PACIFIC JOURNAL OF MATHEMATICS
Vol. 35, No. 2, 1970

# CONCERNING THE DOMAINS OF GENERATORS OF LINEAR SEMIGROUPS

J. W. SPELLMANN

**Let $S$ denote a Banach space over the real numbers. Let $A$ denote the infinitesimal generator of a strongly continuous semigroup $T$ of bounded linear transformations on $S$. It is known that the Riemann integral $\int_a^b T(x)pdx$ is in the domain of $A$ (denoted by $D(A)$) for each $p$ in $S$ and each nonnegative number interval $[a, b]$. This paper develops sufficient conditions on nonnegative continuous functions $f$ and on elements $p$ in $S$ in order that the Riemann integral $\int_a^b T(f(x))pdx$ be an element of the domain of $A$.**

2. A change of variable technique. A change of variable theorem may sometimes be used to transform

$$\int_a^b T(f(x))pdx \quad \text{to} \quad \int_c^d T(x)(f^{-1})'(x)pdx$$

where $f^{-1}$ denotes the inverse of $f$. This motivates the first theorem.

THEOREM 1. *Suppose $p \in S$, $0 \leqq c < d$ and $h$ is a real valued function which has a continuous derivative on $[c, d]$. Then*

$$\int_c^d T(x)h(x)pdx$$

*is in $D(A)$ and*

$$A\int_c^d T(x)h(x)pdx = h(d)T(d)p - h(c)T(c)p - \int_c^d T(x)h'(x)pdx \; .$$

*Proof.*

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \varepsilon^{-1}[T(\varepsilon) - T(0)]\int_c^d T(x)h(x)pdx \\
&= \lim_{\varepsilon\to 0} \varepsilon^{-1}\int_{c+\varepsilon}^{d+\varepsilon} T(x)h(x-\varepsilon)pdx - \lim_{\varepsilon\to 0} \varepsilon^{-1}\int_c^d T(x)h(x)pdx \\
&= \lim_{\varepsilon\to 0} \varepsilon^{-1}\int_d^{d+\varepsilon} T(x)h(x-\varepsilon)pdx - \lim_{\varepsilon\to 0} \varepsilon^{-1}\int_c^{c+\varepsilon} T(x)h(x)pdx \\
&\quad - \lim_{\varepsilon\to 0} \varepsilon^{-1}\int_{c+\varepsilon}^d T(x)[h(x) - h(x-\varepsilon)]pdx \\
&= h(d)T(d)p - h(c)T(c)p - \int_c^d T(x)h'(x)pdx \; .
\end{aligned}$$

The second theorem then follows as an immediate consequence of Theorem 1.

THEOREM 2. *Suppose* $p \in S$, $0 \leqq a < b$, $0 \leqq c < d$ *and* $f$ *is a continuous function from* $[a, b]$ *to* $[0, \infty]$ *so that*

(i) $(f^{-1})''$ *is continuous on* $[c, d]$ *and*

(ii) $\int_a^b T(f(x))pdx = \pm\int_c^d T(x)(f^{-1})'(x)pdx$. *Then* $\int_a^b T(f(x))p$ *is in* $D(A)$ *and*

$$A\int_a^b T(f(x))pdx = \pm\Big[(f^{-1})'(d)T(d)p - (f^{-1})'(c)T(c)p - \int_c^d T(x)(f^{-1})''(x)pdx\Big]$$

EXAMPLE 1. Suppose $0 \leqq a < b$, $m$ and $k$ are real numbers so that $m \neq 0$ and $mx + k \geqq 0$ for all $x \in [a, b]$. Then $\int_a^b T(mx + k)pdx$ is in $D(A)$ and

$$A\int_a^b T(mx + k)pdx = \frac{1}{m}[T(mb + k)p - T(ma + k)p] .$$

It is noted that Theorem 2 says nothing about $\int_a^b T(f(x))pdx$ being in $D(A)$ if $\int_a^b T(f(x))pdx$ does not equal $\pm\int_c^b T(x)(f^{-1})'(x)pdx$ or if $(f^{-1})''$ is not continuous on $[c, , d]$. A different approach is considered in the next section which sometimes allows for such exceptions.

3. **A closed operator technique.** In this section, the restrictions imposed on the function $f$ in the hypothesis of Theorem 2 will be relaxed. In accomplishing this, additional restrictions will be placed on the point $p$ mentioned in Theorem 2. The fact that the infinitesimal generator $A$ of the semigroup $T$ is a closed linear operator implies the next theorem.

