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DERIVATIONS AND ACTIONS

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This paper continues the study of functions which act in a Banach algebra containing compact operators on a Hilbert space. In particular, a strong action of a function f in $\mathfrak{A}$ is considered, that is, an action $T \to f(T)$ from $\mathfrak{A}_U = \{T \in \mathfrak{A}: \sigma(T) \subset U\}$ into $\mathfrak{A}$ such that $f(T)$ commutes with all operators of finite rank which commute with T. Let $E_\lambda(T)$ be the projection of the Hilbert space onto $\ker(\lambda I - T)^{\nu(\lambda)}$ such that $\ker E_\lambda(T) = (\lambda I - T)^{\nu(\lambda)} H$. If f defines a strong action $T \to f(T)$ in $\mathfrak{A}$, then for each $T \in \mathfrak{A}_U$ and $\lambda \in \sigma_0(T)$ there exist $a_0, a_1, \cdots, a_{\nu(\lambda)-1} \in C$ such that $f(T)E_\lambda(T) = \sum_{j=0}^{\nu(\lambda)-1} (1/j!) a_j (T - \lambda I)^j E_\lambda(T)$ (Theorem 3.4). If an algebra $\mathscr{M}$ of functions defines what is called a D-action φ in $\mathfrak{A}$, then, in fact, there exists a system of derivations $\{D_k: 0 \leqq k < m\}$ from $\mathscr{M}$ into the algebra of all functions on U such that $\varphi(f, T)E_\lambda(T) = \sum_{j=0}^{\nu(\lambda)-1} (1/j!) D_j f(\lambda)(T - \lambda I)^j E_\lambda(T)$ for all $f \in \mathscr{M}$, $T \in \mathfrak{A}_U$ and $\lambda \in \sigma_0(T)$ (Theorem 4.2). Finally, it is shown that if $\mathscr{M}$ defines an action φ in $\mathfrak{A}$, the function $x(t) = t$ is in $\mathscr{M}$ and $\varphi(x, \cdot)$ is a D-action of x in $\mathfrak{A}$ which is continuous when restricted to $\{T \in \mathfrak{A}: \sigma(T) \subset [|z| < r]\}$, then for every analytic function $f \in \mathscr{M}$ $\varphi(f, T)$ is defined by a sum involving the Cauchy integral formula and terms of the form $(1/j!) D_j f(\lambda)(T - \lambda I)^j E_\lambda(T)$.

In this paper we continue the study of the concept of what we call the action of functions in Banach algebras, a study which we began in [2]. Generally speaking, let $\mathfrak{A}$ be a Banach algebra, U the open unit disk of the complex plane, $\mathfrak{A}_U$ the set of all elements of $\mathfrak{A}$ with spectrum contained in U and f a complex-valued function with domain U. Then we say f acts in $\mathfrak{A}$ or f defines an action in $\mathfrak{A}$ if there exists a mapping $x \to f(x)$ of $\mathfrak{A}_U$ into $\mathfrak{A}$ such that for every maximal commutative subalgebra $\mathscr{C}$ of $\mathfrak{A}$, a complex homomorphism h on $\mathscr{C}$ and $x \in \mathscr{C} \cap \mathfrak{A}_U$, $f(x)$ is in $\mathscr{C}$ and $h(f(x)) = f(h(x))$. An algebra $\mathscr{M}$ of functions defined on U acts in $\mathfrak{A}$ if there exists a mapping $\varphi: \mathscr{M} \times \mathfrak{A}_U \to \mathfrak{A}$ such that for each $f \in \mathscr{M}, x \to \varphi(f, x)$ is an action of f in $\mathfrak{A}$ and for each $x \in \mathfrak{A}_U$, $f \to \varphi(f, x)$ is an algebra homomorphism.

In addition to defining the concept of functions acting in general Banach algebras and establishing some properties of such actions, we also described the algebra of functions which acts in certain Banach algebras containing only normal compact operators on a Hilbert space H, the functions being determined by the algebra of operators. This paper extends that study of actions of functions in Banach algebras of compact operators to algebras which contain nonnormal operators.

While we do not characterize those functions which can act in such algebras, we do describe the restriction of $f(T)$ to certain finite-dimensional subspaces of H.

Let $\mathfrak{A}$ be a Banach algebra which has as its elements compact operators on a Hilbert space H. Since $T \in \mathfrak{A}$ is compact, the spectrum of T consists only of eigenvalues and each nonzero eigenvalue has finite index $\nu(\lambda)$. There is a projection $E_\lambda(T)$ of H onto $\ker(\lambda I - T)^{\nu(\lambda)}$ such that the kernel of $E_\lambda(T)$ is the range of the operator $(\lambda I - T)^{\nu(\lambda)}$.

If f is analytic on the unit disk U with $f(0) = 0$, then for every $T \in \mathfrak{A}_U$ the Cauchy integral formula defines an operator $f(T) \in \mathfrak{A}$ and the mapping $T \rightarrow f(T)$ from $\mathfrak{A}_U$ into $\mathfrak{A}$ satisfies the definition of f acting in $\mathfrak{A}$ (see p. 568 of [1]). In addition, for each nonzero eigenvalue λ of $T \in \mathfrak{A}_U, f(T)$ can be described on $\ker(\lambda I - T)^{\nu(\lambda)}$ by the equation

$$f(T)E_\lambda(T) = \sum_{k=0}^{\nu(\lambda)-1} \frac{1}{j!} f^{(j)}(\lambda)(T - \lambda I)^j E_\lambda(T)$$

[1, p. 559]. If f and g are both analytic on the unit disk and $T \in \mathfrak{A}_U$, then the coefficient of $(T - \lambda I)^j E_\lambda(T)/j!$ in the sum expressing $(f + g)E_\lambda(T)$ is the constant $f^{(j)}(\lambda) + g^{(j)}(\lambda)$ while the coefficient in the sum expressing $(fg)(T)E_\lambda(T)$ (fg the pointwise product) is the constant $(fg)^{(j)}(\lambda) = \sum_{i=0}^{j} \binom{j}{i} f^{(i)}(\lambda) g^{(j-i)}(\lambda)$.

Now suppose we change the hypotheses somewhat. Suppose f is any function which defines an action $T \rightarrow f(T)$ in our algebra $\mathfrak{A}$ of compact operators. Then is the operator $f(T)$ $(T \in \mathfrak{A}_U)$ still defined on the finite-dimensional subspace $\ker(\lambda I - T)^{\nu(\lambda)} (\lambda \neq 0)$ by a formula of the form

$$f(T)E_\lambda(T) = \sum_{j=0}^{\nu(\lambda)-1} \frac{1}{j!} a_j (T - \lambda I)^j E_\lambda(T)? \tag{0.1}$$

Let us assume for the moment that the answer to our question is yes for some $T \in \mathfrak{A}_U$ and a nonzero eigenvalue λ of T and that a second function g defines an action $S \rightarrow g(S)$ in $\mathfrak{A}$ such that $g(T)E_\lambda(T) = \sum_{j=0}^{\nu(\lambda)-1} (1/j!) b_j (T - \lambda I)^j E_\lambda(T)$. The mappings $S \rightarrow f(S) + g(S)$ and $S \rightarrow f(S)g(S) (S \in \mathfrak{A}_U)$ are actions of the functions $f + g$ and fg, respectively, in $\mathfrak{A}$. A straightforward calculation shows that $(f + g)(T)E_\lambda(T)$ and $(fg)(T)E_\lambda(T)$ are defined by formulas similar to (0.1) but with $a_j + b_j$ and $\sum_{j=0}^{k} \binom{k}{j} a_j b_{k-j}$, respectively, replacing a_j in (0.1). Thus the coefficients of $(T - \lambda I)^j E_\lambda(T)/j!$ in these two sums expressing $(f + g)(T)E_\lambda(T)$ and $(fg)(T)E_\lambda(T)$ bear the same type of relationship to the coefficients in the sums expressing $f(T)E_\lambda(T)$ and $g(T)E_\lambda(T)$ as they did when f and g were analytic. In fact, the coefficient of $(T - \lambda I)^j E_\lambda(T)/j!$ for the product looks very much like the Leibniz rule for higher-order derivatives or for a system of derivations (see [3]). Suppose $\mathscr{M}$ is an algebra of functions which defines an action φ in $\mathfrak{A}$ and suppose $\varphi(f, T)E_\lambda(T)$ ($T \in \mathfrak{A}_U$, λ a nonzero eigenvalue of T) is defined by an equation similar to (0.1). Is there a system of operators $\{D_k: 0 \leqq k < m\}$ (m possibly infinite) defined on $\mathscr{M}$ such that D_k is linear, $D_k(fg) = \sum_{j=0}^{k} \binom{k}{j}(D_j f)(D_{k-j} g)$ for each integer $k < m$ and $f, g \in \mathscr{M}$ and such that $\varphi(f, T)E_\lambda(T)$ is defined by equation (0.1) with $a_j = D_j f(\lambda)$ for $j = 0, 1, \cdots, \nu(\lambda) - 1$?

For that matter, is it possible for an algebra $\mathscr{M}$ of functions to act in $\mathfrak{A}$ if there is a system of derivations defined on $\mathscr{M}$? Is equation (0.1) the guide to defining the action?

These are some of the questions we tackle in this paper. We show that if a function f defines an action $T \to f(T)$ in $\mathfrak{A}$ such that $f(T)$ commutes with every operator of finite rank that commutes with T, then $f(T)E_\lambda(T)$ is, in fact, given by an equation of the form (0.1) (Theorem 3.4). Moreover, if $\mathfrak{A}$ contains an element with nonzero spectrum and the mapping $(f, T) \to f(T)$ is what we call a D-action in $\mathfrak{A}$ of an algebra $\mathscr{M}$ of functions, then there exists a system of derivations $\langle D_k \rangle$ from $\mathscr{M}$ into the algebra of all functions on U such that $f(T)E_\lambda(T)$ is defined by equation (0.1) with $a_j = D_j f(\lambda)$ (Theorem 4.2).

In answer to the last question above we show that if $\mathfrak{A}$ has as its elements only operators of finite rank and $\mathscr{M}$ is an algebra of functions defined on U, then every member of a restricted class of systems of derivations from $\mathscr{M}$ into the algebra of all functions on U determines a different action of $\mathscr{M}$ in $\mathfrak{A}$ (Theorem 2.5). Note that for such an algebra Theorem 4.2 is a converse to this theorem.

