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BAD BEHAVIOR AND INCLUSION RESULTS FOR MULTIPLIERS OF TYPE (p, q)

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To my teachers, Bert Bundesen and Peter Lettice

This paper is concerned with the space of multipliers from $L^p(G)$ to $L^q(G)$ for various pairs of indices p and q, where G is an LCA group. We show that if $1 \leqq p < 2 < q \leqq \infty$, and G is noncompact, then there are multipliers of type (p, q) whose 'Fourier transforms' are not measures. This is an extension of a result of Hörmander, and completes work begun in two earlier papers (this journal, 1966). In the second part, we show that if G is infinite, many of the natural inclusion relations between spaces of multipliers are proper.

In his paper [10], Hörmander established a large number of important results for multipliers from $L^p(R^n)$ to $L^q(R^n)$. Subsequently, many of the results of the early parts of Hörmander's paper have been extended, by using quite different techniques, to the case where R^n is replaced by a general (usually noncompact) LCA group. See Figà-Talamanca [2], Gaudry [5], [6], [7] and Figà-Talamanca and Gaudry [3]. However, some of Hörmander's results (notably the general form of his Theorem 1.9) have remained hitherto inaccessible with only the techniques of the cited papers available.

The main purpose of this paper is to give a simple, all-embracing approach which allows us to complete the process of generalization and, moreover, provides a much simpler approach to many of the results of [2], [3], [5], [6] and [7]. As an extra bonus, we are able to show that the natural inclusion relations between spaces of multipliers are proper whenever the underlying group is infinite. One such result (Theorem 4.1) yields a qualitative extension of Theorem 2.4 of Hörmander's paper.

To set the notation and terminology, G and X will denote LCA groups in duality. For $1 \leqq p \leqq \infty$, write $L^p(G)$ for the usual Lebesgue space constructed relative to Haar measure on G. The spaces $C_c(G)$, $M(G)$ and $M_{bd}(G)$ will be the spaces of continuous functions with compact supports, of Radon measures, and of bounded Radon measures on G respectively. $\hat{S}$ will denote the Fourier transform of the object S whenever it is defined.

For $1 \leqq p \leqq q \leqq \infty$, the space L_q^p of multipliers of type (p, q) is defined as follows. When $p < \infty$, it is the space of continuous linear operators T from L^p to L^q which commute with translations: $T\tau_a = \tau_a T$ for all $a \in G$, where $\tau_a f(x) = f(x - a)$. In case $p = \infty$, it is further required that each T be continuous for the weak* topologies on domain and range spaces. The norm on L_p^q will be denoted $\|T\|_{p,q}$. It can be shown [6] that the space L_p^q is identifiable with a certain subspace of the space of 'quasimeasures' on G (for the definition of the space of quasimeasures, see [5]), each $T \in L_p^q$ being defined by convolution with a unique quasimeasure. Further, the space L_p^q can be identified in a natural way with the space $M_p^q(X)$ of Fourier transforms of representing quasimeasures. The elements of $M_p^q(X)$ are quasimeasures on X. We shall write $\hat{T}$ for the Fourier transform of the quasimeasure which represents T. The results we present in § 3 can be thought of as showing that the elements of $M_p^q(X)$ are in general very for from being 'smooth'.

2. The basic construction. The methods used below center around a construction, for general LCA groups, of analogues of the Rudin-Shapiro polynomials. The latter objects are usually defined on the circle group: see [11, Exercise 6, p. 33]. A construction similar to that given below has been used by Hewitt and Ross [9] in a different but related context.

LEMMA 2.1. *Let G and X be* LCA *groups in duality, with G noncompact. Suppose that Ω is a fixed open relatively compact subset of X and that $\varphi \equiv \varphi_0$ is a nonzero function in $C_c(X)$ with support in Ω and $\hat{\varphi} \in L^1(G)$. Then if $\delta > 0$ is small and arbitrary, and n is an arbitrary positive integer, there is a function $\varphi_n \in C_c(X)$ supported by Ω with the properties:*

(i) $\hat{\varphi}_n \in L^1(G)$;

(ii) $\|\varphi_n\|_\infty \leqq 2^{(n+1)/2}\|\varphi\|_\infty$;

(iii) $\|\hat{\varphi}_n\|_\infty \leqq C(1+\delta)^n$;

(iv) $\|\hat{\varphi}_n\|_2 \geqq D(2-\delta)^{n/2}$,

where C and D are positive constants independent of n and δ. When G is discrete, δ may be taken to be zero.

