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DECOMPOSABLE SYMMETRIC TENSORS

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A *k-field* is a field over which every polynomial of degree less than or equal to k splits completely. The main theorem characterizes the maximal decomposable subspaces of the k^{th} symmetric space $\bigvee_k V$, where V is finite-dimensional vector space over an infinite k-field. They come in three forms:

(1) $\{x_1 \vee \cdots \vee x_k : x_k \in V\}$, $x_1, \cdots, x_{k-1}$ **fixed;**

(2) $\langle a, b\rangle_k = \{x_1 \vee \cdots \vee x_k : x_i \in \langle a, b\rangle\}$**; and**

(3) $\{x_1 \vee \cdots \vee x_{k-r} \vee \langle a, b\rangle_{(r')}\}$, $x_1, \cdots, x_{k-r}$ **fixed;**

where a and b are linearly independent vectors in V and $\langle a, b\rangle$ is the subspace spanned by a and b.

We consider symmetric tensor products of vector spaces and the problem of characterizing their maximal decomposable subspaces. This problem has been resolved in the skew-symmetric case by Westwick [4] using results due to Wei-Liang Chow [1, Lemma 5] when the underlying field is algebraically closed with characteristic zero.

A *k-field* is a field F over which every polynomial of degree at most k splits completely. In this paper we determine the maximal decomposable subspaces in the symmetric case when the underlying vector space is finite-dimensional over an infinite k-field whose characteristic (if any) exceeds the length of the product.

1. Let F be a field and V a vector space over F. The k-fold Cartesian product of V will be denoted by V^k where $1 < k$. A *rank k symmetric tensor space* is a vector space together with a k-multilinear symmetric mapping σ which is universal for k-multilinear symmetric maps of V^k and is spanned by $\sigma(V^k)$. We will use the notation $\bigvee_k V$ for this space. (The anti-symmetric or Grassman space is usually denoted by $\bigwedge^k V$.)

If $\bigvee_k V$ with σ: $V^k \rightarrow \bigvee_k V$ is a symmetric tensor space, the *decomposable symmetric tensors* or "symmetric products" are those elements of $\bigvee_k V$ in the set $\sigma(V^k)$. We will denote $\sigma(x_1, \cdots, x_k)$ by $x_1 \vee \cdots \vee x_k$. A subspace S of $\bigvee_k V$ is decomposable if $S \subseteqq \sigma(V^k)$. *Trivial decomposable subspaces* are the zero subspace and those consisting of scalar multiples of a single product. The *factors* of the product $x_1 \vee \cdots \vee x_k$ are the 1-dimensional subspaces $\langle x_1\rangle, \cdots, \langle x_k\rangle$ of V.

If V is n-dimensional, it is well-known that $\bigvee_k V$ is vector space isomorphic to the space of homogeneous polynomials of degree k over F [3, p. 428]. Any linear mapping f: $V \rightarrow V$ induces a unique linear mapping $\bigvee_k f$: $\bigvee_k V \rightarrow \bigvee_k V$ obtained by extending the mapping

$f^k: V^k \to \bigvee_k V$ defined by $f^k(x_1, \cdots, x_k) = f(x_1) \vee \cdots \vee f(x_k)$. This mapping will be denoted by simply $\bigvee_f$ when the length of the product is not in question.

PROPOSITION 1. *If x and y are decomposable symmetric tensors with k-1 common factors (counting repetitions), then $x + y$ is decomposable.*

Proof. The mapping σ is multilinear.

If U is any subspaces of V and $x_1, \cdots, x_k$ vectors of V then $\{x_1 \vee \cdots \vee x_k \vee u \mid u \in U\}$ is a decomposable subspace of $\bigvee_{k+1} V$ and will be denoted by $x_1 \vee \cdots \vee x_k \vee U$. Clearly,

$$x_1 \vee \cdots \vee x_k \vee U \subseteqq x_1 \vee \cdots \vee x_k \vee V.$$

Decomposable subspaces of the form $x_1 \vee \cdots \vee x_{k-1} \vee V$ will be called *type* 1 *subspaces.*

2. Let x be a product $x_1 \vee \cdots \vee x_k$ in $\sigma(V^k)$. If $w \in V$ then $w \vee x$ denotes the product $w \vee x_1 \vee \cdots \vee x_k$ in $\sigma(V^{k+1})$.

