrm{Re}\,\mu = 0$, we write $I^{\mu} = I^{\mu+1} I^{-1}$ and the result easily follows; thus (2.7) is established for all complex μ with $\mathrm{Re}\,\mu > -\mathrm{Re}\,(1+\alpha)$.

REMARK 1. When μ is a negative integer say $-m$, then (2.7) is written as

$$\left(\frac{d}{dx}\right)^{m} [x^{\alpha} Z_n^{\alpha}(x\,;k)] = \frac{\Gamma(kn+\alpha+1)}{\Gamma(kn+\alpha-m+1)} x^{\alpha-m} Z_n^{\alpha-m}(x\,;k)$$

which can also be proved by direct differentiation provided $\mathrm{Re}\,\alpha > m-1$.

REMARK 2. For $k=1$, (2.7) unifies the results ([3], 10.12(27)) and ([4], 13.1(49)) for Laguerre polynomials.

3. A singular integral equation. We show that the convolution equation

$$(3.1) \qquad \int_0^x (x-t)^{\alpha} Z_n^{\alpha}(\lambda(x-t)\,;k)\, f(t)\, dt = g(x)$$

for $\mathrm{Re}\,\alpha > -1$ admits a locally integrable solution f given by

$$(3.2)\quad f(x)=\frac{n!}{\Gamma(kn+\alpha+1)}\int_0^x (x-t)^{l-\alpha-2}E^{n}_{k,l-\alpha-1}(\lambda(x-t))^k I^{-l}g(t)\,dt$$

provided $I^{-l}g$ exists for $\operatorname{Re} l > \operatorname{Re}\alpha+1$ and is locally integrable in $(0,\delta)$, $0<x<\delta<\infty$.

The function

$$(3.3)\qquad E^{c}_{a,b}(z)=\sum_{j=0}^{\infty}\frac{(c)_j z^j}{\Gamma(aj+b)\,j!}\qquad \operatorname{Re} a>0$$

is a very special case of the generalized hypergeometric functions considered by Wright [16] and is also expressible as a Fox's H-function [5]. On the other hand $E^{c}_{a,b}(z)$ is a most natural generalization of the Mittag-Leffler's function $E_a(z)$ ([3], 18.1; [9]) and also contains the confluent hypergeometric function ${}_1F_1(c;d;z)$ ([3], ch.VI), the Wiman's function $E_{a,b}(z)$ ([3], 18.1(19)) and several other fuctions as special cases. It is an entire function of order $(\operatorname{Re} a)^{-1}$ and indeed has a number of properties which may be of independent interest. A fact of immediate interest to us is that the polynomials $Z^{\alpha}_n(x;k)$ bear to $E^{c}_{a,b}(x)$ a relation which is analogous to that which the Laguerre polynomials $L^{\alpha}_n(x)$ bear to the confluent hypergeometric function ${}_1F_1$; evidently

$$(3.4)\qquad Z^{\alpha}_n(x;k)=\frac{\Gamma(kn+\alpha+1)}{n!}E^{-n}_{k,\alpha+1}(x^k)\,.$$

As usual let

$$(3.5)\qquad L[f(t)]=\hat{f}(p)=\int_0^{\infty}e^{-pt}f(t)\,dt\qquad \operatorname{Re} p>0$$

denote the Laplace transform of f. Then it is easily verified that for $\operatorname{Re}\lambda$, $\operatorname{Re} p>0$,

$$(3.6)\qquad L[t^{b-1}E^{c}_{a,b}(\lambda t)^a]=p^{-b+ac}(p^a-\lambda^a)^{-c}\qquad \operatorname{Re} b>0\,,$$

$$(3.7)\qquad L[t^{\alpha}Z^{\alpha}_n(\lambda t;k)]=\frac{\Gamma(kn+\alpha+1)}{n!\;p^{kn+\alpha+1}}(p^k-\lambda^k)^n\,,\qquad \operatorname{Re}\alpha>-1\,.$$

We next note a general result on the Laplace transform of the r-times repeated indefinite integral as well as the rth order derivative of a function; in fact, we observe that

$$(3.8)\qquad p^{\mu}\hat{f}(p)=L[I^{-\mu}f(t)]$$

for suitable f, complex μ and p with $\operatorname{Re} p>0$. Evidently both ([4], 4.1(8)) and ([4], 4.1(9)) are included in (3.8) as special cases.

