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RITT'S QUESTION ON THE WRONSKIAN

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Among the questions for investigation at the end of his Colloquium Publication, Differential Algebra, J. F. Ritt suggested the study of special differential ideals, in particular those generated by the Wronskians. In this paper we obtain a test for an element to be a member of a certain (algebraic) ideal, and apply this result to the differential ideal generated by the second order Wronskian.

Let $y_i, z_j, i, j \in \{0, 1, 2, \cdots\}$ be independent indeterminants over a field F. We work in the ring $R = F[y_i, z_j]$. Let (a, b), with a and b integers satisfying $0 \leqq a < b$, represent the determinant

$$\begin{vmatrix} y_a & z_a \\ y_b & z_b \end{vmatrix}$$

and call $a + b$ the weight of this determinant. If F is of characteristic zero and $y_i(z_i)$ is considered to be the i^{th} derivative of $y(z)$, then $W = (0, 1)$ is the Wronskian of y and z. Using the Wronskian as a model, we consider ideals

$$I_t = (W_0, W_1, \cdots, W_t) ,$$

where W_i is any fixed linear combination with nonzero coefficients in F, of all determinants of weight $i + 1$. For $P \in R$ we obtain a constructive procedure to determine if $P \in I = I_0 \cup I_1 \cup I_2 \cup \cdots$. In fact, the procedure can be applied directly to polynomials in expressions $P(a_1, b_1) \cdots P(a_n, b_n)$. This work is similar to that of Levi [3] for the differential ideals $[y^p]$ and $[uv]$ as well as [1], [2], [4], [5], and [6]. Our results are a generalization, for $n = 2$, of those in [1] to a general ring.

It is known ([1]) that the exponent of $\{I\}$ with respect to I is infinite. We will see that if $P \in \{I\}$ then $P \cdot Q \in I$ if Q is a power product of sufficient degree in y_i, z_j with small i and j, while if $P \notin I$ then $P \cdot Q \notin I$ for all power products Q if i and j are large. In §2 we obtain a particular basis for R as a vector space over F, a subset of which provides a basis of R modulo I. This leads directly to canonical forms for elements of R and a constructive test for an element of R to be in I. (Although it is known ([7], p. 34) that the Wronskian is zero if and only if y and z are linearly dependent, the Ritt-Randenbush Theorem of Zeros ([7], p. 27) informs us that one cannot distinguish by zeros, elements which are in $\{I\}$ from those in I. Thus a test for membership in I cannot be stated in terms of solutions.)

1. Ordering. We order m-tuples, $X = (x_1, \cdots, x_m)$, with each x_i a rational number, lexicographically, and say that $X' = (x'_1, \cdots, x'_m)$ is higher than X if $x_1 < x'_1$ or $x_i = x'_i$ for $i \leqq h - 1$ and $x_h < x'_h$.

We consider elements of $R - F$, called δ-terms, which are expressed in the form

$$P = y_{i_1} \cdots y_{i_k} z_{j_1} \cdots z_{j_l} (a_1, b_1) \cdots (a_n, b_n)$$

and let $S = \{i_1, \cdots, i_k, j_1, \cdots, j_l, a_1, b_1, \cdots, a_n, b_n\}$ be the set of subscripts of P, $k + n = \deg_y P$, $l + n = \deg_z P$. Comparing only elements with the same set of subscripts, the same degree in y, and the same degree in z, we partially order R by

$$(n+1)^{-1}, a_1 + b_1, a_2 + b_2, \cdots, a_n + b_n, i_1, \cdots, i_k, b_1, \cdots, b_n$$

where we assume $a_1 + b_1 \leqq a_2 + b_2 \leqq \cdots \leqq a_n + b_n$ and $i_1 \leqq i_2 \leqq \cdots \leqq i_k$. (We also assume $a_i < b_i$ for all i.) It is clear that this is indeed a partial ordering and that if $P > P'$ then $PQ > P'Q$ for all $Q \neq 0$.

We say that the δ-term P is *replaceable* if

$$P = \sum c_i Q_i \text{ with } c_i \in F$$

where each Q_i is a δ-term comparable with P and lower than P (in the ordering just described). If for each Q_i the difference with P occurs before b_1, we say that P is *s-replaceable*.