PROPOSITION 2. *If D is a decomposable subspace of $\bigvee_k V$ then $w \vee D$ is a decomposable subspace of $\bigvee_{k+1} V$.*

Proof. We will show that if $x + y = z \in \sigma(V^k)$ and $w \in V$ then $w \vee x + w \vee y = w \vee z$.

Define an injection $i: V^k \to V^{k+1}$ by

$$i_w(v_1, \cdots, v_k) = (w, v_1, \cdots, v_k)\ .$$

The universal property of $\bigvee_k V$ implies there is a unique linear $f: \bigvee_k V \to \bigvee_{k+1} V$ such that

$$f(x_1 \vee \cdots \vee x_k) = w \vee x_1 \vee \cdots \vee x_k\ .$$

The desired result follows because f is linear.

$$\begin{tikzcd}
V^{k+1} \arrow[r, "\sigma"] \arrow[d, "i_w"'] & \bigvee_{k+1} V \\
V^k \arrow[r, "\sigma"] \arrow[ur, dashed, "\sigma \circ i_w"] & \bigvee_k V \arrow[u, "f"']
\end{tikzcd}$$

Clearly f is injective. Moreover the image of a decomposable subspace of $\bigvee_k V$ under f is decomposable.

PROPOSITION 3. *$x_1 \vee \cdots \vee x_k = 0$ if and only if some $x_i = 0$.*

Proof. Suppose $x_1, \cdots, x_k$ are nonzero vectors. Choose any basis $(e_i)_{i \in I}$ of V over a field F. For each x_i assume the p_i^{th} coordinate to be nonzero. Let $p = (p_1, \cdots, p_k)$. Define a multilinear and symmetric mapping $f_p \colon V^k \to F$ by

$$f_p(x_1, \cdots, x_k) = \alpha(1, p_1) \cdots \alpha(k, p_k)$$

where each vector x_i has coordinates $(\alpha(i, j))_{j \in I}$. Then $f_p(x_1, \cdots, x_k)$ is nonzero and since $f_p = \sigma \circ \bar{f}_p$, where $\bar{f}_p$ is the extension of f_p to $\mathbf{V}_k V$, $x_1 \vee \cdots \vee x_k$ could not be zero.

Since σ is multilinear $x_i = 0$ for some $i = 1, \cdots, k$ implies $x_1 \vee \cdots \vee x_k = 0$.

S_k denote the set of $k!$ permutations of $\{1, \cdots, k\}$.

PROPOSITION 4. *Let V be an n-dimensional vector space. The identity*

$$x_1 \vee \cdots \vee x_k = y_1 \vee \cdots \vee y_k \neq 0$$

holds if and only if there is a $\pi \in S_k$ and scalars $\lambda_1, \cdots, \lambda_k$ such that

$$\lambda_1 \lambda_2 \cdots \lambda_k = 1$$

and

$$x_i = \lambda_i y_{\pi(i)} \qquad i = 1, \cdots, k .$$

Proof. This is a result of the fact that the rank k symmetric tensor space is isomorphic to the k^{th} component of the polynomial algebra in n indeterminants over F [3, p. 428]. The latter is a unique factorization domain.

In what follows we will suppose $x = x_1 \vee \cdots \vee x_k$ and $y = y_1 \vee \cdots \vee y_k$ are independent products such that $x + y$ is decomposable, say $x + y = z_1 \vee \cdots \vee z_k$. We will often use the assumption that x and y are nonzero products without explicit mention. The subspace of V spanned by the vectors $x_1, \cdots, x_k$ will be denoted $[x]$ and its dimension by $|x|$. For notational convenience we set

$$x \cap y = [x] \cap [y]$$

$$x \cup y = [x] + [y] .$$

If S is a subspace of V then $S_{(k)}$ is the set $\{x_1 \vee \cdots \vee x_k \mid x_i \in S\}$. In general $S_{(k)}$ is not a subspace. If U is a subspace of $\mathbf{V}_k V$ then the one-dimensional subspace $\langle v \rangle$ of V is a factor of U if

$$U \subseteqq v \vee V \vee \cdots \vee V .$$

We will frequently denote a repeated product $U \vee \cdots \vee U$ by $U_{(k)}$.