THEOREM 3. *Suppose* $p \in D(A)$, $0 \leqq a < b$ *and* $f$ *is a continuous function from* $[a, b]$ *to* $[0, \infty)$. *Then*

$$\int_a^b T(f(x))pdx$$

*is in* $D(A)$ *and*

$$A\int_a^b T(f(x))pdx = \int_a^b T(f(x))Apdx .$$

The fourth theorem follows from Example 1, properties of continuous real valued functions and the fact that the space $S$ is complete.

THEOREM 4. *Suppose* $p \in S$, $0 \leqq a < b$ *and* $f$ *is a continuous function from* $[a, b]$ *to* $[0, \infty)$. *Suppose* $\{f_n\}_{n=1}^{\infty}$ *is a sequence of piecewise linear functions, each from* $[a, b]$ *to* $[0, \infty)$, *which converge uniformly to* $f$ *on* $[a, b]$. *Then* $\int_a^b T(f(x))pdx$ *is in* $D(A)$ *whenever* $\left\{A\int_a^b T(f_n(x))pdx\right\}_{n=1}^{\infty}$ *is a Cauchy sequence in* $S$. *In this case*

$$A\int_a^b T(f(x))pdx = \lim_{n\to\infty} A\int_a^b T(f_n(x))pdx \; .$$

In order to develop useful corollaries to Theorem 4, we make the following definitions.

DEFINITION 1. Suppose $K = \{x_j\}_{j=0}^{n}$ is a partition of $[a, b]$ and $f$ is a continuous real valued function defined on $[a, b]$. Then $[f; K]$ denotes the piecewise linear function defined on $[a, b]$ by the rule

$$[f; K](x) = [f(x_j) - f(x_{j-1})][(x_j - x_{j-1})^{-1}][x - x_{j-1}] + f(x_{j-1})$$

for $x \in [x_{j-1}, x_j]$, $j = 1, 2, \cdots, n$.

DEFINITION 2. Suppose $0 < \alpha \leqq 1$. Then $\Delta(\alpha)$ denotes the subset of $S$ which contains $p$ if and only if for each positive number $r$, there is a positive number $M(r)$ so that

$$\| T(x)p - p \| < x^{\alpha} M(r)$$

for all $x \in [0, r]$.

It is noted that $D(A) \subseteqq \Delta(\alpha)$ for each $\alpha \in [0, 1]$. However, the next example illustrates that $\Delta(1)$ may not be a subset of $D(A)$.

EXAMPLE 2. Let $S$ denote the Banach space of real valued functions which are bounded and uniformly continuous on $[0, \infty)$. For each $f \in S$, let

$$\| f \| = \operatorname*{lub}_{x \geqq 0} | f(x) | \; .$$

Let $T$ be the strongly continuous linear semigroup defined on $S$ by the rule

$$[T(\beta)f](x) = f(\beta + x)$$

for each pair $(\beta, x)$ in $[0, \infty) \times [0, \infty)$. Then $f$ is in $D(A)$ if and only if $f'$ is in $S$.

Let $g$ be the function is $S$ so that

$$g(x) = \begin{cases} 1 - x & \text{if } x \in [0, 1] \\ 0 & \text{if } x \geqq 1 \end{cases} .$$

Then $g$ is in $\Delta(1)$, but $g$ is not in $D(A)$.

DEFINITION 3. Suppose $0 \leqq a < b$ and each of $P_1 = \{x_j\}_{j=0}^n$ and $P_2 = \{t_k\}_{k=0}^m$ is a partition of $[a, b]$. The statement the $P_2$ is a doubling refinement of $P_1$ means that

(1) $m = 2n$ and

(2) $t_{2j} = x_j$ for $j = 0, 1, \cdots, n$.

DEFINITION 4. Suppose $0 \leqq a < b$, $\alpha \in [0, 1]$, $f: [a, b] \to [0, \infty)$ and $P = \{P_n\}_{n=1}^\infty = \{\{a_{nk}\}_{k=0}^{2^n}\}_{n=1}^\infty$ is a sequence of partitions of $[a, b]$ so that $P_{n+1}$ is doubling refinement of $P_n$ for each positive integer $n$. The statement that $f$ satisfies condition $S(\alpha)$ relative to $P$ means that

(1) $\{[f; P_n]\}_{n=1}^\infty$ converges uniformly to $f$ on $[a, b]$,

(2) $f(a_{nk+1}) \neq f(a_{nk})$ for $n = 1, 2, \cdots$ and $k = 0, 1, \cdots, 2^n - 1$,

(3) $\sum_{n=1}^\infty \sum_{k=0}^{2^n-1} |\Delta f_{n,k} - \Delta f_{n+1,2k}||f(a_{n+1,2k+1}) - f(a_{nk})|^\alpha$ converges

and

(4) $\sum_{n=1}^\infty \sum_{k=0}^{2^n-1} |\Delta f_{n,k} - \Delta f_{n+1,2k+1}||f(a_{n,k+1}) - f(a_{n+1,2k+1})|^\alpha$ converges

where $\Delta f_{n,k} = [a_{n,k+1} - a_{n,k}][f(a_{n,k+1}) - f(a_{n,k})]^{-1}$ for $n$ a positive integer, $k$ an integer in the number interval $[0, 2^n - 1]$.