The action of an analytic function f in $\mathfrak{A}$ defined by the Cauchy integral formula has the property that whenever a sequence $\langle T_n \rangle \subset \mathfrak{A}_U$ converges to T, then the sequence $\langle f(T_n) \rangle$ converges to $f(T)$ (see p. 1101 of [1]). We show that if a D-action of a function f has this limit property, then the associated function $D_k f$ are continuous (Theorem 5.4). Finally we prove that if $T \to x(T)$ is a D-action which has this limit property when restricted to the set $\{T \in \mathfrak{A}: \sigma(T) \subset [|z| < r]\}$ for some $0 < r < 1$, then there exists a natural action $(f, T) \to f(T)$ of the algebra of functions analytic on U vanishing at zero (Theorem 6.1). In fact, if $T \in \mathfrak{A}$, $\sigma(T) \subset [|z| < r]$, then the image of T is

$$f(T) = \frac{1}{2\pi i} \int_{|z|=r} f(z)(zI - x(T))^{-1} dz .$$

1. Notation and terminology. Throughout this paper $\boldsymbol{R}$ and $\boldsymbol{C}$ denote the real and complex numbers, respectively. All algebras are complex algebras. If X is a topological space, then $C(X)$ is the algebra of all continuous complex-valued functions defined on X while

$C_0(X)$ is the subalgebra of $C(X)$ containing all functions which "vanish at infinity—f is in $C_0(X)$ if, and only if, for each $\varepsilon > 0$ there exists compact $K \subset X$ such that $|f(t)| < \varepsilon$ for $t \notin K$.

We denote by $[|z - \lambda| < r]$ the open complex disk $\{z \in C: |z - \lambda| < r\}$.

Let H be a Hilbert space, $L(H)$ the Banach algebra of continuous linear operators on H with the operator norm $\|T\|$. We denote by C_∞ the set of all compact operators in $L(H)$, a closed two-sided ideal of $L(H)$ with the algebra of all operators of finite rank dense in it. We denote by C_0 the algebra of operators in $L(H)$ of finite rank. If T is in C_∞, then the spectrum of T, $\sigma(T) = \{\lambda \in C: \lambda I - T$ is not invertible$\}$ is at most countable and zero is the only possible cluster point. Each $\lambda \in \sigma(T)$ is an eigenvector of T [12, p. 219]. For each $T \in L(H)$ we denote by $\sigma_0(T)$ the nonzero elements of $\sigma(T)$.

For each $T \in L(H)$ and $\lambda \in \sigma(T)$, the *index* $\nu_T(\lambda) = \nu(\lambda)$ of λ for T is the smallest integer k (if it exists) such that $\ker(\lambda I - T)^k = \ker(\lambda I - T)^{k+1}$. The index of $\lambda \neq 0$ for compact T is finite, the subspace $\ker(\lambda I - T)^{\nu(\lambda)}$ has finite dimension and H is the direct sum of $\ker(\lambda I - T)^{\nu(\lambda)}$ and $(\lambda I - T)^{\nu(\lambda)}H$ [12, p. 219]. We call the projection $E_\lambda(T)$ of H onto $\ker(\lambda I - T)^{\nu(\lambda)}$ with $\ker E_\lambda(T) = (\lambda I - T)^{\nu(\lambda)}H$ the *Riesz projection onto* $\ker(\lambda I - T)^{\nu(\lambda)}$.

Suppose $T \in C_\infty$ and $\lambda \in \sigma_0(T)$. Of great use in determining the behavior of an operator on $E_\lambda(T)H$ is what we call a *Jordan basis* of $E_\lambda(T)H$. This is a basis $\{x_{ij}: 1 \leqq i \leqq m, 1 \leqq j \leqq r_i\}$ of $E_\lambda(T)H$ such that $\nu(\lambda) = r_1 \geqq r_2 \geqq \cdots \geqq r_m \geqq 1$, $r_1 + r_2 + \cdots r_m = \dim E_\lambda(T)H$ and

$$(T - \lambda I)x_{ij} = \begin{cases} x_{ij+1} & \text{for} \quad 1 \leqq j < r_i, 1 \leqq i \leqq m \\ 0 & \text{for} \quad j = r_i, 1 \leqq i \leqq m \,. \end{cases}$$

Such a basis is constructed in the same manner as the basis for the Jordan canonical form of a matrix (cf. [1], p. 563).

Let $\mathscr{B}$ be a subalgebra of C_∞ such that (i) $\mathscr{B}$ contains C_0; (ii) $\mathscr{B}$ is a Banach algebra with respect to a norm $|\cdot|$ and $\|T\| \leqq |T|$ for all $T \in \mathscr{B}$; (iii) $\sigma_{\mathscr{B}}(T) = \sigma(T)$ for all $T \in \mathscr{B}$. We call such an algebra a *Banach algebra of compact operators* and it is to such algebras that we restrict our attention. The algebras C_p, $1 \leqq p \leqq \infty$, [1, 9] are Banach algebras of compact operators.

If H is finite-dimensional, then $\mathscr{B} = L(H)$ as a consequence of (i). If $\mathscr{B}$ is closed under the usual involution $T \to T^*$, then the condition $\|T\| \leqq |T|$ above is redundant; the identity mapping $T \to T$ of $\mathscr{B}$ into $L(H)$ is a *-homomorphism and hence a norm-reducing mapping [10, p. 208].

For a commutative Banach algebra $\mathfrak{A}$ we denote by $M_{\mathfrak{A}}$ the maximal ideal space of $\mathfrak{A}$ regarded also as the space of nonzero complex homomorphisms on $\mathfrak{A}$ with the relative weak*-topology. The Gelfand mapping $x \to \hat{x}$ of $\mathfrak{A}$ into $C_0(M_{\mathfrak{A}})$ is defined by setting $\hat{x}(h) = h(x)$ for all $h \in M_{\mathfrak{A}}$, $x \in \mathfrak{A}$.

For each set $S \subset \mathfrak{A}$ the commutant of S in $\mathfrak{A}$ is the set $S'_{\mathfrak{A}} = \{y \in \mathfrak{A}: xy = yx \text{ for all } x \in S\}$. If $S = \{x\}$, then we write $\{x\}'_{\mathfrak{A}} = (x)'_{\mathfrak{A}}$. The second commutant of S in $\mathfrak{A}$ is the set $(S'_{\mathfrak{A}})'_{\mathfrak{A}} = S''_{\mathfrak{A}}$.

Throughout the paper the symbol U denotes the open unit disk of the complex plane unless stated otherwise. If $\mathfrak{A}$ is a Banach algebra, $\mathfrak{A}_U$ denotes the set $\{x \in \mathfrak{A}: \sigma_{\mathfrak{A}}(x) \subset U\}$ or, equivalently, the open unit ball in the topology generated by the spectral radius.

2. Functions which act in algebras of compact operators.

We first present the definition basic to this paper.

DEFINITION 2.1. A set $\mathscr{S}$ of complex-valued functions defined on U acts in $\mathfrak{A}$ (or defines an action in $\mathfrak{A}$) if there exists a mapping $\varphi: \mathscr{S} \times \mathfrak{A}_U \to \mathfrak{A}$ such that (i) for every maximal commutative subalgebra $\mathscr{C}$ of $\mathfrak{A}$ and $x \in \mathscr{C} \cap \mathfrak{A}_U$ $f(x)$ is in $\mathscr{C}$ and $[f(x)]^\wedge = f \circ \hat{x}$ on $M_{\mathscr{C}}$; (ii) whenever f, g and $\alpha f + g$ are in $\mathscr{S}$ (or f, g and fg are in $\mathscr{S}$), then $\varphi(\alpha f + g, x) = \alpha\varphi(f, x) + \varphi(g, x)$ (or $\varphi(fg, x) = \varphi(f, x)\varphi(g, x)$) for all $x \in \mathfrak{A}_U$. If $f \in \mathscr{S}$, we say f acts in $\mathfrak{A}$. The mapping $\varphi(f, \cdot)$: $\mathfrak{A}_U \to \mathfrak{A}$ is called an action of f in $\mathfrak{A}$ while φ is called an action of $\mathscr{S}$ in $\mathfrak{A}$. [2].

Definition 2.1 can be set for functions whose domain is an arbitrary set $U \subset \boldsymbol{C}$ but the results and proofs are essentially the same for arbitrary sets as for the open unit disk with the exception of those in §6. In §6 we can replace the open unit disk by a simply-connected open set and obtain similar results. Throughout the paper, therefore, we assume all functions are defined on the open unit disk of the complex plane.

Several examples of functions which define actions in Banach algebras are given in [2]. The next theorems establish the existence of algebras of functions which act in Banach algebras containing compact operators.

Unless stated otherwise $\mathscr{B}$ denotes a Banach algebra of compact operators, $\mathfrak{A}$ a closed subalgebra of $\mathscr{B}$ and $\mathfrak{A}(T)$ the closed subalgebra of $\mathfrak{A}$ generated by $T \in \mathfrak{A}$, or by T and I if $\dim H < \infty$.

THEOREM 2.2. *Let $\mathscr{S}$ be a set of functions analytic on U with $f(0) = 0$ if $\dim H = \infty$. Then $\mathscr{S}$ defines an action φ in any closed subalgebra $\mathfrak{A}$ of $\mathscr{B}$ such that* (1) *for each $T \in \mathfrak{A}_U$, $\varphi(f, T)$ is in $(T)''_{\mathscr{B}}$ and* (2) *if a sequence $\langle T_n \rangle \subset \mathfrak{A}$ converges to $T \in \mathfrak{A}_U$, then for large n $\varphi(f, T_n)$ is defined and $\langle \varphi(f, T_n) \rangle$ converges to $\varphi(f, T)$.*

Proof. If $(f, T) \in \mathscr{S} \times \mathfrak{A}_U$ choose $r < 1$ such that $r > \max\{|\lambda|: \lambda \in \sigma(T)\}$ and define $\varphi(f, T)$ by the Cauchy integral formula

$$\varphi(f, T) = \frac{1}{2\pi i}\int_{|z|=r} f(z)(zI - T)^{-1}dz \,. \tag{2.1}$$

Then $\varphi(f, T)$ is in $\mathfrak{A}(T)$ and in $(T)''_{\mathscr{B}}$ [1, p. 1101]. Since $T \in \mathfrak{A}$, $\mathfrak{A}(T)$ is contained in $\mathfrak{A}$ and consequently φ maps $\mathfrak{A}_U$ into $\mathfrak{A}$. Conditions (i) and (ii) of Definition 2.1 follow quickly from the continuity of homomorphisms and the "homomorphism theorem" [1, p. 568, and 11, p. 203]. The proof of (2) of the theorem can be adapted from the proof in [1] (p. 1101) for sequences in C_p and the details are left to the reader.

If $T \in \mathfrak{A}$, $\lambda \in \sigma_0(T)$ and $f \in \mathscr{S}$, then

$$\varphi(f, T) = \sum_{k=0}^{\nu(\lambda)-1} \frac{1}{k!} f^{(k)}(\lambda)(T - \lambda I)^k E_\lambda(T)$$

[1, p. 559]. Using this equation as a prototype we can define many actions of a function on some Banach algebras by means of higher-order systems of derivations.