REMARK. If $x + y = z$ it is always true that $[z] \subseteqq x \cup y$. For, if some $z_i \notin x \cup y$ and B is a basis of $x \cup y$ we may choose $f \in L(V, V)$ so that

$$f(z_i) = 0$$

$$f(b) = b \qquad b \in B .$$

Then, $x + y = (\mathbf{V} f)\, z = 0$, contradicting our standing assumption that x and y are independent.

PROPOSITION 5. *If B is a basis of $[y]$ and there are i, j such that $B \cup \{x_i, z_j\}$ is an independent set then x and y have a common factor.*

Proof. Choose $f \in L(V, V)$ so that

$$f(x_i) = x_i$$

$$f(z_j) = 0$$

$$f(b) = b \qquad b \in B .$$

Then,

$$f(x_1) \vee \cdots \vee x_i \vee \cdots \vee f(x_k) = -y_1 \vee \cdots \vee y_k .$$

Proposition 4 now implies $\langle x_i \rangle$ is also a factor of y.

PROPOSITION 6. *If x and y have no common factors and $[y] \not\subseteqq [x]$ then for all $i = 1, \cdots, k$*

$$y_i \notin [x] \text{ and } z_i \notin [x] .$$

Proof. Let $y_j \notin [x]$. If B is any basis of $[x]$ we may complete the independent set $B \cup \{y_j\}$ to a basis of V. Consequently there is $f \in L(V, V)$ such that

$$f(y_j) = 0$$

$$f(b) = b \qquad b \in B .$$

If some $z_i \in [x]$ we have

$$x_1 \vee \cdots \vee x_k = f(z_1) \vee \cdots \vee z_i \vee \cdots \vee f(z_k) .$$

Proposition 4 implies $\langle z_i \rangle$ is then a factor of x. The choice of any $g \in L(V, V)$ with $\ker g = \langle z_i \rangle$ together with Proposition 4 shows $\langle z_i \rangle$ is also a factor of y. We have shown that if x and y have no common factors then no $z_i \in [x]$.

Choose some z_i and complete the independent set $B \cup \{z_i\}$ to a basis. Define $h \in L(V, V)$ by

$$h(z_i) = 0$$
$$h(b) = b \qquad b \in B .$$

Then

$$x_1 \vee \cdots \vee x_k = -h(y_1) \vee \cdots \vee h(y_k)$$

and we obtain a common factor whenever some $y_i \in [x]$ since then $h(y_i) = y_i$.

PROPOSITION 7. *If B is any basis of $[y]$ and for some i and j $B \cup \{x_i, x_j\}$ is an independent set then x and y have a common factor.*

Proof. Choose $f \in L(V, V)$ such that either $f(x_i) = 0$ or $f(x_j) = 0$ and $f(b) = b$ for every $b \in B$. Then

$$y_1 \vee \cdots \vee y_k = f(z_1) \vee \cdots \vee f(z_k) .$$

If some $z_i \in [y]$ then it is a common factor. Assume no $z_i \in [y]$. We claim one of the following is the zero subspace:

$$[y] \cap \langle x_i, z_1 \rangle$$
$$[y] \cap \langle x_j, z_1 \rangle .$$

For, if both are nonzero there are scalars α, β such that

$$z_1 = \alpha x_i + y' = \beta x_j + y'' \qquad \text{where } y', y'' \in [y] .$$

Hence,

$$\alpha x_i - \beta x_j \in [y] .$$

Since $z_1 \notin [y]$, both α and β are nonzero. But this violates the hypothesis. If $[y] \cap \langle x_i, z_1 \rangle = 0$ we apply Proposition 5 to $B \cup \{x_i, z_1\}$ and conclude x and y have a common factor.