Proof. Define the sequences $(\varphi_k)_0^n$, $(\psi_k)_0^n$ inductively as follows. Choose $\varphi_0 = \psi_0 = \varphi \in C_c(X)$. For $k > 0$ define

$$(1) \qquad \begin{cases} \varphi_k = \varphi_{k-1} + \chi_{k-1}\psi_{k-1} \\ \psi_k = \varphi_{k-1} - \chi_{k-1}\psi_{k-1} \end{cases}$$

where the χ_{k-1} are characters chosen (by using the noncompactness of G) so that

$$(2) \qquad \begin{cases} \|\hat{\varphi}_k\|_\infty \leqq (1+\delta)\max(\|\hat{\varphi}_{k-1}\|_\infty, \|\hat{\psi}_{k-1}\|_\infty) \\ \|\hat{\psi}_k\|_\infty \leqq (1+\delta)\max(\|\hat{\varphi}_{k-1}\|_\infty, \|\hat{\psi}_{k-1}\|_\infty) \end{cases}$$

and

$$\begin{cases} \|\hat{\varphi}_k\|_2^2 \geqq (2-\delta)\min(\|\hat{\varphi}_{k-1}\|_2^2, \|\hat{\psi}_{k-1}\|_2^2) \\ \|\hat{\psi}_k\|_2^2 \geqq (2-\delta)\min(\|\hat{\varphi}_{k-1}\|_2^2, \|\hat{\psi}_{k-1}\|_2^2)\,. \end{cases} \tag{3}$$

It is not difficult to check that φ_n so defined does indeed satisfy the conclusions of the lemma (observe that

$$|\varphi_k|^2 + |\psi_k|^2 = 2(|\varphi_{k-1}|^2 + |\psi_{k-1}|^2) = \cdots = 2^{k+1}|\varphi_0|^2)$$

with $C = \|\varphi_0\|_1$, $D = \|\varphi_0\|_2$.

REMARK. If G is discrete, $X = \Omega$ and $\varphi_0 = 1$, the function $\hat{\varphi}_n$ takes the values $0, \pm 1$ and has precisely 2^n points in its support.

3. Multipliers whose transforms are not measures. Hörmander [10, Th. 1.9] proved that when $G = R^n$ and $1 \leqq p < 2 < q \leqq \infty$, there are multipliers of type (p, q) whose Fourier transforms are not measures. His proof depends on the crucial Lemma 1.2, which is in turn heavily dependent on the fact that the underlying group is R^n. Our first main result is that Hörmander's theorem continues to hold when R^n is replaced by any noncompact LCA group. This has previously been established ([3, Th. 2.5]; [7, Th. 6.6]) in the case where $p = 1$ by methods which are quite different from those employed here to establish the more general result.

Before proceeding, it will be useful to recall that if p' denotes the index conjugate to p, then $L_p^q = L_{q'}^{p'}$ with equality of norms. Further, in the triangle $x \geqq y$, $0 \leqq x \leqq 1$, $\log \|T\|_{p,q}$ is a convex function of $(1/p, 1/q)$. For these facts we refer the reader to [10, Th. 1.3] or [1, Chapter 16].

THEOREM 3.1. *Let G be a noncompact* LCA *group, $1 \leqq p < 2 < q \leqq \infty$. Then there are multipliers of type (p, q) whose Fourier transforms are not measures.*

Proof. Suppose the contrary: then if Ω is any open relatively compact set in X, the mapping $T \to \hat{T}|\Omega$ (the restriction of $\hat{T}$ to Ω) carries $L_p^q(G)$ into $M(\Omega)$, the space of (bounded) Radon measures on Ω. Then by the closed graph theorem, there is a constant K with

$$\int_\Omega d|\hat{T}|\Omega| \leqq K\|T\|_{p,q} \tag{4}$$

for all $T \in L_p^q(G)$. To show that the graph of the mapping is closed, one uses for example the fact that the mapping $T \to \hat{T}$ is continuous from $L_p^q(G)$ into the space of quasimeasures on X, the latter space being endowed with its weak* topology. For this, see [6]. By the duality result referred to above, we may suppose without loss of generality that the point $(1/p, 1/q)$ lies in the triangle bounded by the lines $x = 1, y = 1/2, x + y = 1$. Join the points $(1/p, 1/q)$ and $(1/2, 1/2)$ by a straight line, and let it meet the line $x = 1$ in the point $(1, 1/s)$. Then $2 < s$. By convexity,

$$\| T \|_{p,q} \leqq \| T \|_{2,2}^{\alpha} \| T \|_{1,s}^{1-\alpha}$$

where $1/p = \alpha/2 + (1 - \alpha)/1$, $1/q = \alpha/2 + (1 - \alpha)/s$. But it is known [10, Ths. 1.4, 1.5] that $L_2^2(G) = L^\infty(X)$ with $\| T \|_{2,2} = \| \hat{T} \|_\infty$ and that $L_1^s(G) = L^s(G)$ with $\| T \|_{1,s} = \| T \|_{L^s(G)}$ if $1 < s \leqq \infty$, in the first case T being defined by pointwise multiplication of Fourier transforms by the element of $L^\infty(G)$ with which it is identified, and in the second case the operation being ordinary convolution. So (4) yields

$$\int_\Omega d|\hat{T}|\Omega| \leqq K \| \hat{T} \|_\infty^{\alpha} \| T \|_{L^s(G)}^{1-\alpha} \tag{5}$$

for all $T \in L^1(G)$ with $\hat{T} \in C_c(X)$ say.