2. Basis.

DEFINITION. The δ-term P is called a λ-term if
(1) $n = 0$ or $a_1 \leqq a_2 \leqq \cdots \leqq a_n$ and $b_1 \leqq b_2 \leqq \cdots \leqq b_n$;
(2) $i_1 \leqq \cdots \leqq i_k \leqq j_1 \leqq \cdots \leqq j_l$;
(3) $a_n \leqq i_1$ and $a_n \leqq j_1$.

In this section we show that the set of λ-terms is a basis of R.

LEMMA 1. *If P is a δ-term which fails to satisfy* (1) *of the definition of a λ-term, then P is s-replaceable.*

Proof. Assume $a_1 < a_2$ and $b_2 < b_1$, and consider the fourth order determinant

$$D = \begin{vmatrix} y_{a_1} & y_{b_1} & y_{a_2} & y_{b_2} \\ z_{a_1} & z_{b_1} & z_{a_2} & z_{b_2} \\ 0 & y_{b_1} & y_{a_2} & y_{b_2} \\ 0 & z_{b_1} & z_{a_2} & z_{b_2} \end{vmatrix} .$$

Subtracting the third row from the first, the fourth from the second and then expanding by minors of the first two rows, we see that $D = 0$. Expanding D (in the original form) by minors of the first two rows and using $D = 0$ we find:

$$(a_1, b_1)(a_2, b_2) = (a_1, a_2)(b_1, b_2) + (a_1, b_2)(a_2, b_1) .$$

Now, since $a_1 < a_2 < b_2 < b_1$, it follows that $a_1 + a_2$ and $a_1 + b_2$ are both less than $a_1 + b_1$ and $a_2 + b_2$. Thus each product on the right side of the equation is lower than $(a_1, b_1)(a_2, b_2)$. It follows that P is s-replaceable.

LEMMA 2. *If P is a δ-term which fails to satisfy* (2) *of the definition of a λ-term, then P is s-replaceable.*

Proof. Assume $i_k > j_1$ and let $a = i_k$, $b = j_1$. Note that $y_a z_b = -(b, a) + y_b z_a$, and each term on the right is lower than $y_a z_b$. It follows that P is s-replaceable.

LEMMA 3. *If P is a δ-term which fails to satisfy* (3) *of the definition of a λ-term, then P is s-replaceable.*

Proof. Assume $i_1 < a_n$ and consider the third order determinant

$$D = \begin{vmatrix} y_c & y_a & y_b \\ y_c & y_a & y_b \\ z_c & z_a & z_b \end{vmatrix}$$

where $c = i_1$, $a = a_n$, and $b = b_n$. Expanding D by minors of the first row and using $D = 0$, we find

$$y_c(a, b) = y_a(c, b) - y_b(c, a) .$$

Again, since $c < a$, each term on the right is lower than P and it follows that P is s-replaceable. (The other case $j_1 < a_n$ is treated similarly.)

The three lemmas show that if P is a δ-term which is not a λ-term, then P is replaceable. Since the number of δ-terms with a fixed set of subscripts is finite, this replacement process must terminate. Thus we have proved

THEOREM 1. *The λ-terms span R.*

We now complete the proof that the λ-terms are a basis of R.

THEOREM 2. *The λ-terms are linearly independent over F.*

Proof. Assume the λ-terms are dependent and let

$$\sum c_i P_i = 0 \tag{1}$$

where the P_i are λ-terms and $c_i \in F$, with some $c_i \neq 0$. It is clear that we may assume that each P_i has the same set of subscripts, S, and the same degree, d, in y. Let d be minimal; that is, we assume the λ-terms with degree in y less than d are linearly independent. (Clearly, the λ-terms of degree zero in y are independent.) We rewrite (1) in the form

$$\sum c_i P_i = c_0 P_0 \tag{2}$$

where for each P_i on the left the number of determinants in P_i is positive, while P_0 is a power product of y's and z's. Of all the terms on the left with $c_i \neq 0$, let $b = \max b_i$ where (a_i, b_i) is the determinant of minimum weight in P_i. We note that for all i, $a_i = a =$ minimum number in S.