3. F is a k-field if every polynomial over F of degree at most k splits completely over F. Let L_k denote $\{x \in \bigvee_k V : |x| = 1\}$. L_k is composed of all products $\alpha x_1 \vee \cdots \vee x_1$ where $\alpha \in F$, $x_1 \in V$. If F is a k-field then in particular

$$\alpha x_1 \vee \cdots \vee x_1 = (\alpha^{1/k} x_1) \vee \cdots \vee (\alpha^{1/k} x_1) .$$

However L_k need not be a subspace unless $k = p^r$ where r is a positive integer and p is the prime characteristic of F. That it *is* a subspace in this case is apparent because $\binom{p^k}{m}$ for $m = 1, \cdots, p^k - 1$ and so

$$x_1 \vee \cdots \vee x_1 + y_1 \vee \cdots \vee y_1 = (x_1 + y_1) \vee \cdots \vee (x_1 + y_1) .$$

PROPOSITION 8. *If F has prime characteristic p and $k = p^r$, r a positive integer, then* $\dim L_k = \dim V$.

Proof. Under these conditions it is not difficult to show that $x_1, \cdots, x_m$ are linearly independent in V if and only if $x_1 \vee \cdots \vee x_1, \cdots, x_m \vee \cdots \vee x_m$ are linearly independent in L_k.

PROPOSITION 9. *L_k is a decomposable subspace if and only if F has characteristic p and $k = p^m$, m a positive integer.*

Proof. We have seen that this condition is sufficient. If u, v are independent vectors in V then $u_{(k)} = u \vee \cdots \vee u$, $v_{(k)} = v \vee \cdots \vee v$ are in L_k and part of a basis for $\bigvee_k V$ by Proposition 8. Since L_k is decomposable there is a nonzero scalar γ and vector w such that

$$u_{(k)} + v_{(k)} = \gamma w_{(k)} . \tag{1}$$

The remark preceeding Proposition 5 implies there are scalars α, β such that $w = \alpha u + \beta v$. By induction,

$$\begin{aligned} w_{(k)} = \alpha^k u_{(k)} &+ \binom{k}{1} \alpha^{k-1} u_{(k-1)} \vee v + \cdots \\ &+ \binom{k}{r} \alpha^{k-r} \beta^r u_{(k-r)} \vee v_{(r)} + \cdots \\ &+ \beta^k v_{(k)} . \end{aligned}$$

Since the products $u_{(k-r)} \vee v_{(r)}$ are part of a basis of $\bigvee_k V$ we obtain

$$\gamma\alpha^k = \gamma\beta^k = 1$$

$$\gamma \binom{k}{r} \alpha^{k-r} \beta_r = 0 \qquad r = 1, \cdots, k-1 .$$

Because both α and β are nonzero $\alpha^{k-r} \beta^r$ is and so

$$\binom{k}{r} . 1 = 0 \qquad r = 1, \cdots, k-1 .$$

Hence F has characteristic p and

$$p \,\Big|\, \binom{k}{r} \qquad r = 1, \cdots, k-1 .$$

It is not difficult to show that this implies k is a power of p.

4. If a and b are two independent vectors in V then the set $\{x_1 \vee \cdots \vee x_k \mid x_i \in \langle a, b\rangle\}$ is denoted by $\langle a, b\rangle_{(k)}$. Let $F[\alpha]$ denote the polynomial algebra in one variable over F and define a linear mapping $g : \langle a, b\rangle \to F[\alpha]$ by $g(a) = \alpha$, $g(b) = 1$. *If* $f : V \to \langle a, b\rangle$ is a projection on $\langle a, b\rangle$ then $\bigvee_k g \circ f : \bigvee_k V \to F[\alpha]$ is a linear mapping obtained by extending $(g \circ f)^k : V^k \to F[\alpha]$ defined by

$$(g \circ f)^k (v_1, \cdots, v_k) = \prod_{i=1}^{k} g \circ f(v_i) . \qquad v_i \in V .$$

If

$$t = \prod_{i=0}^{k} \gamma_i \, a_{(k-i)} \vee b_i \qquad \gamma_i \in F$$

is any element of $\langle a, b\rangle_{(k)}$ then

$$(2) \qquad (\bigvee_k g \circ f)\, t = \gamma_0 + \gamma_1 \alpha + \cdots + \gamma_k \alpha^k .$$

The equality (2) implies that the restriction of $\bigvee_k g \circ f$ to $\langle a, b\rangle_{(r)}$ is a linear isomorphism onto $F[\alpha]$ which preserves "products", i.e., a decomposable tensor corresponds to a product of k linear polynomials.