Choose $\varphi = \varphi_0$ to be a function in $C_c(X)$ with support in Ω, $\|\varphi\|_\infty = 1$, and $\hat{\varphi} \in L^1(X)$. For each positive integer n, let φ_n be the function defined in Lemma 2.1 with $\delta = \delta_n = 1/n$. Since $\hat{\varphi}_n \in L^1 \cap L^\infty(G)$, it follows that $\varphi_n \in M_p^q(X)$. Define $\rho_n = \varphi_n/2^{(n+1)/2}$. Then (a) $\|\rho_n\|_\infty \leqq 1$; (b) $\|\hat{\rho}_n\|_\infty \leqq C(1 + 1/n)^n 2^{-1/2(n+1)} \to 0$ as $n \to \infty$; and (c)

$$\|\hat{\rho}_n\|_2 \geqq (1 - 1/2n)^{n/2} D 2^{-1/2} \to \exp(-1/4) D 2^{-1/2} \neq 0$$

as $n \to \infty$. Since ρ_n is supported by the fixed relatively compact set Ω, it follows from (a) and (c), Plancherel's theorem and Hölder's inequality that $\int_\Omega |\rho_n(\chi)| d\chi$ does not tend to zero as $n \to \infty$. On the other hand, since $\|\hat{\rho}_n\|_\infty \to 0$ and $\|\hat{\rho}_n\|_2$ is bounded, it follows from Hölder's inequality that $\|\hat{\rho}_n\|_s \to 0$. Substituting $\hat{T} = \rho_n$ in (5) and letting $n \to \infty$, we have a contradiction.

REMARK 3.2. If $1 \leqq p \leqq q \leqq 2$ (equivalently $2 \leqq p \leqq q \leqq \infty$), it is easy to show that $M_p^q \subset L_{\mathrm{loc}}^{q'}(X)$ (resp. $L_{\mathrm{loc}}^p(X)$). For this see [10, Th. 1.6].

4. Proper inclusion relations. It is known [10, § 1.2] that $L_1^1 = M_{bd}(G)$, that $L_2^2(G) = L^\infty(X)$, and that if $1 < p_1 < p_2 < 2$, then $L_1^1 \subset L_{p_1}^{p_1} \subset L_{p_2}^{p_2} \subset L_2^2$. It has recently been shown ([3], [12]) that when G is infinite, the above inclusions are all strict. In this final section, we wish to prove results of a similar type. It should be noted that the elementary techniques used below can be applied to establish the results on proper inclusions contained in [3] and [12].

Young's inequality, restated, yields the information that if $1 <$

$p < q < \infty$ and $1/p - 1/q = 1 - 1/r$, then $L^r(G) \subset L_p^q(G)$. By using the fact that $L_p^q = L_{q'}^{p'}$ for all pairs of indices (p, q) and applying the Riesz convexity theorem, we see that if $1 < p_0 < q_0 < \infty$, $p_0 = q_0'$, $1/p_0 - 1/q_0 = 1 - 1/r$, $1 < p_1 < p_2 < p_0 < 2$ and $1/p_i - 1/q_i = 1 - 1/r$ $(i = 1, 2)$, then $L^r \subset L_{p_1}^{q_1} \subset L_{p_2}^{q_2} \subset L_{p_0}^{q_0}$. It is our aim to show that whenever G is infinite, the inclusions are all strict.

Let us remark once for all that since for all pairs of indices (p, q), $L_p^q = L_{q'}^{p'}$, there is another set of results which can be obtained immediately from those below simply by passing to the conjugate pairs of indices.

The proof that all the inclusions are strict proceeds step by step as follows.

THEOREM 4.1. *Let G be an infinite* LCA *group*, $1 < p < q < \infty$. *and* $1/p - 1/q = 1 - 1/r$. *Then the inclusion* $L^r(G) \subset L_p^q(G)$ *is strict.*

Proof. The proof divides into two cases.