The next theorem is a useful corollary to Theorem 4.

THEOREM 5. *Suppose* $0 \leqq a < b$, $0 < \alpha \leqq 1$,

$$P = \{P_n\}_{n=1}^\infty = \{\{a_{nk}\}_{k=0}^{2^n-1}\}_{n=1}^\infty$$

*a sequence of partitions of* $[a, b]$ *so that* $P_{n+1}$ *is a doubling refinement of* $P_n$ *for each positive integer* $n$. *Suppose* $f: [a, b] \to [0, \infty)$ *is continuous and satisfies condition* $S(\alpha)$ *relative to* $P$. *Then if*

$$p \in \Delta(\alpha), \int_a^b T(f(x))pdx$$

*is in* $D(A)$ *and*

$$A\int_a^b T(f(x))pdx = \lim_{n\to\infty} A\int_a^b T([f; P_n](x))pdx .$$

*Proof.* The proof of Theorem 5 follows from Example 1 and Theorem 4.

The next theorem relaxes conditions on the function $f$ mentioned in Theorem 2. The conditions imposed on point $p$, however, will be more restrictive.

THEOREM 6. *Suppose* $0 \leqq a < b$, $p \in \Delta(1)$, $f\colon [a, b] \to [0, \infty)$ *so that*
(1) $f'$ *is continuous on* $[a, b]$
(2) $|f'(x)| > 0$ *for all* $x \in [a, b]$
(3) $f''$ *is bounded on* $[a, b]$.
*Then* $\int_a^b T(f(x))p\,dx$ *is in* $D(A)$.

*Proof.* Let $P = \{P_n\}_{n=1}^{\infty} = \{\{a_{nk}\}_{k=0}^{2^n}\}_{n=1}^{\infty}$ be a sequence of partitions of $[a, b]$ so that $a_{n,k} = a + (k2^{-2})(b - a)$. Then $P_{n+1}$ is a doubling refinement of $P_n$ for each positive integer $n$ and $\{[f; P_n]\}_{n=1}^{\infty}$ converges uniformly to $f$ on $[a, b]$. The mean value theorem and the hypothesis on $f$ imply $f$ satisfies conditive $S(1)$ relative to $P$. An application of Theorem 4 completes the proof.

It is noted that the same sequence $P$ of partitions used in the proof of Theorem 6 may be used to show that $\int_a^b T(f(x))p\,dx$ *is* $D(A)$ whenever $p \in \Delta(1)$ and $f$ is a nonconstant and nonnegative polynomial whose coefficients are either all positive or all negative.

The next example shows that hypothesis (i) of Theorem 2 is not a necessary condition for $\int_a^b T(f(x))p\,dx$ to be in $D(A)$.

EXAMPLE 3. Suppose $0 < b$, $\beta > 0$, $m$ is a positive integer, $1 - 1/m < \alpha \leqq 1$ and $p \in \Delta(\alpha)$. Let $f(x) = \beta x^m$ for $x \geqq 0$. Then $(f^{-1})''$ is not continuous at 0. However, using the same sequence $P$ as in the proof of Theorem 6, $\int_a^b T(f(x))p\,dx$ may be shown to be in $D(A)$.

The fourth example will indicate that hypothesis (ii) of Theorem 2 is not necessary for $\int_a^b T(f(x))p\,dx$ *to be in* $D(A)$.