DEFINITION 2.3. A set of $m + 1$ linear operators $\{D_0, D_1, \cdots, D_m\}$ from an algebra $\mathscr{M}$ into an algebra $\mathscr{L}$ is a system of derivations of order m (from $\mathscr{M}$ to $\mathscr{L}$) if for every pair $x, y \in \mathscr{M}$ and integer $k = 0, 1, \cdots, m$ $D_k(xy)$ satisfies the Leibniz rule

$$D_k(xy) = \sum_{j=0}^{k} \binom{k}{j}(D_j x)(D_{k-j} y) \,. \tag{2.2}$$

A sequence of linear operators $D_j: \mathscr{M} \to \mathscr{L}$, $j = 0, 1, 2, \cdots$, is a system of derivations of infinite order if $\{D_0, D_1, \cdots, D_k\}$ is a system of order k for each integer $k \geqq 1$. [3].

EXAMPLE 2.4. Let

$$\mathscr{M} = C^n(\boldsymbol{R}) = \{f \in C(\boldsymbol{R}): f^{(j)} \in C(\boldsymbol{R}), j = 0, 1, 2 \cdots, n\} ,$$

where $f^{(k)} = d^k f/dt^k$. Choose functions $h_1, h_2 \in C(\boldsymbol{R})$ and set $D_0 f = f$, $D_1 f = h_1 f^{(1)}$ and $D_2 f = h_1^2 f^{(2)} + h_2 f^{(1)}$. It is easily checked that (D_0, D_1, D_2) is a system of derivations of order two from $C^n(\boldsymbol{R})$ into $C(\boldsymbol{R})$. (For further examples and properties see [3].).

THEOREM 2.5. *Let $\mathscr{M}$ be an algebra of functions defined on U and $\mathfrak{A}$ a subalgebra of $\mathscr{B}$ which contains only operators of finite rank. Set* $m = \sup\{\nu_T(\lambda): T \in \mathfrak{A}, \lambda \in \sigma(T)\}$. *Then every system of* *derivations* $\{D_k: 0 \leqq k < m\}$ *of order* $m-1$ (*infinite order if* $m = \infty$) *from* $\mathscr{M}$ *into the algebra of all functions on* U *such that* D_0 *is the identity operator on* $\mathscr{M}$ *defines an action* φ *of* $\mathscr{M}$ *in* $\mathfrak{A}$ *if* $\dim H < \infty$ *or of* $\mathscr{M}_0 = \{f \in \mathscr{M}: f(0) = 0\}$ *in* $\mathfrak{A}$ *if* $\dim H = \infty$. *For each* $(f, T) \in \mathscr{M} \times \mathfrak{A}_U$, $\varphi(f, T)$ *is in the second commutant of* T *in* $\mathscr{B}$.

Proof. Suppose $\{D_k: 0 \leqq k < m\}$ is a system of derivations from $\mathscr{M}$ into the algebra of all functions on U with D_0 the identity operator. For each $(f, T) \in \mathscr{M} \times \mathfrak{A}_U$ set

$$\varphi(f, T) = \sum\left\{\frac{1}{j!}D_j f(\lambda)(T - \lambda I)^j E_\lambda(T): 0 \leqq j < \nu(\lambda), \lambda \in \sigma(T)\right\}. \tag{2.3}$$

That $\varphi(f, T)$ is well-defined and an element of $\mathfrak{A}$ if $\dim H < \infty$ and $f \in \mathscr{M}$ or if $\dim H = \infty$ and $f \in \mathscr{M}_0$ follows from the fact that $\nu(\lambda) \leqq m$ for $\lambda \in \sigma(T)$, $(T - \lambda I)^j E_\lambda(T)$ is in $\mathfrak{A}$ for $0 \leqq j < \nu(\lambda)$ and $E_\lambda(T) \in \mathfrak{A}$ if $\dim H < \infty$. Since $E_\lambda(T)$ is in the second commutant of T in $\mathscr{B}$, $\varphi(f, T)$ also must be. Condition (ii) of Definition 2.1 follows from the properties of systems of derivations (linearity and the Leibniz rule).

It is shown in §4 that what we call a D-action in $\mathfrak{A}$ of an algebra $\mathscr{M}$ of functions determines a higher-order system of derivations on $\mathscr{M}$. The action φ can then be described on $E_\lambda(T)H$ $(T \in \mathfrak{A}_U, \lambda \in \sigma_0(T))$ by equation (2.3).

EXAMPLE 2.6. Let $\mathfrak{A} = L(C^N)$. Denote by $\mathscr{M}$ the algebra of all polynomials in the functions $y_1, y_2, \cdots, y_n$ defined on U. For each polynomial f in n variables denote by f_j the partial derivative of f with respect to the j-th variable. Define $D_k: \mathscr{M} \to \mathscr{M}$, $k = 0, 1, \cdots, m \geqq N-1$ by setting $D_0 f = f$, $D_1 f = \sum_{j=1}^n f_j(y_1, y_2, \cdots, y_n)$ and $D_k f = D_1(D_{k-1} f)$ for each $f \in \mathscr{M}$. It is easily checked that $(D_0, D_1, \cdots, D_m)$ is an m-th order system of derivations on $\mathscr{M}$. Thus for each $(f, T) \in \mathscr{M} \times \mathfrak{A}_U$ we define $\varphi(f, T)$ by formula (2.3) and the mapping φ is an action of $\mathscr{M}$ in $\mathfrak{A}$.

EXAMPLE 2.7. Although Definition 2.1 is stated for the unit disk, we present this example of an action of a function defined on an interval (a, b) because it is well-known.

Let $\mathfrak{A} = L(C^N)$ and $\mathscr{M} = C^m(a, b)$, $m \geqq N-1$, with $D_k f = f^{(k)}$ for $f \in \mathscr{M}$ and $k = 0, 1, \cdots, m$. An action φ of $\mathscr{M}$ in $\mathfrak{A}$ is defined by equation (2.3). If we regard the operators on C^N as $N \times N$ matrices, we find that $\varphi(f, T)$ is the matrix corresponding to $f \in C^m$ (a, b) and $T \in \mathfrak{A}_U$ defined by Gantmacher [4].

For Banach algebras of compact operators on infinite-dimensional Hilbert spaces we have the following generalization of Theorems 2.2 and 2.5. Note that C_0 is the set of operators of finite rank.

THEOREM 2.8. *Let $\mathscr{M}$ be an algebra of functions defined on U such that $f \in \mathscr{M}$ is analytic on a disk $[|z| < \delta_f]$ and $f(0) = 0$. Suppose $(D_0, D_1, \cdots)$ is an infinite order system of derivations from $\mathscr{M}$ into the algebra of all functions on U such that D_0 is the identity operator on $\mathscr{M}$ and for each $f \in \mathscr{M}$, and $|z| < \delta_f$, $D_k f(z) = f^{(k)}(z)$. Then $\mathscr{M}$ acts in every closed subalgebra of $\mathscr{B}$.*

Proof. Let $\mathfrak{A}$ be a closed subalgebra of $\mathscr{B}$. If $T \in \mathfrak{A} \cap \mathscr{C}_0$, then each operator in $\mathfrak{A}(T)$ is of finite rank so that the mapping φ_0: $\mathscr{M} \times \mathfrak{A}(T)_U \to \mathfrak{A}$ defined by equation (2.3) is an action of $\mathscr{M}$ in the algebra $\mathfrak{A}(T)$ (Theorem 2.5). For each $f \in \mathscr{M}$ the mapping $\varphi_1(f, \cdot)$: $\{T \in \mathfrak{A}: \sigma(T) \subset [|z| < \delta_f]\} \to \mathfrak{A}$ defined by the Cauchy integral formula (2.1) ($\lim_{n\to\infty} \| T^n \|^{1/n} < r < \delta_f$) is an action in $\mathfrak{A}$ of the restriction of f to $[|z| < \delta_f]$ (Theorem 2.2). Note that if $T \in \mathfrak{A}$, $\sigma(T) \subset [|z| < \delta_f]$ and σ is a finite subset of $\sigma(T)$, then

$$\begin{aligned}(2.4) \qquad \varphi_1(f, T)\sum_{\lambda \in \sigma} E_\lambda(T) &= \sum_{\lambda \in \sigma} \sum_{j=0}^{\nu(\lambda)-1} \frac{1}{j!} f^{(j)}(\lambda)(T - \lambda I)^j E_\lambda(T) \\ &= \sum_{\lambda \in \sigma} \varphi_0(f, TE_\lambda(T))E_\lambda(T)\end{aligned}$$

($D_j f = f^{(j)}$ on $[|z| < \delta_f]$). Moreover, if $T \in \mathfrak{A}_U$ and $P = \sum \{E_\lambda(T)$: $|\lambda| \geqq \delta_f\}$, then $\varphi_1(f, T - TP) = \varphi_1(f, T - TP)(I - P)$ as a result of the homomorphism theorem for the Cauchy integral formula [1, p. 568].

Finally, we define φ: $\mathscr{M} \times \mathfrak{A}_U \to \mathfrak{A}$ as follows: for $(f, T) \in \mathscr{M} \times \mathfrak{A}_U$ set $P = \sum \{E_\lambda(T): |\lambda| \geqq \delta_f\}$ and $\varphi(f, T) = \varphi_1(f, T - TP) + \varphi_0(f, TP)P$. The fact that φ satisfies conditions (i) and (ii) of Definition 2.1 follows from Theorems 2.2 and 2.5. The proofs that φ is linear and preserves products are essentially the same so we prove that φ preserves products.

Suppose $f, g \in \mathscr{M}$ and $T \in \mathfrak{A}_U$. Set $\varepsilon = \min(\delta_f, \delta_g, \delta_{fg})$, $P = \sum \{E_\lambda(T): |\lambda| \geqq \varepsilon\}$ $P_f = \sum \{E_\lambda(T): |\lambda| \geqq \delta_f\}$ and $P_g = \sum \{E_\lambda(T): |\lambda| \geqq \delta_g\}$. We assume, without loss of generality that $\varepsilon = \delta_{fg} \leqq \delta_f \leqq \delta_g$. Then, in view of the definition of φ and equation (2.4) we have

$$\begin{aligned}&\varphi(f, T)\varphi(g, T) \\ &\quad = [\varphi_1(f, T - TP_f) + \varphi_0(f, TP_f)P_f][\varphi_1(g, T - TP_g) + \varphi_0(g, TP_g)P_g] \\ &\quad = [\varphi_1(f, T - TP) + \varphi_0(f, TP)P][\varphi_1(g, T - TP) + \varphi_0(g, TP)P] \\ &\quad = \varphi_1(f, T - TP)\varphi_1(g, T - TP) + \varphi_0(f, TP)\varphi_0(g, TP)P \\ &\quad = \varphi_1(fg, T - TP) + \varphi_0(fg, TP)P = \varphi(fg, T) .\end{aligned}$$

The action defined in the proof of Theorem 2.8 is the action defined by the Cauchy integral formula when $\mathscr{M}$ is an algebra of functions analytic on U which vanish at zero and $D_k f = f^{(k)}$ for $k = 0, 1, 2, \cdots$.