(i) G is discrete. If $L_p^q(G) = \ell^r(G)$, then there is a constant K with

$$||\hat{t}||_r \leqq K ||\hat{t}||_{p,q} \tag{6}$$

for all trigonometric polynomials t on X (closed graph theorem). We suppose without loss of generality that the point $(1/p, 1/q)$ lies in the triangle bounded by the lines $x = 1, x + y = 1, x = y$. Define the point $(1, 1/s)$ as in the proof of Theorem 3.1. Then by convexity,

$$||\hat{t}||_{p,q} \leqq ||t||_\infty^\alpha ||\hat{t}||_s^{1-\alpha}$$

where $1/q = \alpha/2 + (1 - \alpha)/s$, so that (6) implies

$$||\hat{t}||_r \leqq K ||t||_\infty^\alpha ||\hat{t}||_s^{1-\alpha} \tag{7}$$

for all trigonometric polynomials t on X. Since G is discrete, we may define the Rudin-Shapiro sequence (φ_n) on X as in the lemma with $X = \Omega$ so as to satisfy $||\hat{\varphi}_n||_\infty = 1$, $||\varphi_n||_\infty \leqq 2^{(n+1)/2}$, $||\hat{\varphi}_n||_s = 2^{n/s}$, $||\hat{\varphi}_n||_r = 2^{n/r}$, each φ_n being a trigonometric polynomial. Substituting in (7), we get

$$2^{n/r} \leqq K 2^{n\alpha/2} 2^{\alpha/2} 2^{(1-\alpha)n/s} = K 2^{n/q} 2^{\alpha/2} . \tag{8}$$

But $1/p + 1/r - 1 = 1/q$ and $p > 1$. So $1/q < 1/r$, and (8) is contradicted when $n \to \infty$.

(ii) G is nondiscrete; i.e., X is noncompact. As before, if we assume that $L_p^q = L^r(G)$, we get an inequality

$$||\varphi||_r \leqq K ||\hat{\varphi}||_\infty^\alpha ||\varphi||_s^{1-\alpha} \tag{9}$$

for all $\varphi \in C_c(G)$ say. Now manufacture a sequence (φ_n) of functions, supported this time by a fixed open relatively compact subset of G, with $\|\varphi_n\|_\infty$ bounded, $\|\hat{\varphi}_n\|_\infty \to 0$ and $\|\varphi_n\|_2 \not\to 0$. Each φ_n is in $C_c(G)$. For such a sequence (φ_n), $\|\varphi_n\|_r$ does not tend to zero if $r < 2$ since if it did, Hölder's inequality would imply that $\|\varphi_n\|_2 \to 0$. Again, if $r \geqq 2$, $\|\varphi_n\|_r$ does not tend to zero since all the functions φ_n are supported by a fixed compact set and $\|\varphi_n\|_2$ does not tend to zero. In either case, we have a contradiction of (9).

COROLLARY 4.2. *If G is an infinite* LCA *group and*

$$1 < p < q < \infty,\ 1/p - 1/q = 1 - 1/r\ ,$$

then $L^p * L^r(G) \neq L^q(G)$.

Proof. Actually more than this is true. For if the space $A_p^{r'}$ is defined as in [4], then the dual of $A_p^{r'}$ is $L_{p'}^{r'}$ [4, Th. 2]. However, $L^p * L^r(G) \subset A_p^{r'} \subset L^q(G)$, and since $L_{p'}^{r'}(G) \neq L^{q'}(G)$, it is easy to deduce that $A_p^{r'}(G) \neq L^q(G)$.

REMARKS 4.3. (i) When $G = R^n$, there are well-known examples of functions which are in L_p^q but not in $L^r(G)$, for example $\varphi(x) = (1 + |x|)^{-n/r}$. See [10, Th. 2.4].

(ii) Corollary 4.2 is a strong form of a special case of a theorem due to Yap [14].

The second step in the program is to show that if $1 < p_1 < p_0 < q_0 < \infty$, $p_0 = q_0'$, and $1/p_i - 1/q_i = 1 - 1/r$ $(i = 0, 1)$, and G is infinite, then $L_{p_1}^{q_1} \neq L_{p_0}^{q_0}$. We shall treat the noncompact case first, since it is simpler.

THEOREM 4.4. *Let G be an infinite (noncompact)* LCA *group,* $1 < p_0 < q_0 < \infty$, $p_0 = q_0'$, $q_1 < q_0$, $1/p_i - 1/q_i = 1 - 1/r$ $(i = 0, 1)$. *Then* $L_{p_1}^{q_1} \subsetneqq L_{p_0}^{q_0}$.