We are now prepared to solve (3.1). From (3.1), (3.4) and using ([4], 4.1(20)), we have

$$(3.9)\qquad \frac{\Gamma(kn+\alpha+1)}{n!}(p^k-\lambda^k)^n\,p^{-kn-\alpha-1}\hat{f}(p)=\hat{g}(p)\,.$$

For $\operatorname{Re} l > \operatorname{Re}(\alpha+1)$, (3.9) can be written (compare with [1]) as

$$(3.10)\qquad \hat{f}(p)=\frac{n!}{\Gamma(kn+\alpha+1)}\{(p^k-\lambda^k)^{-n}\,p^{-l+kn+\alpha+1}\}\{p^l\,\hat{g}(p)\}$$

and we finally get

$$f(x)=\frac{n!}{\Gamma(kn+\alpha+1)}\int_0^x (x-t)^{l-\alpha-2}E^n_{k,l-\alpha-1}(\lambda(x-t))^k\,I^{-l}g(t)\,dt$$

using ([4], 4.1(20)), (3.6) and (3.8).

4. A general class of generating functions. For arbitrary λ, we prove the generating relation

$$(4.1)\qquad (1-t)^{-\lambda}E^{\lambda}_{k,\alpha+1}\left(\frac{-x^k t}{1-t}\right)=\sum_{n=0}^{\infty}\frac{(\lambda)_n Z^{\alpha}_n(x;k)\,t^n}{\Gamma(kn+\alpha+1)}\,.$$

From (2.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(\lambda)_n Z^{\alpha}_n(x;k)\,t^n}{\Gamma(kn+\alpha+1)}&=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(-1)^m(\lambda)_n\,x^{km}\,t^n}{\Gamma(km+\alpha+1)\,(n-m)!\,m!}\\&=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m(\lambda)_{n+m}\,x^{km}\,t^{n+m}}{\Gamma(km+\alpha+1)n!\,m!}\\&=\sum_{m=0}^{\infty}\frac{(\lambda)_m(-x^k t)^m}{\Gamma(km+\alpha+1)\,m!}\sum_{n=0}^{\infty}\frac{(\lambda+m)_n\,t^n}{n!}\\&=(1-t)^{-\lambda}E^{\lambda}_{k,\alpha+1}\left(\frac{-x^k t}{1-t}\right).\end{aligned}$$

For $k=1$, $\lambda=1+\alpha$, (4.1) yields the well-known generating function ([3], 10.12(7)) for the Laguerre polynomials.

From (4.1) we obtain, on applying Taylor's theorem

$$(4.2)\qquad \frac{(\lambda)_n Z^{\alpha}_n(x;k)}{\Gamma(kn+\alpha+1)}=\frac{1}{2\pi i}\int_C (1-t)^{-\lambda}E^{\lambda}_{k,\alpha+1}\left(\frac{-x^k t}{1-t}\right)t^{-n-1}\,dt\,,$$

C being a closed contour surrounding $t=0$ and lying within the disk $|t|<1$. Putting $u=x^k/1-t$,

$$(4.3)\quad x^{k\lambda-k}Z^{\alpha}_n(x;k)=\frac{\Gamma(kn+\alpha+1)}{(\lambda)_n 2\pi i}\int_{C'}\frac{u^{n+\lambda-1}E^{\lambda}_{k,\alpha+1}(x^k-u)\,du}{(u-x^k)^{n+1}}$$

where C' is a circle $|u-x^k|=\rho$ of small radius ρ.

Choosing $\lambda=1$, we have in terms of Wiman's function

$$(4.4) \qquad Z_n^\alpha(x\,;\,k) = \frac{\Gamma(kn+\alpha+1)}{n!\;2\pi i}\int_{C'} \frac{u^n\,E_{k,\,\alpha+1}(x^k-u)\,du}{(u-x^k)^{n+1}}\,.$$

Also evaluating the integral (4.3) by the Cauchy's residue theorem, we obtain for arbitrary λ with $\operatorname{Re}\lambda > 0$,

$$(4.5) \quad Z_n^\alpha(x\,;\,k) = \frac{\Gamma(kn+\alpha+1)}{(\lambda)_n\,n!}\,x^{k-k\lambda}\,\frac{\partial^n}{\partial u^n}\,[u^{\lambda+n-1}\,E_{k,\,\alpha+1}^{\lambda}(x^k-u)]_{u=x^k}\,.$$

Since $E_{1,b}^{b}(z) = (1/\Gamma(b))e^z$, for $k = 1$ and $\lambda = \alpha+1$, (4.5) reduces to the Rodrigues for the Laguerre polynomials.

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