PROPOSITION 10. *F is a k-field if and only if each $\langle a, b\rangle_{(k)}$ is a decomposable subspace of $\bigvee_k V$.*

Proof. Assume F is a k-field. If x and y are products in $\langle a, b\rangle_{(k)}$ let $P(\alpha) = (\bigvee_k g \circ f)(x + y)$. There are elements r_i in F such that $P(\alpha) = r_0(\alpha - r_1) \cdots (\alpha - r_k)$. Consider

$$z = r_0(a - r_1 b) \vee \cdots \vee (a - r_k b) \in \langle a, b\rangle_{(k)} .$$

Clearly, $P(\alpha) = \bigvee_k (g \circ f) z$ which implies $x + y = z$ because the restriction of $\bigvee_k g \circ f$ to $\langle a, b\rangle_{(k)}$ is injective. Therefore $\langle a, b\rangle_{(k)}$ is decomposable.

Conversely if $\langle a, b\rangle_{(k)}$ is decomposable and

$$P(\alpha) = \gamma_0 + \gamma_1 \alpha + \cdots + \gamma_k \alpha^k \in F[\alpha]$$

then (2) implies $P(\alpha) = (\bigvee_k g \circ f)\, t$ for some $t \in \langle a, b\rangle_{(k)}$.
But t is a product, say

$$t = (\lambda_1 a + \mu_1 b) \vee \cdots \vee (\lambda_k a + \mu_k b) .$$

Hence

$$P(\alpha) = (\lambda_1 + \mu_1 \alpha) \cdots (\lambda_k + \mu_k \alpha) .$$

LEMMA 11. *If F is infinite and $\langle x, y\rangle \subseteqq \sigma(V^k)$ then $|x| > 2$ implies x and y a common factor.*

Proof. Assume x_1, x_2, x_3 are independent and are contained in a basis B of V. For every $\lambda \in F$ there is a product $z(\lambda) = z_1(\lambda) \vee \cdots \vee z_k(\lambda)$ such that $x + \lambda y = z(\lambda)$. Define three linear mappings of V by

$$f_i(x_i) = 0 \qquad i = 1, 2, 3 .$$

$$f(b) = b \in B - \{x_1, x_2, x_3\}$$

Extending each mapping to $\bigvee_k V$ we obtain for each $\lambda \in F$:

$$(3) \qquad (\bigvee f_i)y = (\bigvee f_i)z(\lambda) \qquad i = 1, 2, 3 .$$

If (3) is zero for some i we infer from Proposition 3 that $f_i(y_j) = 0$ for some $j = 1, \cdots, k$. This means that $\langle x_i\rangle = \langle y_j\rangle$ is a common factor of x and y. For each λ, the vectors $z_1(\lambda), \cdots, z_k(\lambda)$ may be chosen so that (3) and Proposition 4 imply

$$(4) \qquad f_1(y_j) = f_1(z_j(\lambda)) \qquad j = 1, \cdots, k .$$

Let $z_i(\lambda)$ and y_j have coordinates $(\alpha_{ib}(\lambda): b \in B)$ and $(\beta_{jb}: b \in B)$ respectively. For each $\lambda \in F$ (4) implies

$$(5) \qquad \alpha_{jb}(\lambda) = \beta_{jb} \qquad b \neq x_1 .$$

If $i = 2$ then (3) and Proposition 4 implies for each $\lambda \in F$

$$f_2(z_j(\lambda)) = c_j(\lambda) f_2(y_{\pi(j)}) \qquad j = 1, \cdots, k .$$

where $\pi \in S_k$ and the $c_j(\lambda)$ are scalars such that $\prod_{j=1}^k c_j(\lambda) = 1$. Therefore,

$$(6) \qquad \alpha_{jb}(\lambda) = c_j(\lambda) \beta_{\pi(j)b} \; b \neq x_2 \qquad j = 1, \cdots, k .$$

If for some j, $\alpha_{jb}(\lambda) = 0$ for every $b \neq x_2$ then $\langle z_k\rangle = \langle x_2\rangle$ is a common factor of x and $z(\lambda)$; hence a common factor of x and y. Accordingly, we may assume for each j there is a basis element $b(j) \neq x_2$ such that $\beta\pi_{(j)b(j)} \neq 0$. If for some j $b(j) \neq x_1$ as well, then (5) and (6) imply

$$(7) \qquad c_j(\lambda) = \beta_{jb(j)} \beta^{-1}_{\pi(j)b(j)} .$$

On the other hand, suppose $b(j) = x_1$ for some j and $\beta_{\pi(j)b} = 0$ for all b distinct from x_1 and x_2. From (3) with $i = 3$ we obtain

$$(8) \qquad \alpha_{jb}(\lambda) = d_j(\lambda) \beta_{\omega(j)b} \qquad j = 1, \cdots, k .$$

where $\omega \in S_n$ and the $d_i(\lambda)$ are scalars such that $\prod_{j=1}^k d_j(\lambda) = 1$.