EXAMPLE 2.9. Let $\mathscr{M}$ be an algebra of functions which are analytic on the disk $[|z| < r]$ for some $r < 1$ and have infinitely differentiable real and imaginary parts defined on U (considered as a subset of the real plane $\boldsymbol{R}^2$). If $f \in \mathscr{M}$ and $f(z) = u(x, y) + iv(x, y)$, set $D_k f = f^{(k)}(z)$ for $|z| < r$ and $D_k f(z) = (\partial^k/\partial x^k)(u + iv)(x, y)$ for $r \leqq |z| < 1$ $(z = x + iy)$. Then $\{D_k: k = 0, 1, 2, \cdots\}$ is a system of derivations on $\mathscr{M}$. The action of $\mathscr{M}$ in $\mathfrak{A}$ of Theorem 2.8 is then a generalization of the action defined in Theorem 2.5.

3. Form of the operator $f(T)E_\lambda(T)$.

We have seen that an algebra $\mathscr{M}$ of functions with domain U and a system of derivations of order m from $\mathscr{M}$ into the algebra of all functions on U can define an action of $\mathscr{M}$ in subalgebras of $L(\boldsymbol{C}^N)$, $N \leqq m + 1$, via equation (2.3). In the next sections we show that under certain conditions an action of $\mathscr{M}$ in $\mathfrak{A}$ determines a higher-order system of derivations. As a first step we prove that for each $T \in \mathfrak{A}$ and $\lambda \in \sigma_0(T)$, there exist constants a_k, $0 \leqq k < \nu(\lambda)$, such that

$$f(T)E_\lambda(T) = \sum_{j=0}^{\nu(\lambda)-1} \frac{1}{j!} a_j (T - \lambda I)^j E_\lambda(T) . \tag{3.1}$$

In order to do so we need operators which may not be elements of $\mathfrak{A}$ although they always belong to $\mathscr{B}$ since they have finite rank. Thus we note that each of the actions defined in §2 has the property for $T \in \mathfrak{A}_U$ $f(T)$ is in the second commutant of T in $\mathscr{B}$ and set the following definition.

DEFINITION 3.1. Let $\mathscr{B}$ be a Banach algebra, $\mathfrak{A}$ a closed subalgebra of $\mathscr{B}$. A set $\mathscr{S}$ of functions defined on U acts strongly (defines a strong action) in $\mathfrak{A}$ if $\mathscr{S}$ acts in $\mathfrak{A}$ in the usual sense and if the action φ has the additional property that for every $(f, x) \in \mathscr{S} \times \mathfrak{A}_U$, $\varphi(f, x) \in (x)''_{\mathscr{B}}$.

If $\mathscr{B}$ is a Banach algebra of compact operators, then $\mathscr{B}$ contains all operators of finite rank and as a result has no nonzero central idempotents [10, p. 165]. Thus if $\dim H = \infty$ and f acts strongly in $\mathfrak{A}$, then $f(0) = 0$ [2].

In order to obtain equation (3.1) not only for $T \in \mathfrak{A}$ and $\lambda \in \sigma_0(T)$ but also for $T \in \mathfrak{A} \cap C_0$ and all $\lambda \in \sigma(T)$ we make use of the following lemma.

LEMMA 3.2. *If T is a nonzero operator of finite rank and $\sigma(T) = \{0\}$, then for any nonzero $\lambda \in \boldsymbol{C}$, there exists an operator S of finite rank such that $\lambda \in \sigma_0(S)$ and $T = (S - \lambda I)E_\lambda(S)$.*

Proof. Since $\dim TH < \infty$, the index of zero must be a finite integer n; otherwise for $k > (\dim TH) + 1$ there would be $x \in \ker T^k - \ker T^{k-1}$ and the set $\{Tx, T^2x, \cdots, T^{k-1}x\}$ would be linearly independent.

The spaces TH and $\ker T^*$ are orthogonal with $TH \subset \ker T^{n-1}$ and $H = TH \oplus \ker T^*$. Therefore we can choose by induction elements $y_1, y_2, \cdots$ satisfying the following conditions: $y_1 \in \ker T^*$, $T^{n-1}y_1 \neq 0$; $y_j \in \ker T^{r_j}$, where $r_1 = n$ and $r_j, j \geqq 2$, is the largest integer such that $\ker T^{r_j} \oplus \mathrm{sp}\,\{T^{k-1}y_i: 1 \leqq i < j, 1 \leqq k < r_i - r_{j-1}\}$ is a proper subspace of H. This process must end after a finite number—say m—of steps. Set $x_{ij} = T^{j-1}y_i$. Either $\ker T \oplus \mathrm{sp}\,\{x_{ij}: 1 \leqq i \leqq m; 1 \leqq j < r_i\} = H$ or there exists a largest integer k $(2 < k \leqq r_m)$ such that $\ker T^{k-1} \oplus \mathrm{sp}\,\{x_{ij}: 1 \leqq i \leqq m; 1 \leqq j < r_i\}$ is not all of H and a vector y orthogonal to this subspace such that $T^k y = 0$ and $T^{k-1}y \neq 0$. Then the vectors $y, Ty, \cdots, T^{k-1}y$ and $\{x_{ij}: 1 \leqq i \leqq m, 1 \leqq j \leqq r_j\}$ form a linearly independent set. The vector Ty is in TH and hence a linear combination of the vectors x_{ij}, a contradiction.

Let P be the projection onto $H_0 = \mathrm{sp}\,\{x_{ij}: 1 \leqq i \leqq m, 1 \leqq j \leqq r_i\}$ such that $\ker P$ is the orthogonal complement in $\ker T$ of $\mathrm{sp}(x_{1r_1}, \cdots, x_{mr_m})$. Then $S = (T + \lambda I)P$ is the desired operator.

In an attempt to simplify Lemma 3.2 the orthogonal projection onto TH was considered but if $H = C^4$ with the usual inner product and if $T(a, b, c, d) = (\frac{1}{2}(-a + b + c + d), c - d, \frac{1}{2}(a - b + c + d), 0)$, then the orthogonal projection P onto the range of T is given by $P(a, b, c, d) = (a, b, c, 0)$. It is easily checked that for $\lambda \neq 0$ $TP - \lambda(P - I) \neq T$.

If $\dim H < \infty$, then the set $\{x_{ij}: 1 \leqq i \leqq m, 1 \leqq j \leqq r_i\}$ in the proof above can be extended to a Jordan basis $\{x_{ij}: 1 \leqq i \leqq m_0, 1 \leqq j \leqq r_i\}$ for H such that $\sum_{i=1}^{m_0} r_i = \dim H$, $r_i = 1$ and $Tx_{i1} = 0$ for $m < i \leqq m_0$. If T_0 is the matrix for T with respect to this basis, then we obtain the matrix for S by substituting λ for zero on the diagonal of each nonzero block of T_0.

COROLLARY 3.2.1. *If $T \in C_0$ and $0 \in \sigma(T)$, then for nonzero $\lambda \in \boldsymbol{C}$, there exists an operator $S \in C_0$ such that $\lambda \in \sigma_0(S)$, $\ker E_0(T) \subset \ker E_\lambda(S)$, $E_\lambda(S)H \subset E_0(T)H$ and $TE_0(T) = (S - \lambda I)E_\lambda(S)$.*

Proof. The operator $T_0 = TE_0(T)$ satisfies the hypotheses of Lemma 3.2. Therefore there exists $S_0 \in L(E_0(T)H)$ such that $\lambda \in \sigma_0(S_0)$,

S_0 has finite rank and $T_0 = (S_0 - \lambda I)E_\lambda(S_0)$. Define S by setting $SE_0(T) = S_0$ and $S(I - E_0(T)) = 0$.

As a first step toward equation (3.1) we determine the form on $E_\lambda(T)H$ of any operator which commutes with T.

LEMMA 3.3. *Suppose* $T \in C_\infty$ *and* $\lambda \in \sigma_0(T)$ *with* $\{x_{jk}: 1 \leqq j \leqq m, 1 \leqq k \leqq r_j\}$ *a Jordan basis for* $\ker(\lambda I - T)^{\nu(\lambda)}$. *If* S *and* T *commute, then there exist constants* b_j^{ks}, $j = 1, 2, \cdots, r_s$; $k, s = 1, 2, \cdots, m$, *such that*

$$b_j^{ks} = 0, \quad \text{for} \quad s < k, 1 \leqq j \leqq r_s - r_k \tag{3.2}$$

$$Sx_{ij} = \sum_{s=1}^{m} \sum_{k=j}^{r_s} b_{k-j+1}^{is} x_{sk} \quad \text{for} \quad 1 \leqq i \leqq m, 1 \leqq k \leqq r_j \,. \tag{3.3}$$

If constants b_j^{ks}, $1 \leqq j \leqq r_s$, $1 \leqq k, s \leqq m$ *satisfy* (3.2), *then there exists* $S \in C_\infty$ *such that* S *commutes with* T *and is defined on* $E_\lambda(T)H$ *by* (3.3).

Proof. If $T \in C_\infty$, $\lambda \in \sigma_0(T)$ and if S commutes with T, then $SE_\lambda(T) = E_\lambda(T)S$ so that there exist constants b_j^{ks} such that $Sx_{k1} = \sum_{s=1}^{m} \sum_{j=1}^{r_s} b_j^{ks} x_{sj}$. Since S and $T - \lambda I$ commute and $(T - \lambda I)^k x_{si} = 0$ for $r_s - r_k < i \leqq r_s$, $k \leqq s \leqq m$, and $x_{ki} = (T - \lambda I)^{i-1} x_{k1}$ for $1 \leqq i \leqq r_k$, we have $0 = S(T - \lambda I)^{r_k} x_{ki} = \sum_{s=1}^{k-1} \sum_{j=1}^{r_s - r_k} b_j^{ks} x_{s, r_k + j}$. Therefore, $b_j^{ks} = 0$ for $1 \leqq j \leqq r_s - r_k$, $1 \leqq s < k$, and Sx_{ki} is given by (3.3).

Now let b_j^{ks}, $1 \leqq j \leqq r_s$, $1 \leqq k, s \leqq m$, be constants satisfying (3.2). Set $S(I - E_\lambda(T)) = 0$ and define Sx_{ki} by (3.3). It is a straightforward computation to show that S and T commute.

COROLLARY 3.3.1. *Let* $T = (R - \mu I)E_\mu(R)$, *where* $R \in C_0$ *and* $\mu \in \sigma_0(R)$, *and let* $\{x_{kj}: 1 \leqq j \leqq r_k, 1 \leqq k \leqq m\}$ *be a Jordan basis for* $E_\mu(R)H$. *Then for every operator* S *which commutes with both* T *and* $E_\mu(R)$ *there exist constants* b_j^{ks}, $1 \leqq j \leqq r_s$, $1 \leqq k, s \leqq m$, *satisfying* (3.2) *such that* S *is defined on* $E_\mu(R)H$ *by equation* (3.3). *Conversely, if* $\{b_j^{ks}: 1 \leqq j \leqq r_s, 1 \leqq k, s \leqq m\}$ *satisfies* (3.2), *there exists an operator* S *which commutes with* T *and* $E_\mu(R)$ *such that* Sx_{ij}, $1 \leqq i \leqq m$, $1 \leqq j \leqq r_i$, *is defined by* (3.3).