Proof. If the two spaces in question were identical, an application of the closed graph theorem would yield the existence of a positive constant K for which

$$\|T\|_{p_1,q_1} \leqq K\|T\|_{p_0,q_0}\ . \tag{10}$$

We now interpolate in much the same kind of way as we did in the proof of Theorem 4.1. Define α by the relations

$$\begin{aligned} 1/p_0 &= \alpha/2 + (1 - \alpha)/1 \\ 1/q_0 &= \alpha/2\ . \end{aligned} \tag{11}$$

Then by convexity [10, Th. 1.3] for all $\varphi \in C_c(G)$, we have

$$||\varphi||_{p_0,q_0} \leqq ||\hat{\varphi}||_\infty^\alpha ||\varphi||_\infty^{1-\alpha} . \tag{12}$$

(Recall that for $1 < s \leqq \infty$, $L_1^s(G) = L^s(G)$, with equality of norms.) (10) and (12) taken together yield the inequality

$$||\varphi * f||_{q_1} \leqq K||\hat{\varphi}||_\infty^\alpha ||\varphi||_\infty^{1-\alpha} ||f||_{p_1} \tag{13}$$

for all $\varphi, f \in C_c(G)$. Choose and fix $\varphi, f \in C_c(G)$. We now manufacture two sequences (φ_n) and (ψ_n) of functions much as in the proof of Lemma 2.1, but with a few modifications. Define

$$\begin{cases} \varphi_0 = \psi_0 = \varphi \\ \varphi_n = \varphi_{n-1} + \psi_{n-1,x_{n-1}} \\ \psi_n = \varphi_{n-1} - \psi_{n-1,x_{n-1}} \end{cases} \qquad (n = 1, 2, \cdots) \tag{14}$$

where the point x_{n-1} is chosen so that the supports of φ_{n-1} and $\psi_{n-1,x_{n-1}}$ are disjoint ($\psi_{n-1,x_{n-1}}$ is the x_{n-1}-translate of ψ_{n-1}) *and* so that the supports of $f * \varphi_{n-1}$ and $(f * \psi_{n-1})_{x_{n-1}}$ are disjoint. Then (14) leads to the further relation

$$\begin{cases} f * \varphi_n = f * \varphi_{n-1} + (f * \psi_{n-1})_{x_{n-1}} \\ f * \psi_n = f * \varphi_{n-1} - (f * \psi_{n-1})_{x_{n-1}} . \end{cases} \tag{15}$$

Arguing as in the proof of Lemma 2.1, we get that

$$||\varphi_n * f||_{q_1} = 2^{n/q_1}||\varphi * f||_{q_1} ;$$

$||\hat{\varphi}_n||_\infty \leqq 2^{(n+1)/2}||\hat{\varphi}||_\infty$; $||\varphi_n||_\infty = ||\varphi||_\infty$. Then substituting φ_n for φ in (13), we deduce

$$2^{n/q_1}||\varphi * f||_{q_1} \leqq K'||\hat{\varphi}||_\infty^{\alpha n/2}||\varphi||_\infty^{1-\alpha}||f||_{p_1} . \tag{16}$$

Now $\alpha n/2 = n/q_0$ and $1/q_0 < 1/q_1$. So we have a contradiction when $n \longrightarrow \infty$.

In order to be able to establish Theorem 4.4 for the case of a general infinite compact Abelian group, we shall construct modified Rudin-Shapiro polynomials for the group $G = \Pi_1^\infty Z(r)$ (complete direct product) where r is a prime, and $Z(r)$ is as usual the cyclic group of order r. Our construction is itself a modification of an argument due to Daniel Rider [13]. {We are grateful to Alessandro Figà-Talamanca for drawing our attention to Rider's paper.}

LEMMA 4.5. *Let r be a prime integer, $G = \Pi_1^\infty Z(r)$, and $X = {}^*\Pi_1^\infty Z(r)$ (weak direct product). Write $\chi_0, \chi_1, \cdots$ for the characters of G induced by the elements* $(0, 0, \cdots)$, $(1, 0, 0, \cdots)$, $(0, 1, 0, 0, \cdots)$, $\cdots$ *of X. Write $\zeta = \exp(2\pi i/r)$. Then there exists a sequence $(\varphi_k)_0^\infty$*

of trigonometric polynomials on G with the following properties.

(i) $\hat{\varphi}_k$ *has precisely* r^k *points in its support.*

(ii) $\hat{\varphi}_k$ *is supported by the subgroup of* X *generated by* $\chi_0, \cdots, \chi_k$.

(iii) $\hat{\varphi}_k$ *takes only the values* $0, 1, \zeta, \cdots, \zeta^{r-1}$.

(iv) $\|\varphi_k\|_\infty \leqq r^{(k+1)/2}$.

Moreover, $\varphi_k^{*(r-1)} = \varphi_k * \cdots * \varphi_k$*, the* $(r-1)$*th convolution power of* φ_k*, also has the properties* (i)–(iv).