Were $\beta_{\omega(j)x_2} = 0$ then $\langle z_j(\lambda)\rangle = \langle x_1\rangle$ would be a common factor of x and $z(\lambda)$, hence a factor of y as well. If $\beta_{\omega(j)x_2} \neq 0$ then (5) together with $b = x_2$ in (8) imply

$$(9) \qquad d_j(\lambda) = \beta_{jx_2}\beta^{-1}_{\omega(i)x_2} \, .$$

From (5) we know that for any $\lambda \in F$ all coordinates of $z(\lambda)$ except $b = x_1$ are in the finite set $C_1 = \{\beta_{jb} : j = 1, \cdots, k;\ b \in B\}$. For each $i = 1, \cdots, k$ we have from (6)

$$(10) \qquad \alpha_{jx_1}(\lambda) = c_j(\lambda)\beta_{\pi(j)x_1}$$

and from (8) we obtain

$$\alpha_{jx_1}(\lambda) = c_j(\lambda)\beta_{\pi(j)x_1} \, .$$

Now if $b(j) \neq x_1$ then (7) and (10) imply

$$\alpha_{jx_1}(\lambda) = \beta_{jb(j)}\beta^{-1}_{\pi(j)b(j)}\beta_{\pi(j)x_1}$$

and if $b(j) = x_1$ then (8) and (9) imply

$$\alpha_{jx_1}(\lambda) = \beta_{jx_2}\beta^{-1}_{\omega(j)x_2}\beta_{\omega(j)x_1} \, .$$

We conclude that for any $\lambda \in F$ the coordinates of each $z_j(\lambda)$ are contained in the finite set

$$C_1 \cup \{\beta_{jb(j)}\beta^{-1}_{\pi(j)b(j)}\beta_{\pi(j)x_1}, \beta_{jx_2}\beta^{-1}_{\omega(j)x_2}\beta_{\omega(j)x_1} : j = 1, \cdots, k\} \, .$$

Accordingly, the number of vectors $z_j(\lambda)$ is finite and there are only a finite number of distinct products $z(\lambda) = z_1(\lambda) \vee \cdots \vee z_k(\lambda)$. But F is infinite. Hence there are distinct scalars λ, λ' such that $x + \lambda y = x + \lambda' y$ which implies $y = 0$. This contradicts our standing assumption that x and y are nonzero products and completes the proof.

We need the following lemma in order to prove Theorem 13.

LEMMA 12. *Let V be a finite-dimensional vector space over a field F and $\mathscr{C}$ any collection of proper subspaces of V. If $V = \bigcup \mathscr{C}$ then* $\mathrm{Card}\, F \leqq \mathrm{Card}\, \mathscr{C}$.

Proof. When $\dim V = 1$, V has no proper subspaces and the conclusion is vacuously true.

If $b_1, \cdots, b_n$ is any basis of V denote the (n-1)-dimensional subspace $\langle b_1, \cdots, b_{n-2}, b_{n-1} + \lambda b_n\rangle$ by S_λ, where λ is a scalar. Then Card $\{S_\lambda : \lambda \in F\} =$ Card F. For, if $S_\lambda = S_{\lambda'}$ then in particular

$$b_{n-1} + \lambda b_n = \alpha_1 b_1 + \cdots + \alpha_{n-2} b_{n-2} + \alpha_{n-1}(b_{n-1} + \lambda' b_n)$$

for some scalars $\alpha_1, \cdots, \alpha_{n-1}$. Thus $\alpha_i = 0$ for $i = 1, \cdots, n-2$. and $\alpha_{n-1} = 1$ which implies $\lambda = \lambda'$.