THEOREM 3.4. *Let* f *be a function with domain* U *and a strong action* $T \to f(T)$ *in a closed subalgebra* $\mathfrak{A}$ *of a Banach algebra* $\mathscr{B}$ *of compact operators. Then for each* $T \in \mathfrak{A}_U$ *and* $\lambda \in \sigma_0(T)$, *or* $T \in \mathfrak{A}_U \cap C_0$ *and* $\lambda \in \sigma(T)$, *there exist constants* $a_0, a_1, \cdots, a_{n-1}$ (*n be the index of* λ *for* T) *such that* $a_0 = f(\lambda)$ *and*

$$f(T)E_\lambda(T) = \sum_{k=1}^{n-1} \frac{1}{k!} a_k (T - \lambda I)^k E_\lambda(T) \,. \tag{3.4}$$

Proof. Choose $T \in \mathfrak{A}_U$ and $\lambda \in \sigma_0(T)$ and let n be the index of λ for T. Let $\{x_{kj}: 1 \leqq j \leqq r_k, 1 \leqq k \leqq m\}$ be a Jordan basis for $E_\lambda(T)H$. Set $E = E_\lambda(T)$. Since $f(T)$ and T commute, there exist constants $b_i^{ks}, 1 \leqq j \leqq r_s, 1 \leqq k, s \leqq m$, satisfying (3.2) such that $f(T)x_{ki}$ is defined by (3.3).

Let $P_k, 1 \leqq k \leqq m$, be the projection onto the span of $\{x_{ki}: 1 \leqq i \leqq r_k\}$, such that $\ker P_k = \ker E_\lambda(T) \oplus \mathrm{sp}\,\{x_{ij}: 1 \leqq i \leqq m, 1 \leqq j \leqq r_i, i \neq k\}$. Since T and $P_k (1 \leqq k \leqq m)$ commute, $f(T)$ and P_k commute. Thus $f(T)x_{ki} = \sum_{j=1}^{r_k} b_{j-i+1}^{kk} x_{kj}$.

As a consequence of Lemma 3.3 the operator S defined by setting $S(I - E) = 0$ and $Sx_{ki} = \sum_{s=k}^{m} x_{si}$ commutes with T. But then S and $f(T)$ commute so that for $1 \leqq k \leqq m, 1 \leqq i \leqq r_k$,

$$f(T)Sx_{ki} = \sum_{s=k}^{m} \sum_{j=1}^{r_s} b_{j-i+1}^{ss} x_{sj}$$

and $Sf(T)x_{ki} = \sum_{j=1}^{r_k} \sum_{s=k}^{m} b_{j-i+1}^{kk} x_{sj}$ are equal. Thus we must have $b_{p-i+1}^{kk} = b_{p-i+1}^{ss}$ for $k \leqq s \leqq m, i \leqq p \leqq r_s$.

Set $a_k = k!\, b_k^{11}, k = 0, 1, \cdots, n-1$. It is easily checked that $f(T)E_\lambda(T) = \sum_{j=0}^{n-1} (1/j!) a_j (T - \lambda I)^j E_\lambda(T)$. If $\mathscr{C}$ is a maximal commutative subalgebra of $\mathfrak{A}$, $T \in \mathscr{C}$ and h is a $\boldsymbol{C}$-homomorphism on $\mathscr{C}$ such that $h(T) = \lambda$, then

$$f(\lambda) = h(f(T)E_\lambda(T)) = h\Big(\sum_{j=0}^{n-1} \frac{1}{j!} a_j (T - \lambda I)^j E_\lambda(T)\Big) = a_0 \,.$$

If $T \in \mathfrak{A}_U \cap C_0$ and $\lambda = 0$ is in $\sigma(T)$, then there exists $R \in C_0$ and $\mu \in \sigma_0(R)$ such that $TE_0(T) = (R - \mu I)E_\mu(R)$, $\ker E_0(T) \subset \ker E_\mu(R)$, $E_\mu(R)H \subset E_0(T)H$ (Lemma 3.2). The proof of the existence of constants $a_0, a_1, \cdots, a_{n-1}$ (n the index of 0 for T) such that $a_0 = f(0)$ and $f(T)E_\mu(R) = \sum_{j=0}^{n-1} (1/j!) a_j T^j E_\mu(R)$ is similar to the proof for $T \in \mathfrak{A}_U$ $\lambda \in \sigma(T)$. Since $f(T) \in \mathfrak{A} \cap (T)''_{\mathscr{B}}$ and $\mathscr{B}$ contains C_0, we have $f(T)E_0(T)(I - E_\mu(R)) = 0$, $f(T)E_0(T) = f(T)E_\mu(R)$ while $TE_0(T) = TE_\mu(R)$.

When f is analytic on U and the action of f in $\mathfrak{A}$ is defined by the Cauchy integral formula, then for each $T \in \mathfrak{A}_U$ and $\lambda \in \sigma(T)$, we have $f(T)E_\lambda(T) = \sum_{j=0}^{\nu(\lambda)-1} (1/j!) f^{(j)}(\lambda)(T - \lambda I)^j E_\lambda(T)$ (cf. §2). The coefficients of $(T - \lambda I)^j E_\lambda(T), 0 \leqq j < \nu(\lambda)$, depend only on λ and the function f—i.e., for any $S \in \mathfrak{A}_U$ with $\lambda \in \sigma_0(S)$, $(1/j!) f^{(j)}(\lambda)$ is the coefficient of $(S - \lambda I)^j E_\lambda(S)$ in the sum expressing $f(S)E_\lambda(S)$. The following example for 2×2 matrices demonstrates that in general the coefficient of $(T - \lambda I)^j E_\lambda(T)$ in the sum expressing $f(T)E_\lambda(T)$ depends not only on λ and f but also on T.

EXAMPLE 3.5. Let $\mathfrak{A}$ be the full algebra of 2×2 matrices, regarded as operators on two-dimensional Hilbert space, and f a function defined on U. If $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\sigma(T) = \{\lambda\} \subset U$, define $f(T)$ by

$$(3.5)\quad f(T) = f(\lambda)I + b(T - \lambda I) = \begin{pmatrix} f(\lambda) + b(a - \lambda) & b^2 \\ bc & f(\lambda) + (d - \lambda)b \end{pmatrix} .$$

If $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $\sigma(T) = \{\lambda, \mu\} \subset U\,(\lambda \neq \mu)$, set

$$f(T) = \frac{1}{\lambda - \mu}[f(\lambda)(T - \mu I) - f(\mu)(T - \lambda I)] .$$

It is easily checked that the mapping $T \to f(T)$ defined in this way is an action of f in $\mathfrak{A}$. Obviously, the coefficient of $(T - \lambda I)$ in (3.5) depends on T.

There are actions of analytic functions other than the one defined by the Cauchy integral formula or the action described above which are of interest to us. They are important because for these actions we can define a higher-order system of derivations which in turn determines the behavior of $f(T)$ on the spaces $\ker(\lambda I - T)^{\nu(\lambda)}$, $\lambda \in \sigma_0(T)$. Consequently, we make the following definition.

DEFINITION 3.6. A strong action $T \to f(T)$ of a function f in $\mathfrak{A}$ is a D-action if it satisfies the following condition for every pair of operators $T_1, T_2 \in \mathfrak{A}_U$:

(D) If $T_1, T_2 \in \mathfrak{A}_U$ and $\lambda \in \sigma_0(T_1) \cap \sigma_0(T_2)$, or $T_1, T_2 \in \mathfrak{A}_U \cap C_0$ and $\lambda \in \sigma(T_1) \cap \sigma(T_2)$, and if $f(T_i)E_\lambda(T_i) = \sum_{j=0}^{\nu_i - 1} (1/j!) a_{ij}(T_i - \lambda I)^j E_\lambda(T_i)$ $(i = 1, 2;\ \nu_i$ the index of λ for $T_i)$, then $a_{1j} = a_{2j}$ for $j = 0, 1, \cdots, \min(\nu_1, \nu_2) - 1$. We say that a strong action $\varphi\colon \mathscr{S} \times \mathfrak{A}_U \to \mathfrak{A}$ of a set $\mathscr{S}$ of functions, each defined on U, is a D-action of $\mathscr{S}$ in $\mathfrak{A}$ if for each $f \in \mathscr{S}$, the mapping $\varphi(f, \cdot)\colon \mathfrak{A}_U \to \mathfrak{A}$ is a D-action of f in $\mathfrak{A}$.

It is possible to prove that if an action of a function satisfies certain conditions involving invertible operators and projections, then the action is a D-action. This proof is omitted here because it does not contribute to the general purpose of the paper which is to show the relationship between systems of derivations and actions.

4. *D-actions and systems of derivations.* In §2 we showed that if $\mathscr{M}$ is an algebra of functions defined on U, $\{D_k\colon 0 \leqq k < m\}$ is a system of derivations from $\mathscr{M}$ to the algebra of all functions on U, and D_0 is the identity operator, then there exists an action of $\mathscr{M}$ in any algebra $\mathfrak{A}$ such that $\mathfrak{A} \subset C_0$ and $\sup\{\nu_T(\lambda)\colon T \in \mathfrak{A}, \lambda \in \sigma(T)\} \leqq m$. This action, defined by equation (2.3), satisfies condition (D) of Definition 3.6. In this section we prove that every D-action defined by $\mathscr{M}$ in a closed subalgebra $\mathfrak{A}$ of an algebra of compact operators has associated with it a higher-order system of derivations.

Throughout this section we assume $\mathfrak{A}$ contains an element with nonzero spectrum and $m = \sup\{\nu_T(\lambda): T \in \mathfrak{A}, \lambda \in \sigma_0(T)\}$.

PROPOSITION 4.1. *If f defines a D-action $T \to f(T)$ $(T \in \mathfrak{A}_U)$ in $\mathfrak{A}$, then there exists a family $\{D_k f: 0 \leq k < m\}$, of functions defined on U such that for each $T \in \mathfrak{A}_U$ and $\lambda \in \sigma_0(T)$, or $T \in \mathfrak{A}_U \cap C_0$ and $\lambda \in \sigma(T)$,*

$$(4.1) \qquad f(T)E_\lambda(T) = \sum_{j=0}^{\nu(\lambda)-1} \frac{1}{j!} D_j f(\lambda)(T - \lambda I)^j E_\lambda(T) .$$

Proof. Suppose first that $\dim H < \infty$ and $\mathfrak{A}$ contains an identity. In this case $m \leq \dim H$ and there exists $T \in \mathfrak{A}$ and $\lambda \in \sigma(T)$ such that $\nu_T(\lambda) = m$. For each $z \in U$, set $T_z = \sum_{\lambda \in \sigma(T)} (T - (\lambda - z)I)E_\lambda(T) = T - \sum_{\lambda \in \sigma(T)} (\lambda - z)E_\lambda(T)$. Then T_z has the single eigenvalue z, the index of z for T_z is m and $T_z \in \mathfrak{A}_U$. Therefore there exist constants $a_0, a_1, \cdots, a_{m-1}$ such that $a_0 = f(z)$ and $f(T_z) = \sum_{j=0}^{m-1} (1/j!)a_j(T_z - zI)^j$ (Theorem 3.4). Set $D_j f(z) = a_j$, $j = 0, 1, \cdots, m-1$. That $D_j f: U \to \boldsymbol{C}$ is well-defined is an immediate consequence of the definition of a D-action.