Proof. Observe first of all that if s is an integer, then

$$\sum_{j=0}^{r-1} \zeta^{sj} = \begin{cases} r & (s \equiv 0 \bmod r) \\ 0 & \text{otherwise.} \end{cases}$$

Therefore if $c_0, \cdots, c_{r-1}$ are arbitrary complex numbers,

$$\sum_{s=0}^{r-1} \left| \sum_{j=0}^{r-1} \zeta^{sj} c_j \right|^2 = r \sum_{j=0}^{r-1} |c_j|^2 . \tag{17}$$

Now define the sequence $\{P_k^0, \cdots, P_k^{r-1}\}$ $k = 0, 1, \cdots$ of r-tuples of polynomials on G as follows. Define $P_0^0 = \cdots = P_0^{r-1} = 1$. Then define P_{k+1}^s inductively as follows:

$$P_{k+1}^s = \Sigma_{j=0}^{r-1} \chi_{k+1}^j \zeta^{sj} P_k^j \qquad (s = 0, 1, \cdots, r-1) .$$

It is easy to check that each of the functions P_k^s $(s = 0, 1, \cdots, r-1)$ has as its spectrum the subgroup of X generated by $\chi_0, \cdots \chi_k$ and has Fourier coefficients taking the values $0, 1, \cdots, \zeta^{r-1}$ only. Now by virtue of (17),

$$\Sigma_{s=0}^{r-1} |P_{k+1}^s|^2 = r \Sigma_{s=0}^{r-1} |P_k^s|^2$$

since $|\chi_{k+1}| = 1$. Therefore $|P_{k+1}^s| \leqq r^{(k+2)/2}$ $(s = 0, 1, \cdots, r-1; k = 0, 1, \cdots)$. Define $\varphi_k = P_k^0$. Then the sequence (φ_k) enjoys properties (i)–(iv).

Now it is not hard to see that the $(r-1)$th convolution power $\varphi_k * \cdots * \varphi_k$ of φ_k also satisfies conditions (i)–(iv). For its Fourier transform is just the complex conjugate of that of φ_k.

We shall need one further result, namely a simple lemma relating the space of multipliers on a group G to the corresponding space of multipliers on a quotient group of G.

LEMMA 4.6. *Let G be a compact Abelian group with dual X. Suppose that X_0 is a subgroup of X, and that $1 \leqq p \leqq q \leqq \infty$. Let ψ be a bounded function on X_0, and ψ' the function on X which coincides with ψ on X_0, and is zero off X_0. Then $\psi \in M_p^q(X_0)$ if and only if $\psi' \in M_p^q(X)$.*

Proof. Write G_0 for the annihilator of X_0 in G, so that the dual of G/G_0 is X_0. Suppose first that $\psi \in M_p^q(X_0)$. If $f' \in L^p(G)$, then it is well known that the function ν on X_0 obtained by restricting $\hat{f}'$ to X_0 is the Fourier transform of a function in $L^p(G/G_0)$. On the other hand, it is easy to see that the function on X obtained by extending $\psi\nu$ so that it is 0 off X_0, is the Fourier transform of a function (constant on cosets of G_0) in $L^q(G)$. We conclude that $\psi' \in M_p^q(X)$. The converse is proved in a similar manner.

The compact case of Theorem 4.4 now follows.

THEOREM 4.7. *Let G be an infinite compact Abelian group,*

$$1 < p_0 < q_0 < \infty,\ p_0 = q_0',\ q_1 < q_0,\ 1/p_i - 1/q_i = 1 - 1/r \qquad (i = 0, 1)\ .$$

Then $L_{p_1}^{q_1} \subsetneqq L_{p_0}^{q_0}$.

Proof. Lemma 4.6 shows that it suffices to prove the theorem for a suitable quotient of G.

There are several cases to consider, depending on the group theoretic structure of X, the dual of G.

Case (i). X is not a torsion group. Then X contains a copy of Z, the additive group of the integers. To establish this case, it suffices to prove the theorem when $G = T$, the circle group.

If $L_{p_1}^{q_1} = L_{p_0}^{q_0}$, it follows that $L_{p_0}^{q_0} = L_{q'_1}^{p'_1}$, and as in the proof of Theorem 4.4, we deduce the inequality

$$(13') \qquad \|\varphi * f\|_{p'_1} \leqq K\|\hat{\varphi}\|_\infty^\alpha \|\varphi\|_\infty^{1-\alpha}\|f\|_{q'_1}$$

for all trigonometric polynomials f and φ, where $1/q_0 = \alpha/2$. Now manufacture a sequence (φ_n) of Rudin-Shapiro polynomials with $\hat{\varphi}_n(k) = \pm 1$ for $0 \leqq k \leqq 2^n - 1$, $\hat{\varphi}_n(k) = 0$ for $k \geqq 2^n$, and $\|\varphi_n\|_\infty \leqq 2^{(n+1)/2}$. Replace both f and φ in (13') by φ_n. Now $|\varphi_n * \varphi_n - \exp(2^n - 1)ix| = |D_{2^{n-1}-1}|$, where D_k denotes the Dirichlet kernel of order k. Now [1, Exercise 7.5] $\|D_k\|_{p'_1} \sim k^{1/p_1}$ as $k \to \infty$ when $1 < p_1 < \infty$. (13') yields the estimate