Consider $\mathscr{C}_\lambda = \{S_\lambda \cap T : T \in \mathscr{C}\}$. Because $V = \bigcup \mathscr{C}$ we have $S_\lambda = \bigcup \mathscr{C}_\lambda$. The set mapping from $\mathscr{C}$ to $\mathscr{C}_\lambda$ defined by $T \to S_\lambda \cap T$ is onto. Consequently, $\operatorname{Card} C_\lambda \leqq \operatorname{Card} \mathscr{C}$. Since $\dim S_\lambda = n-1$ induction yields $\operatorname{Card} F \leqq \operatorname{Card} \mathscr{C}_\lambda$, completing the proof.

If D is a decomposable subspace of $\bigvee_k V$ and $v \in V$ then $D(v)$ denotes $\{t \in D \mid \langle v \rangle$ is a factor of $t\}$. Any $D(v)$ is a subspace of D and is the zero subspace when v is a factor of no product in D. A nontrivial decomposable subspace can have at most k-1 factors. We have already remarked that any decomposable subspace with exactly k-1 factors (counting repetitions) is contained in a type 1 subspace. At the other extreme we have:

LEMMA 13. *If V is finite dimensional over an infinite k-field either without characteristic or with characteristic $p > k$ then the only maximal nontrivial decomposable subspaces of $\bigvee_k V$ without factors are those of the form $\langle a, b \rangle_{(k)}$.*

Proof. Let D be a maximal decomposable subspace without factors. If Char $F = p$ then Proposition 8 and $p > k$ imply L_k is not a subspace. Thus, we can assume $D \neq L_k$; i.e., D contains at least one product x with $|x| > 1$. We proceed by showing first that D cannot contain a product x with $|x| > 2$:

Assume, on the contrary, that $x = x_1 \vee \cdots \vee x_k$ is such a product of D.

For every product $y \in D$ we have $\langle x, y \rangle \subseteqq D \subseteqq \sigma(V^k)$. Lemma 11 implies each nonzero $y \in D$ must have a factor in common with x. Hence $D = \bigcup_{i=1}^k D(x_i)$, where each $D(x_i)$ must be a proper subspace since D is without factors. Since V is finite-dimensional Lemma 12 implies $\operatorname{Card} F < k$, contrary to hypothesis. Accordingly $|x| \leqq 2$ for every $x \in D$. Since D is not L_k, D contains a product x with $|x| = 2$. In what follows we suppose x_1, x_2 are independent.

Were $y \in D$ and $|y| = 1$ then $y = \alpha y_1 \vee \cdots \vee y_1$. If $y_1 \notin [x]$ Proposition 7 implies x and y have a common factor and so $y_1 \in [x]$, a contradiction. Therefore $[y] \subseteqq [x]$ for every $y \in D$ with $|y| = 1$.

Suppose $y \in D$, $|y| = 2$ but $[y] \nsubseteqq [x]$. The rest of the proof is in two parts and we consider first such y with no factors in common with x:

Complete x_1, x_2 to a basis B and define $f \in L(V, V)$ by

$$f(x_i) = x_1 \qquad i = 1, 2 \tag{11}$$
$$f(b) = b \qquad b \in B - \{x_1, x_2\}\,.$$

Were $(\mathbf{V}_F)y = 0$ then some $y_i \in [x]$, contrary to Proposition 6. If $|(\mathbf{V}_F)y| = 1$ then

$$\alpha x_1 \vee \cdots \vee x_1 + \beta f(y_1) \vee \cdots \vee f(y_k) = (\mathbf{V}_F) z \neq 0 \tag{12}$$

would imply (as in § 3) that the underlying field has characteristic p and $k = p^r$ for some prime p and positive integer r, contrary to hypothesis. (If $(\mathbf{V}_F)z = 0$ then some $z_i \in [x]$, again contradicting Proposition 6.) The remaining alternative is $|(\mathbf{V}_F)y| = 2$. Since we are assuming x and y have no common factors, (12) and Proposition 7 imply for some $i = 1, \cdots, k$

$$\langle x_1 \rangle = \langle f(y_i) \rangle . \tag{13}$$

But (11) and (13) imply $y_i \in [x]$, a contradiction of Proposition 6 again.