Now suppose $\dim H < \infty$ and $\mathfrak{A}$ does not contain an identity or $\dim H = \infty$. Choose an integer k such that $0 \leq k < m$ and $T \in \mathfrak{A}$, $\lambda \in \sigma_0(T)$ such that $\nu_T(\lambda) = n > k$. For $z \in U$ set $T_z = [T - (\lambda - z)I]E_\lambda(T)$. Then $T_z \in \mathfrak{A}$, $\sigma(T_z) = \{0, z\}$ and n is the index of z for T_z. If $z \neq 0$, then $E_\lambda(T) = E_z(T_z)$, while if $z = 0$, then $T_0 = (T - \lambda I)E_\lambda(T)$. Thus there exist constants a_k, $k = 0, 1, \cdots, n-1$, such that $a_0 = f(z)$ and $f(T_z)E_z(T_z) = \sum_{j=0}^{n-1} (1/j!)a_j(T_z - zI)^j E_\lambda(T)$ (Theorem 3.4). Set $D_k f(z) = a_k$. As a consequence of the definition of a D-action the function $D_k f: U \to \boldsymbol{C}$ is well defined and $f(T)E_\lambda(T)$ is defined by equation (4.1) for $T \in \mathfrak{A}_U$ and $\lambda \in \sigma_0(T)$, or $T \in \mathfrak{A}_U \cap C_0$ and $\lambda \in \sigma(T)$.

Using the algebra homomorphism property of an action of an algebra of functions in $\mathfrak{A}$, we obtain the following theorem relating D-actions and systems of derivations.

THEOREM 4.2. *If $\varphi: \mathscr{M} \times \mathfrak{A}_U \to \mathfrak{A}$ is a D-action in $\mathfrak{A}$ of an algebra $\mathscr{M}$ of functions with domain U, then there exists a system of derivations $\{D_k: 0 \leq k < m\}$ of order $m - 1$ if $m < \infty$, or of infinite order if $m = \infty$, from $\mathscr{M}$ into the algebra of all functions defined on U and D_0 is the identity operator. For $T \in \mathfrak{A}_U$ and $\lambda \in \sigma_0(T)$, or for $T \in \mathfrak{A}_U \cap C_0$ and $\lambda \in \sigma(T)$, $\varphi(f, T)E_\lambda(T)$ is defined by equation (4.1).*

Proof. Since φ is a D-action of $\mathscr{M}$ in $\mathfrak{A}$, for each $f \in \mathscr{M}$ there exist functions $D_k f$, $0 \leq k < m$, such that $\varphi(f, T)E_\lambda(T)$ is defined by

(4.1) for each $T \in \mathfrak{A}_U$ and $\lambda \in \sigma_0(T)$, or $T \in \mathfrak{A}_U \cap C_0$ and $\lambda \in \sigma(T)$ (Proposition 4.1). Define D_k to be the mapping $f \to D_k f$ ($f \in \mathscr{M}$, $0 \leqq k < m$).

Choose an integer $k < m$ and $T \in \mathfrak{A}$, $\lambda \in \sigma_0(T)$ such that $\nu_T(\lambda) = n > k$. Define T_z, $z \in U$, as in the proof of Proposition 4.1. As a consequence of Definition 2.1 we obtain for each $z \in U$, $\alpha \in C$, and $f, g \in \mathscr{M}$,

$$\sum_{j=0}^{n-1} \frac{1}{j!} D_j(\alpha f + g)(z)(T_z - zI)^j E_\lambda(T)$$
$$= \sum_{j=0}^{n-1} \frac{1}{j!}(\alpha D_j f(z) + D_j g(z))(T_z - zI)^j E_\lambda(T)$$

and

$$\sum_{j=0}^{n-1} \frac{1}{j!} D_j(fg)(z)(T_z - zI)^j E_\lambda(T)$$
$$= \sum_{k=0}^{n-1} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} D_j f(z) D_{k-j} g(z)(T_z - zI)^k E_\lambda(T) .$$

The operators $(T_z - zI)^j E_\lambda(T)$, $0 \leqq j < n$, form a linearly independent set so that we must have $D_k(\alpha f + g)(z) = \alpha D_k f(z) + D_k g(z)$ and $D_k(fg)(z) = \sum_{j=0}^{k} \binom{k}{j} D_j f(z) D_{k-j} g(z)$. Thus $\{D_k : 0 \leqq k < m\}$ is a system of derivations from $\mathscr{M}$ to the algebra of all functions defined on U.

We say that the system of derivations of Theorem 4.2 is the system of derivations associated with the D-action φ of $\mathscr{M}$ in $\mathfrak{A}$.

5. **Continuous actions.** Suppose f is a function analytic on U and $T \to f(T)$ is the strong action of f in $\mathfrak{A}$ defined by the Cauchy integral formula. This action has the additional property that if a sequence $\langle T_n \rangle$ converges to $T \in \mathfrak{A}_U$, then $f(T_n)$ is defined for large n and $\langle f(T_n) \rangle$ converges to $f(T)$ (Theorem 2.2). Suppose f is any other function which defines an action $T \to f(T)$ in $\mathfrak{A}$ with this limit property—that is, whenever a sequence $\langle T_n \rangle \subset \mathfrak{A}_U$ converges to $T \in \mathfrak{A}_U$, then $\lim_{n\to\infty} f(T_n) = f(T)$. Then is f continuous? If the action of f in $\mathfrak{A}$ is a D-action, then is the associated family of functions in $C(U)$? These are the questions we consider in this section.

DEFINITION 5.1. Let $\mathfrak{A}$ be a Banach algebra. A function f defines a continuous action $x \to f(x)$ in $\mathfrak{A}$ if the mapping $x \to f(x)$ ($x \in \mathfrak{A}_U$) is an action of f in $\mathfrak{A}$ and for each sequence $\langle x_n \rangle \subset \mathfrak{A}_U$ with limit $x \in \mathfrak{A}_U$, $\lim_{n\to\infty} f(x_n) = f(x)$.

PROPOSITION 5.2. *If a Banach algebra $\mathfrak{A}$ contains an element* *with nonzero spectrum and f defines a continuous action in $\mathfrak{A}$, then f is continuous.*

Proof. For $y \in \mathfrak{A}$, set $r(y) = \max \{|\lambda| : \lambda \in \sigma(y)\}$. Choose $x \in \mathfrak{A}$ such that $r(x) = 1$ and $1 \in \sigma(x)$. If $\zeta_0 \in U$ and $\langle \zeta_n \rangle \subset U$ is a sequence with limit ζ_0, set $x_n = \zeta_n x$ $(n = 0, 1, 2, \cdots)$. Since $|\zeta_n| = r(x_n)$, x_n is in $\mathfrak{A}_U$ for each n and $\lim_{n\to\infty} x_n = x_0$. The action of f in $\mathfrak{A}$ is continuous so $\lim_{n\to\infty} f(x_n) = f(x_0)$. If $\mathscr{C}$ is a maximal commutative subalgebra of $\mathfrak{A}$ such that $x \in \mathscr{C}$, then $\langle x_n \rangle \subset \mathscr{C}$ and hence $\langle f(x_n) \rangle \subset \mathscr{C}$. Therefore $f(\zeta_n) - f(\zeta_0)$ is in the spectrum of $f(x_n) - f(x_0)$ (Definition 2.1) and $0 = \lim_{n\to\infty} \|f(x_n) - f(x_0)\| \geqq \lim_{n\to\infty} |f(\zeta_n) - f(\zeta_0)|$.

If U is an arbitrary set in the complex plane such that $0 \in U$ if $\mathfrak{A}$ does not have an identity and if $\mathfrak{A}$ contains a nonzero idempotent, then we can prove that every function which defines a continuous action in $\mathfrak{A}$ is continuous. The proof is similar to that of Proposition 5.2 but with the nonzero idempotent replacing x.

It is not true, however, that a continuous function defines a continuous action in $\mathfrak{A}$. Even for an analytic function there exists an action in an algebra $\mathfrak{A}$ which is not continuous.

EXAMPLE 5.3. Let $\mathfrak{A}$ be the algebra of 2×2 matrices and f a function analytic on U. Choose $g \in C(U)$ such that g is not identically one. Define $f_g(T)$, $T \in \mathfrak{A}_U$, as follows: if $\sigma(T) = \{\lambda, \mu\} \subset U$ $(\lambda \neq \mu)$, set $f_g(T) = (\lambda - \mu)^{-1}[f(\lambda)(T - \mu I) - f(\mu)(T - \lambda I)]$ while if $\sigma(T) = \{\lambda\} \subset U$, set $f_g(T) = f(\lambda)I + f'(\lambda)g(\lambda)(T - \lambda I)$. This mapping is an action of f in $\mathfrak{A}$ [2]. Choose $\lambda \in U$ such that $g(\lambda) \neq 1$ and sequences $\langle a_n \rangle, \langle b_n \rangle, \langle d_n \rangle$ from C such that $a_n \neq d_n$ $(n = 1, 2, \cdots)$, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} d_n = \lambda$ and $\lim_{n\to\infty} b_n = 1$. Set $T_n = \begin{pmatrix} a_n & b_n \\ 0 & d_n \end{pmatrix}$ and $T = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$. Then $\sigma(T_n) = \{a_n, d_n\}$ and $\sigma(T) = \{\lambda\}$, while $f_g(T) = \begin{pmatrix} f(\lambda) & f'(\lambda)g(\lambda) \\ 0 & f(\lambda) \end{pmatrix}$ and

$$f_g(T_n) = \begin{pmatrix} f(a_n) & b_n(a_n - d_n)^{-1}[f(a_n) - f(d_n)] \\ 0 & f(d_n) \end{pmatrix}.$$

Consequently, the limit of the sequence $f_g(T_n)$ is $\begin{pmatrix} f(\lambda) & f'(\lambda) \\ 0 & f(\lambda) \end{pmatrix} \neq f_g(T)$. Therefore, the action $T \to f_g(T)$ of f in $\mathfrak{A}$ is not continuous.