$$\begin{aligned}\|D_{2^{n-1}-1}\|_{p'_1} &\leqq K' \cdot 1 \cdot 2^{n(1-\alpha)/2} 2^{n/2} + 1 \\ &= K' 2^{n/q'_0} + 1 \\ &= K' 2^{n/p_0} + 1\ ,\end{aligned}$$

so that as $n \to \infty$, $2^{n/p_1} \leqq M 2^{n/p_0}$ for some constant M. But $1/p_1 > 1/p_0$; so we have a contradiction.

Case (ii). X is a torsion group, but contains elements of arbitrarily large order. We may therefore assume that there is a sequence $X_1, X_2, \cdots$ of cyclic subgroups of X of orders $n_1, n_2, \cdots$ where $n_k \to \infty$. For each k, define the positive integer s_k by the condition $2^{s_k} \leqq n_k < 2^{s_k+1}$. Now manufacture a trigonometric polynomial t_k on G (as in the proof of Lemma 2.1) with $t_k = \sum_{j=0}^{2^{s_k}-1} c_j \chi_k^j$ where $c_j = \pm 1$ and χ_k is the generator of X_k. Notice that the support of $\hat{t}_k$ contains precisely 2^{s_k} points.

If $L_{p_0}^{q_0} = L_{p_1}^{q_1}$, we deduce (13′) as before. Now $(t_k * t_k)^\wedge$ is the characteristic function of the subset $\{\chi_k^j : j = 0, \cdots, 2^{s_k} - 1\}$ of X_k. If G_k is the annihilator of X_k, then there are precisely n_k cosets of G_k in G, each having measure $1/n_k$. On one of these cosets, namely G_k itself, $t_k * t_k = 2^{s_k}$. Therefore

$$\begin{aligned} \|t_k * t_k\|_{p_1'} &\geqq \left[\frac{2^{s_k p_1'}}{n_k}\right]^{1/p_1'} \\ &\geqq \left[\frac{2^{s_k p_1'}}{2^{s_k+1}}\right]^{1/p_1'} \\ &= 2^{s_k(1-1/p_1')} 2^{-1/p_1'} \\ &= 2^{s_k/p_1} 2^{-1/p_1'} . \end{aligned}$$

On the other hand, $\|t_k\|_{q_1'} \leqq 2^{(s_k+1)/2}$, $\|\hat{t}_k\|_\infty = 1$, and $\|t_k\|_\infty \leqq 2^{(s_k+1)/2}$. Substituting $t_k = \varphi = f$ in (13′), we derive the inequality

$$\begin{aligned} 2^{s_k/p_1} 2^{-1/p_1'} &\leqq \|t_k * t_k\|_{p_1'} \\ &\leqq K' 2^{s_k(1-\alpha+1)/2} \\ &= K' 2^{s_k(1-1/q_0)} \\ &= K' 2^{s_k/p_0} \end{aligned} \tag{18}$$

since $1/q_0 = \alpha/2$ and $p_0 = q_0'$. Now if $1/q_0 < 1/q_1$, it follows that $1/p_0 < 1/p_1$. Since $s_k \to \infty$ as $k \to \infty$, (18) leads to a contradiction.

Case (iii). X is a group of bounded order. In this case, appeal to a known structure theorem [8, A.25] allows us to claim that X contains a subgroup isomorphic to the weak direct product ${}^*\Pi_1^\infty Z(r)$ where r is a prime integer. It therefore suffices to prove the theorem in the case where $X = {}^*\Pi_1^\infty Z(r)$. We seek, as before, to contradict the inequality (13′). By Lemma 4.5, there exists a sequence (φ_k) of trigonometric polynomials on G, having the properties (i)–(iv); further, the sequence of $(r-1)$th powers $(\varphi_k * \cdots * \varphi_k)$ also have properties (i)–(iv). Observe now that the Fourier transform of the rth convolution power φ_k^{*r} is precisely the characteristic function of the group X_k generated $\chi_0, \cdots, \chi_k$. Denote by G_k the annihilator of X_k in G. Then G_k has precisely r^k distinct cosets in G, each of measure $1/r^k$. Substituting $\varphi = \varphi_k$, $f = \varphi_k^{*(r-1)}$ in (13′), we derive the inequality

$$
\begin{aligned}
r^{k/p_1} &= r^{k(1-1/p_1')} = \|\varphi_k * (\varphi_k^{*r-1})\|_{p_1} \\
&\leqq K\|\varphi_k^{*r-1}\|_{q_1'} \|\hat{\varphi}_k\|_\infty^\alpha \|\varphi_k\|_\infty^{1-\alpha} \\
&\leqq Mr^{k/2} r^{k(1-\alpha)/2} = Mr^{k(1-\alpha/2)} \\
&= Mr^{k/p_0}
\end{aligned} \tag{19}
$$

where M is a constant, since φ_k and φ_k^{*r-1} both have property (iv) of Lemma 4.5, $1/q_0 = \alpha/2$, and $p_0 = q_0'$. Since $1/p_1 > 1/p_0$, (19) is contradicted when $k \to \infty$. The proof is now complete.