EXAMPLE 4. Let $C$ denote Cantor's ternary set (see p. 329 of [3]). For each $x$ in the interval $[0, 1]$, let

$$C_x = \operatorname{lub}(C \cap [0, x]) .$$

Let $w$ be the function defined on $[0, 1]$ by the rule

$$w(x) = {}_2(C_x \cdot 2^{-1})$$

where ${}_2(C_x \cdot 2^{-1})$ denotes the binary form of $(C_x \cdot 2^{-1})$. Hille and Tamarkin, in [2], have shown $w$ to be continuous, nondecreasing and to have a zero derivative almost everywhere on $[0, 1]$. Let $f$ be the function so that

$$f(x) = x + w(x) \text{ for } x \in [0, 1] .$$

Then $f$ is a strictly increasing function which fails to be absolutely continuous on $[0, 1]$. Thus, one would not expect the second condition of the hypothesis of Theorem 2 to hold. However, $\int_0^1 T(f(x))p$ is in $D(A)$ whenever $p$ is in $\Delta(1)$. This is seen by using Theorem 4 and proper choice of partitions of $[0, 1]$. Let

$$\begin{aligned} M_0 &= \{0\},\ N_0 = \{1\} \\ M_1 &= \{{}_3.022\cdots\}^1 \\ N_1 &= \{{}_3.200\cdots\} \\ Q_1 &= \{{}_3.111\cdots\} . \end{aligned}$$

For each integer $m \geqq 2$, let

$$\begin{aligned} M_m &= \{{}_3.a_1 \cdots a_{m-1}022 \cdots\} \\ N_m &= \{{}_3.a_1 \cdots a_{m-1}200 \cdots\} \\ Q_m &= \{{}_3.a_1 \cdots a_{m-1}111 \cdots\} \end{aligned}$$

where $a_i \in \{0, 2\}$ for $i = 1, 2, \cdots, m-1$.

For each nonnegative integer $n$, let $P_{2n}$ and $P_{2n+1}$ denote the following partitions of $[0, 1]$.

$$\begin{aligned} P_{2n} &= \left\{\bigcup_{k=0}^{n} [M_k \cup N_k]\right\} \cup \left\{\bigcup_{k=0}^{n} Q_k\right\} \\ P_{2n+1} &= P_{2n} \cup Q_{n+1} . \end{aligned}$$

Then if $p \in \Delta(1)$, it may be shown that

$$(1)\quad \left\|A\int_0^1 T([f; P_{2n}](x))pdx - A\int_0^1 T([f; P_{2n+1}](x))pdx\right\| = 0$$

$$(2)\quad \left\|A\int_0^1 T([f; P_{2n+1}](x))pdx - A\int_0^1 T([f; P_{2n+2}](x)pdx\right\| \leqq \frac{M^2 2^{n+1}}{2^n + 3^n} .$$

Where $M$ is a number so that

(3) $Mx \geqq \|T(x)p - p\| \qquad x \in [0, 2]$ and

(4) $M \geqq \|T(x)p\| \qquad x \in [0, 2]$.

Thus $\left\{A\int_0^1 T([f; P_n](x))pdx\right\}_{n=1}^{\infty}$ is a Cauchy sequence in $S$. Theorem 4 implies $\int_0^1 T(f(x))pdx$ is in $D(A)$ since $\{[f; P_n]\}_{n=1}^{\infty}$ converges uniformly to $f$ on $[a, b]$.

REMARK ON EXAMPLE 4. If $t \in ([0, 1] - C) \cup (\bigcup_{n=0}^{\infty} P_n)$, the above technique may be used to show that $\int_0^t T(f(x))pdx$ is in $D(A)$. This is done by defining the following partitions $P_n'$ of $[0, t]$. Let

[1] $({}_3.0222\cdots)$ denotes the triadic respresentation of 1/3, etc.

$$P'_n = (P_n \cap [0, t]) \cup \{t\}$$

for each nonnegative integer $n$. If $t \in (C - \bigcup_{n=0}^{\infty} P_n)$ the following theorem may be used to show that $\int_0^t T(f(x))pdx$ is in $D(A)$.

THEOREM 7. *Suppose* $0 \leqq a < b$, $p \in \Delta(1)$ *and* $f$ *is a continuous, nonnegative, strictly monotone real valued function defined on* $[a, b]$. *Then there is a number* $M$ *so that*

$$\left\| A \int_c^d T([f; P](x))pdx \right\| \leqq (d - c)M$$

*for each partition* $P$ *of each subinterval* $[c, d]$ *of* $[a, b]$.

*Proof.* The proof of Theorem 7 follows from Example 1, the fact that $T(x)p$ is a continuous function of $x$, on $(0, \infty)$ and the fact that the infinitesimal generator $A$ is linear.

## REFERENCES

1. E. Hille and J. D. Tamarkin, *Remarks on a known example of monotone continuous function*, Amer. Math. Monthly **36** (1929), 255-64.
2. E. C. Titchmarsh, *The Theory of Functions*, 2nd edition, Oxford University Press, Amen House, London, 1939.

Received Feburary 26, 1970.

TEXAS A & M UNIVERSITY