Now let us return to the setting of a closed subalgebra $\mathfrak{A}$ of a Banach algebra $\mathscr{B}$ of compact operators. We assume again that $\mathfrak{A}$ contains an element with nonzero spectrum and set $m = \sup \{\nu_T(\lambda) : T \in \mathfrak{A}, \lambda \in \sigma_0(T)\}$. Each function which defines a continuous D-action in $\mathfrak{A}$ is continuous. If the action is both continuous and a D-action, then are the functions $D_k f$, $0 \leqq k < m$, associated with the action also continuous?

THEOREM 5.4. *Let $T \to f(T)$ be a continuous D-action of f in $\mathfrak{A}$. Then the functions $D_j f$, $0 \leqq j < m$, associated with the action are continuous.*

Proof. We prove the proposition by induction. Since $\mathfrak{A}$ contains an element with nonzero spectrum, $f = D_0 f$ is continuous on U (Proposition 5.2). Suppose $D_j f$ is continuous for $0 \leqq j < k$ $(k < m)$.

Choose $z_0 \in U$ and a sequence $\langle z_n \rangle \subset U$ converging to z_0. Choose $T \in \mathfrak{A}$ and $\lambda \in \sigma_0(T)$ such that $\nu_T(\lambda) = \nu > k$. Set $T_n = [T - (\lambda - z_n)I]E_\lambda(T)$ for $n = 0, 1, 2, \cdots$. Then $\lim_{n\to\infty} T_n = T_0$ so that $\lim_{n\to\infty} f(T_n) = f(T_0)$.

Let $\{x_{ij}: 1 \leqq i \leqq p, 1 \leqq j \leqq r_i\}$ be a Jordan basis for $E_\lambda(T)H$. Then

$$[f(T_n) - f(T_0)]x_{1,\nu-k} = \sum_{j=0}^{k} \frac{1}{j!} D_j f(z_n) x_{1,\nu-k+j} - \sum_{j=0}^{k} \frac{1}{j!} D_j f(z_0) x_{1,\nu-k+j}$$

so that

$$\text{(5.1)} \qquad |D_k f(z_n) - D_k f(z_0)| \\ \leqq \frac{k!}{\|x_\nu\|} \|f(T_n) - f(T_0)\| \|x_{1,\nu-k}\| + \sum_{j=0}^{k-1} \frac{1}{j!} |D_j f(z_n) - D_j f(z_0)| \|x_{1\nu-k+j}\| .$$

As a result of the induction hypothesis the functions $D_j f$, $0 \leqq j < k$, are continuous. Hence the limit of the expression on the right of (5.1) is zero so that $\lim_{n\to\infty} D_k f(z_n) - D_k f(z_0) = 0$. This proves that $D_k f$ is continuous at z_0.

6. Continuous D-actions and analytic functions.

Suppose an algebra $\mathscr{M}$ of functions defines a strong action φ in $\mathfrak{A}$ and $\mathscr{M}$ contains the function x defined by $x(t) = t$ $(t \in U)$. Then the action of a polynomial $f = \sum_{k=0}^{n} a_k x^k$ ($a_0 = 0$ if $\dim H = \infty$) can be described in terms of the action of x in $\mathfrak{A}$—that is, $\varphi(f, T) = \sum_{k=0}^{n} a_k [\varphi(x, T)]^k$ for all $T \in \mathfrak{A}_U$. An analytic function $f \in \mathscr{M}$ is the limit of polynomials in the topology of uniform convergence on compact sets. Is it possible then to describe the operators $\varphi(f, T)$, $f \in \mathscr{M}$ analytic and $T \in \mathfrak{A}_U$, in terms of $\varphi(x, T)$, perhaps by means of the Cauchy integral formula?

In this section we show that if the action φ of $\mathscr{M}$ in $\mathfrak{A}$ has the property that $\varphi(x, \cdot)$ is a D-action of x in $\mathfrak{A}$ and there exists $0 < r < 1$ such that the restriction of each mapping $\varphi(f, \cdot)$ $(f \in \mathscr{M})$ to the set $\{T \in \mathfrak{A}: \sigma(T) \subset [|z| < r]\}$ is continuous, then, in fact, we can describe the action of an analytic function $f \in \mathscr{M}$ in terms of the Cauchy integral formula and terms of the form $\sum_{j=0}^{\nu(\lambda)-1} (1/j!) D_j f(\lambda)(T - \lambda I)^j E_\lambda(T)$.

Let us set the stage for this theorem. Let $\mathfrak{A}$ be the closed subalgebra of a Banach algebra $\mathscr{B}$ of compact operators such that $\mathfrak{A}$ contains an element with nonzero spectrum. Set $m = \sup\{\nu_T(\lambda): T \in \mathfrak{A}, \lambda \in \sigma_0(T)\}$. Let $\mathscr{M}$ be an algebra of functions defined on U and φ a strong action of $\mathscr{M}$ in $\mathfrak{A}$. We assume the function x defined by $x(t) = t$ $(t \in U)$ is in $\mathscr{M}$ and denote by $P(x)$ the subalgebra of $\mathscr{M}$ consisting of all polynomials in x (if $\dim H = \infty$, then these are the polynomials in x without a constant term). The mapping $\varphi(x, \cdot)$: $\mathfrak{A}_U \to \mathfrak{A}$ is a D-action of x in $\mathfrak{A}$ such that the associated functions $D_j x$ $(0 \leqq j < m)$ are continuous. The action φ of $\mathscr{M}$ in $\mathfrak{A}$ has the additional property that there exists an open disk $V = [|z| < r]$ contained in U such that the restriction of $\varphi(f, \cdot)$ $(f \in \mathscr{M})$ to $\mathfrak{A}_V$ is continuous.

Since x defines a D-action in $\mathfrak{A}$, each polynomial $f \in P(x)$ also defines a D-action in $\mathfrak{A}$. Thus there exists a system of derivations $\{D_k: 0 \leqq k < m\}$ from $P(x)$ into $C(U)$ such that D_0 is the identity operator and $\varphi(f, T)E_\lambda(T) = \sum_{j=0}^{\nu(\lambda)-1} (1/j!)D_j f(\lambda)(T - \lambda I)^j E_\lambda(T)$ for each $f \in P(x)$, $T \in \mathfrak{A}_U$ and $\lambda \in \sigma_0(T)$ (or $T \in \mathfrak{A}_U \cap C_0$ and $\lambda \in \sigma(T)$) (Theorem 4.2). There exist functions $C_{kj} \in C(U)$, $1 \leqq j \leqq k < m$, such that $D_k f = \sum_{j=1}^{k} C_{kj} f^{(j)}$ for $f \in P(x)$, $0 \leqq k < m$ [3]. Thus the functions $D_k f$, $f \in P(x)$, $0 \leqq k < m$, are continuous.

Associated with this system of derivations is a family of multiplicative seminorms (m-seminorms) defined on $P(x)$ for each integer n, $0 \leqq n < m$, and compact set $K \subset U$ by

$$p_{nK}(f) = \sum_{j=0}^{n} \frac{1}{j!} \sup_{z \in K} |D_j f(z)| \,. \tag{6.1}$$

The completion $A(U)$ of $P(x)$ with respect to this family of m-seminorms is the algebra of analytic functions on U if $\dim H < \infty$ or the algebra of analytic functions on U which vanish at zero if $\dim H = \infty$ [3]. Each operator D_k, $0 \leqq k < m$, can be extended uniquely to an operator $\bar{D}_k$ from $A(U)$ into $C(U)$ such that $\{\bar{D}_k: 0 \leqq k < m\}$ is a system of derivations. Furthermore, for each $g \in A(U)$, $\bar{D}_k g$ is defined by the equation $\bar{D}_k g = \sum_{j=1}^{k} C_{kj} g^{(j)}$.

Since $f \in A(U)$ is analytic, we have

$$\lim_{n\to\infty} \sup_{z \in K} \left| f^{(p)}(z) - \sum_{j=0}^{n} \frac{1}{(j-p)!} f^{(j)}(0) z^{j-p} \right| = 0$$

for each compact set $K \subset U$. From this limit we obtain

$$p_{qK}\Big(f - \sum_{j=0}^{n} \frac{1}{j!} f^{(j)}(0) x^j\Big) = \sum_{k=0}^{q} \frac{1}{k!} \sup_{z \in K} \left| D_k\Big(f - \sum_{j=0}^{n} \frac{1}{j!} f^{(j)}(0) x^j\Big)(z) \right|$$

$$= \sum_{k=0}^{q} \frac{1}{k!} \sup_{z \in K} \left| \sum_{p=0}^{k} C_{kp}(z) \left[f^{(p)}(z) - \sum_{j=p}^{n} \frac{1}{(j-p)!} f^{(j)}(0) z^{j-p} \right] \right|$$

$$\leqq \sum_{k=0}^{q} \frac{1}{k!} \sum_{p=0}^{k} \sup_{z \in K} | C_{kp}(z) | \sup_{z \in K} \left| f^{(p)}(z) - \sum_{j=p}^{n} \frac{1}{(j-p)!} f^{(j)}(0) z^{j-p} \right|$$

so that

$$\lim_{n \to \infty} p_{qK}\Big(f - \sum_{j=0}^{n} \frac{1}{j!} f^{(j)}(0) x^j \Big) = 0 \tag{6.2}$$

for each $0 \leqq q < m$, compact $K \subset U$ and $f \in A(U)$.

As a consequence of the definition of the m-seminorms p_{nK} $(0 \leqq n < m, K \subset U$ compact) on $P(x)$ and hence on $A(U)$ the algebra homomorphism $\varphi(\cdot, T)\colon P(x) \to \mathfrak{A}$ is continuous for each $T \in \mathfrak{A}_U \cap C_0$. Thus $\varphi(f, T)$ $(T \in \mathfrak{A}_U \cap C_0)$ can be extended to a continuous algebra homomorphism $\bar{\varphi}(\cdot, T)\colon A(U) \to \mathfrak{A}$. Thus, since $\sigma(\varphi(x, T)) = \sigma(T)$ if $T \in \mathfrak{A}_U$ [2], we obtain for each $T \in \mathfrak{A}_V \cap C_0$ and $f \in A(U)$,

$$\begin{aligned}
\bar{\varphi}(f, T) &= \lim_{n \to \infty} \varphi\Big(\sum_{j=0}^{n} \frac{1}{j!} f^{(j)}(0) x^j, T \Big) = \lim_{n \to \infty} \sum_{j=0}^{n} \frac{1}{j!} f^{(j)}(0) [\varphi(x, T)]^j \\
&= \lim_{n \to \infty} \sum_{j=0}^{n} \frac{1}{j!} f^{(j)}(0) \frac{1}{2\pi i} \int_{|z|=r} z^j [zI - \varphi(x, T)]^{-1} dz \\
&= \lim_{n \to \infty} \frac{1}{2\pi i} \int_{|z|=r} \sum_{j=0}^{n} \frac{1}{j!} f^{(j)}(0) z^j [zI - \varphi(x, T)]^{-1} dz \\
&= \frac{1}{2\pi i} \int_{|z|=r} f(z) [zI - \varphi(x, T)]^{-1} dz \,.
\end{aligned}$$

Moreover, if $T \in \mathfrak{A}_U \cap C_0$, then $T_r = T(\sum_{|\lambda| < r} E_\lambda(T))$ is in $\mathfrak{A}_V \cap C_0$ so that for each $f \in A(U)$

$$\begin{aligned}
\bar{\varphi}(f, T) &= \bar{\varphi}(f, T_r) + \sum_{|\lambda| \geqq r} \sum_{j=0}^{\nu(\lambda)-1} \frac{1}{j!} \bar{D}_j f(\lambda) (T - \lambda I)^j E_\lambda(T) \\
&= \frac{1}{2\pi i} \int_{|z|=r} f(z) [zI - \varphi(x, T_r)]^{-1} dz \\
&\quad + \sum_{|\lambda| \geqq r} \sum_{j=0}^{\nu(\lambda)-1} \frac{1}{j!} \bar{D}_j f(\lambda) (T - \lambda I)^j E_\lambda(T) \,.
\end{aligned} \tag{6.3}$$

If $\mathfrak{A} \cap C_0 = \mathfrak{A}$ (hence for $\dim H < \infty$), we are finished, for this is the desired result. If $\dim H = \infty$, then we use the hypothesis that $\mathfrak{A} \cap C_0$ is dense in $\mathfrak{A}$ to obtain (6.3) for all $T \in \mathfrak{A}_U$, $f \in A(U) \cap \mathscr{M}$.