The final step in the train of argument puts Theorems 4.1, 4.4 and 4.7 together and interpolates so as to give the complete chain of proper inclusions.

THEOREM 4.8. *Let G be any infinite* LCA *group, $1 < p_1 < p_2 < p_0 < 2$, $p_0 = q_0'$, $1/p_i - 1/q_i = 1 - 1/r$ $(i = 0, 1, 2)$. Then the inclusions $L^r \subset L_{p_1}^{q_1} \subset L_{p_2}^{q_2} \subset L_{p_0}^{q_0}$ are all proper.*

Proof. We have already shown that $L^r \subsetneqq L_{p_1}^{q_1}$ and that $L_{p_2}^{q_2} \subsetneqq L_{p_0}^{q_0}$. It remains to show that $L_{p_1}^{q_1} \subsetneqq L_{p_2}^{q_2}$.

If the last two spaces are equal, then the topologies on them must be the same. (The spaces are both Banach, and [10, Th. 1.3] shows that the embedding of $L_{p_1}^{q_1}$ into $L_{p_2}^{q_2}$ is continuous.) Since, however, $1 < p_1 < p_2 < p_0$, there is an index α with $0 < \alpha < 1$, such that

$$1/p_2 = \alpha/p_1 + (1 - \alpha)/p_0 ,$$

and

$$\|T\|_{p_2,q_2} \leqq \|T\|_{p_1,q_1}^\alpha \|T\|_{p_0,q_0}^{1-\alpha} \tag{20}$$

for all $T \in L_{p_1}^{q_1}$, by the Riesz convexity theorem. On the other hand, since $L_{p_1}^{q_1} \neq L_{p_0}^{q_0}$, there is a sequence (T_n) of elements of $L_{p_1}^{q_1}$ with $\|T_n\|_{p_0,q_0} \to 0$ and $\|T_n\|_{p_1,q_1} = 1$; substituting in (20), we deduce that $\|T_n\|_{p_2,q_2} \to 0$. Since $\|T_n\|_{p_1,q_1} = 1$ and the norms on the spaces $L_{p_1}^{q_1}$ and $L_{p_2}^{q_2}$ are equivalent, we have arrived at a contradiction.

REFERENCES

1. R. E. Edwards, *Fourier Series: A Modern Introduction*, Vol. I and Vol. II, Holt, Rinehart and Winston, New York, 1967 and 1968.
2. Alessandro Figà-Talamanca, *On the subspace of L^p invariant under multiplication of transform by bounded continuous functions*, Rend. Sem. Mat. Univ. Padova (1965), 176–189.
3. Alessandro Figà-Talamanca and Garth I. Gaudry, *Multipliers and sets of uniqueness of L^p*, Michigan Math. J, **17** (1970), 179–191.
4. Alessandro Figà-Talamanca and Garth I. Gaudry, *Density and representation theorems for multipliers of type (p, q)*, J. Austral. Math. Soc., **7** (1967), 1–6.

5. G. I. Gaudry, *Quasimeasures and operators commuting with convolution*, Pacific J. Math. **18** (1966), 461-476.
6. ———, *Multipliers of type* (p, q), Pacific J. Math. **18** (1966), 477-488.
7. ———, *Topics in harmonic analysis*, Yale lecture notes (mimeographed), 1969.
8. Edwin Hewitt and Kenneth A. Ross, *Abstract Harmonic Analysis*, Vol. I, Springer Verlag, Berlin, 1963.
9. Edwin Hewitt and Kenneth A Ross, *Abstract Harmonic Analysis*, Vol. II, Springer Verlag, Berlin, 1970.
10. Lars Hörmander, *Estimates for translation invariant operators in* L^p *spaces*, Acta Math. **104** (1960), 93-140.
11. Y. Katznelson, *An Introduction to Harmonic Analysis*, Wiley, New York, 1968.
12. J. F. Price, *Some strict inclusions between spaces of* L^p *multipliers* (to appear)
13. Daniel Rider, *Transformations of Fourier coefficients*, Pacific J. Math. **19** (1966), 347-355.
14. Leonard Y. H. Yap, *On Young's inequality on locally compact groups* (to appear)

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