It remains to consider those $y \in D$ with $|y| = 2$ which have factors in common with x. If for such y, $[y] \neq [x]$ then $x \cap y$ is 1-dimensional. Let $x \cap y = \langle u \rangle$ and assume $\langle u \rangle$ occurs at least r times as a factor of both x and y. Consider the products

$$\bar{x} = x_1 \vee \cdots \vee x_{k-r}$$

$$\bar{y} = y_1 \vee \cdots \vee y_{k-r}$$

in $\sigma(V^{k-r})$. We may suppose that $\bar{x}$ and $\bar{y}$ have no common factors. Since $x + y \in \sigma(V^k)$ and iterations of the mapping f in (0) are also injective we have $\bar{x} + \bar{y} \in \sigma(V^{k-r})$. If either $|\bar{x}| = 2$ or $|\bar{y}| = 2$ then Lemma 10 implies

$$[\bar{x}] \subseteqq [\bar{y}] \quad \text{or} \quad [\bar{y}] \subseteqq [\bar{x}] . \tag{14}$$

Either statement in (14) implies $[x] = [y]$.

If $|\bar{x}| = |\bar{y}| = 1$ then either $[\bar{x}] = [\bar{y}]$ or $\bar{x} \cap \bar{y} = 0$. We will show $\bar{x} \cap \bar{y} = 0$ is contradictory:

Let $\bar{x} = \alpha x_1 \vee \cdots \vee x_1 = (\alpha^{1/r} x_1) \vee \cdots \vee (\alpha^{1/r} x_1)$

$$\bar{y} = \beta y_1 \vee \cdots \vee y_1 = (\beta^{1/r} y_1) \vee \cdots \vee (\beta^{1/r} y_1) .$$

This is possible since F is an r-field for every positive $r \leqq k$. Replace u and v by $\alpha^{1/r} x_1$ and $\beta^{1/r} w_1$ in (1). Then Char F is a prime p and $r = p^m$ for some positive integer m. But by hypothesis $p > k > r$, a contradiction.

We conclude $[y] \subseteqq [x]$ in all cases. Thus, $D \subseteqq \langle a, b \rangle_{(k)}$ where $\{a, b\}$ is any basis of $[x]$. Since D was assumed maximal the proof is complete.

THEOREM. *If V is finite-dimensional over an infinite k-field F either without characteristic or with characteristic $p > k$ then the maximal nontrivial decomposable subspaces of $\bigvee_k V$ are:*

(i) *type 1 subspaces*

and for every independent pair of vectors a, b in v:

(ii) $\langle a, b\rangle_{(k)}$

(iii) $x_1 \vee \cdots \vee x_{k-r} \vee \langle a, b\rangle_{(r)}$ *where $x_i \notin \langle a, b\rangle$ for every $i=1,\cdots, k-r$ and $1 < r < k$.*

Proof. Lemma 13 states that the only decomposable subspace without factors are those of the form (ii). The image of a decomposable subspace under the mapping f in (0) is a decomposable subspace with at least one factor. Iterations of f in (0) yield decomposable subspaces in spaces of greater length. Thus, when F is a k-field, $\langle a, b\rangle_{(r)}$ is a decomposable subspace of $\bigvee_r V$ for every $1 < r < k$ and subspaces of the form

$$x_1 \vee \cdots \vee x_{k-r} \vee \langle a, b\rangle_{(r)}$$

are decomposable. If x_{k-r}, say, is in $\langle a, b\rangle$ then

$$x_1 \vee \cdots \vee x_{k-r} \vee \langle a, b\rangle_{(r)} \subsetneqq x_1 \vee \cdots \vee x_{k-r-1} \vee \langle a, b\rangle_{(r+1)} .$$

Accordingly, subspaces of this type could be maximal only when $x_i \notin \langle a, b\rangle$ for each $i = 1, \cdots, k-r$.

Conversely, if a decomposable subspace has exactly $k - r$ factors it is the image of a decomposable subspace of $\bigvee_r V$ without factors under a composition of $k - r$ mappings f in (0). Lemma 13 states that subspace must be of the form $\langle a, b\rangle_{(r)}$. Hence (ii) and (iii) are the only types of decomposable subspaces with factors.

Routine arguments show that a space of one type cannot be properly contained in another of the same type or a different type. Since every decomposable subspace is contained in a maximal decomposable subspace the proof is completed.

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