Suppose $T \in \mathfrak{A}_U$, $P = \sum_{|\lambda| \geqq r} E_\lambda(T)$ and $T_r = T - TP$. Then $\sigma(T_r) \subset V$ and there exists a sequence $\langle T_n \rangle \subset \mathfrak{A} \cap C_0$ such that $\lim_{n \to \infty} | T_n - T_r | = 0$ and $\sigma(T_n) \subset V$ [1, p. 568]. For each $f \in A(U) \cap \mathscr{M}$ the restriction of $\varphi(f, \cdot)$ to $\mathfrak{A}_V$ is continuous so that $\langle \varphi(f, T_n) \rangle$ converges to $\varphi(f, T_r)$. But

$$\varphi(f, T_n) = \frac{1}{2\pi i}\int_{|z|=r} f(z)[zI - \varphi(x, T_n)]^{-1}dz$$

and

$$\lim_{n\to\infty} [zI - \varphi(x, T_n)]^{-1} = [zI - \varphi(x, T_r)]^{-1}$$

(this last a consequence of the continuity of the action $\varphi(x, \cdot)$ on $\mathfrak{A}_V$ and of inversion). Thus $\varphi(f, T_r) = 1/(2\pi i)\int_{|z|=r} f(z)[zI - \varphi(x, T_r)]^{-1}dz$.

In order to complete the proof we need to show that $\varphi(f, T_r) = \varphi(f, T_r)(I - P)$. To see this choose a sequence $\langle T_n\rangle \subset C_0 \cap \mathfrak{A}$ with limit T and set $P_n = \sum \{E_\lambda(T_n): \lambda \in V \cap \sigma(T_n)\}$. Since $\langle T_n\rangle \subset C_0$, we have $\varphi(f, T_nP_n) = \sum_{|\lambda|<r} \sum_{j=0}^{\nu(\lambda)-1} (1/j!)D_jf(\lambda)(T_nP_n - \lambda I)^jE_\lambda(T_n)$ so that $\varphi(f, T_nP_n) = \varphi(f, T_n)P_n$. Since the characteristic function of V is analytic in a neighborhood of $\sigma(P_n)$ $(n = 1, 2, \cdots)$ and of $\sigma(I - P)$, the sequence $\langle P_n\rangle$ converges to $I - P$ [cf. 1, p. 1101]. Therefore $\lim_{n\to\infty} T_nP_n = T_r$ and $\lim_{n\to\infty} \varphi(f, T_r) = \lim_{n\to\infty} \varphi(f, T_nP_n)P_n = \varphi(f, T_r)(I - P)$. Combining these results we obtain

$$\begin{aligned} \varphi(f, T) &= \varphi(f, T_r) + \sum_{|\lambda|\geq r} \varphi(f, T)E_\lambda(T) \\ &= \frac{1}{2\pi i}\int_{|z|=r} f(z)[zI - \varphi(x, T_r)]^{-1}dz \\ &\quad + \sum_{|\lambda|\geq r} \sum_{j=0}^{\nu(\lambda)-1} \frac{1}{j!}\bar{D}_jf(\lambda)(T - \lambda I)^jE_\lambda(T)\,. \end{aligned} \tag{6.4}$$

Finally we show that the action φ can be extended from $A(U) \cap \mathscr{M}$ to all of $A(U)$. This follows if $\varphi(\cdot, T): A(U) \cap \mathscr{M} \to \mathfrak{A}$ is continuous for each $T \in \mathfrak{A}_U$. Suppose $\langle f_n\rangle \subset A(U) \cap \mathscr{M}$ converges to $f \in A(U) \cap \mathscr{M}$, $T \in \mathfrak{A}_U$, $P = \sum_{|\lambda|\geq r} E_\lambda(T)$ and $T_r = T - TP$. Then the sequence $\langle f_n\rangle$ converges to f uniformly on compact sets and $|[zI - \varphi(x, T_r)]^{-1}|$ is uniformly bounded on V [cf. 1, p. 1101] so that

$$\begin{aligned} |\varphi(f_n, T_r) - \varphi(f, T_r)| &= \left|\frac{1}{2\pi i}\int_{|z|=r} (f_n(z) - f(z))[zI - \varphi(x, T_r)]^{-1}dz\right| \\ &\leqq r \max_{|z|=r} |f_n(z) - f(z)| \max_{|z|=r} |[zI - \varphi(x, T_r)]^{-1}|\,. \end{aligned}$$

Set $q = \max \{\nu_T(\lambda) - 1: \lambda \in \sigma(T)\backslash V\}$ and $K = \sigma(T) \cap V$. Then

$$\begin{aligned} \varphi(f_n - f, T)P &= \sum_{|\lambda|\geq r} \sum_{j=0}^{q} \frac{1}{j!}\bar{D}_j(f_n - f)(\lambda)(T - \lambda I)^jE_\lambda(T) \\ &\leqq \max\{|(T - \lambda I)^jE_\lambda(T)|: \lambda \in \sigma(T)\backslash V, 0 \leqq j \leqq q\}p_{qK}(f_n - f)\,. \end{aligned}$$

Therefore, $\lim_{n\to\infty} |\varphi(f_nT) - \varphi(f, T)| = 0$.

Let $\psi: A(U) \times \mathfrak{A}_U \to \mathfrak{A}$ be the extension of φ from $(A(U) \cap \mathscr{M}) \times \mathfrak{A}_U$ to $A(U) \times \mathfrak{A}_U$. As a consequence of the continuity of $\varphi(\cdot, T)$ for each $T \in \mathfrak{A}_U$, $\psi(f, T)$ also is defined by equation (6.4) if $f \in A(U)$. This completes the proof of Theorem 6.1.

THEOREM 6.1. *Let $\mathscr{M}$ be an algebra of functions which defines a strong action φ in $\mathfrak{A}$ such that for some open disk $V = [|z| < r] \subset U$ the restriction of $\varphi(f, \cdot)$ to $\mathfrak{A}_V$ is continuous for all $f \in \mathscr{M}$. Suppose $x \in \mathscr{M}$ $(x(t) = t)$ and $\varphi(x, \cdot)$ is a D-action of x in $\mathfrak{A}$ such that the associated functions $D_k x$, $0 \leqq k < m$, are continuous. If* $\dim H < \infty$ *or if $\dim H = \infty$ and $\mathfrak{A} \cap C_0$ is dense in $\mathfrak{A}$, then the algebra $A(U)$ of all functions analytic on U which vanish at zero if* $\dim H = \infty$ *defines a D-action ψ in $\mathfrak{A}$ such that for all $T \in \mathfrak{A}_U$* (i) *$\psi(f, T) = \varphi(f, T)$ if $f \in A(U) \cap \mathscr{M}$ and* (ii) *$\psi(f, T)$ is defined by* (6.4) *for all $f \in A(U)$.*

Theorem 6.1 remains true when we replace the open unit disk by a simply-connected open set U (with $0 \in U$ if $\dim H = \infty$). The proof requires only minor changes.

It is obvious that the identity mapping $T \rightarrow T$ is a continuous D-action in $\mathfrak{A}$ of the function x. The corresponding action of $A(U)$ in $\mathfrak{A}$ is the action defined by the usual Cauchy integral formula. The following example shows that there are actions $T \rightarrow x(T)$ of x in $\mathfrak{A}$ such that $x(T) \neq T$ for some $T \in \mathfrak{A}_U$. It is true, however, that for each such action there exists $\delta > 0$ such that if $T \in \mathfrak{A}_U$, then $[T - x(T)][I - \sum \{E_\lambda(T): |\lambda| \geqq \delta\}] = 0$. It is not known if there is an algebra $\mathfrak{A}$ containing an operator of infinite rank and an action $T \rightarrow x(T)$ of x in $\mathfrak{A}$ such that for each $\delta > 0$ the set $\{[x(T) - T]E_\lambda(T): T \in \mathfrak{A}_U, \lambda \in \sigma_0(T), |\lambda| < \delta\}$ is different from $\{0\}$.

EXAMPLE 6.2. Let $0 < \delta < 1$ be chosen and $g_1, g_2, \cdots$ be continuous functions on U such that for $|z| < \delta$ $g_1(z) = 1$ and $g_j(z) = 0$ $(j = 2, 3, \cdots)$. Set $g_0 = x$. For each $T \in \mathfrak{A}_U$ define $x(T)$ by

$$\begin{aligned} x(T) = {} & T(I - \sum \{E_\lambda(T): |\lambda| \geqq \delta\}) \\ & + \sum\left\{\frac{1}{j!}g_j(\lambda)(T - \lambda I)^j E_\lambda(T): 0 \leqq j < \nu(\lambda), |\lambda| \geqq \delta\right\}. \end{aligned}$$

The mapping $T \rightarrow x(T)$ is a D-action of x in $\mathfrak{A}$ which is the identity mapping when restricted to the set $\{T \in \mathfrak{A}: \sigma(T) \subset [|z| < \delta]\}$. If $\mathscr{M} = P(x)$, then we obtain an action of $A(U)$ in $\mathfrak{A}$ which is distinct from the Cauchy integral formula action if one of the functions g_j is not constant on U. In fact, for each $\delta > 0$ and each choice of functions g_j continuous on U and identically one if $j = 1$ or identically zero if $j \geqq 2$ on the disk $[|z| < \delta]$, we obtain a different action of $A(U)$ in $\mathfrak{A}$.

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