In (2), let $y_i = y_a$ and $z_i = z_a$ for $i < b$. If, by this substitution, P_i becomes $\bar{P}_i$, we see that although some $\bar{P}_i$ may be zero, not all of them are. Also each $\bar{P}_i$ which is not zero is a λ-term, and has (a, b) as the determinant of lowest weight. Then, with $\bar{P}_i = (a, b)\bar{Q}_i$ we have

$$(a, b)\sum c_i \bar{Q}_i = c_0 \bar{P}_0 \tag{2}$$

and if $T = \sum c_i \bar{Q}_i$,

$$(a, b)T = c_0 \bar{P}_0 \,. \tag{3}$$

But on the left side of (3) is the expression $y_b z_a T$ which cannot appear on the right since $a < b$ and $\bar{P}_0$ is a λ-term. Thus $T = 0$. But $T = \sum c_i \bar{Q}_i$, some $c_i \bar{Q}_i \neq 0$, and each nonzero $\bar{Q}_i$ is a λ-term of degree $d - 1$ in y. However, d was the minimum degree in y for which λ-terms were dependent. This contradiction completes the proof of Theorem 2, and also concludes the proof that the λ-terms are a basis of R.

3. Canonical forms.

DEFINITION. Let P be a λ-term. P is called a β-term if:
(1) $a_1 > 0$
(2) $a_i < a_{i+1}$ for all i
(3) $b_i < b_{i+1}$ for all i.

LEMMA 4. *If the λ-term P is not a β-term, then P is replaceable, modulo I*

Proof. If $a_1 = 0$, expand $P(a_1, b_1)^{-1} W_{b_1 - 1} \equiv 0 \pmod{I}$ and solve for P. Similarly, if $a_{k-1} = a_k$, or if $b_k = b_{k+1}$, expand $P(a_k, b_k)^{-1} W_h \equiv 0 \pmod{I}$ where $h = a_k + b_k - 1$ and solve for P. In each case it is easy to see that every λ-term obtained is lower than P, and, since every term which is not a λ-term is s-replaceable, it follows that P is itself replaceable. Again, because there are a finite number of λ-terms with a given set of subscripts, this process must terminate. Thus we have proved half of

THEOREM 3. *Every element in R is expressible as a linear combination, with coefficients in F, of a finite number of distinct terms*

$$PW_a W_b \cdots W_r \tag{*}$$

where P is a β-term or 1. *This expression, which may be of degree zero in the W's, is unique.*

Proof. For each term A of the form (*) we will obtain the highest λ-term, B, in the expression for A as a linear combination of λ-terms. The correspondence $A \to B$ is one-to-one, hence no linear combination of terms A of the form (*) can vanish, since the highest B cannot cancel.

Let A be a fixed term of the form (*). With our standard notation for P, and with $V_i = a_i + b_i$, we define a determinant C_h for every W_h in (*). If $S = 1$, $n = 0$, or $h + 1 < V_1$, let $C_h = (0, h + 1)$. If $V_k \leqq h + 1 < V_{k+1}$, let $C_h = (a_k, h + 1 - a_k)$. Finally, if $V_n \leqq h + 1$, let $C_h = (a_n, h + 1 - a_n)$. It is easy to see that $B = PC_a C_b \cdots C_r$ has the properties described above and this completes the proof of the theorem.

COROLLARY 1. *The β-terms form a basis of R* mod *I.*

COROLLARY 2. *A necessary and sufficient condition for an element of R to be in I is that none of the terms (*) of its canonical form is of degree zero in the W's.*

COROLLARY 3. *If P is a β-term of degrees d_1 and d_2 in y and z respectively, and of degree n in 2^{nd} order determinants, then the weight of $P \geqq n(d_1 + d_2 + 2 - n)$.*

Proof. The β-term of minimal weight and the desired degrees is $y_n^{d_1 - n} z_n^{d_2 - n}(1, 2)(2, 3) \cdots (n, n + 1)$.

An equivalent statement of Corollary 3 is

COROLLARY 3′. *If P is a λ-term of degree d_1 and d_2 in y and z respectively and of degree n in 2^{nd} order determinants and the weight of $P < n(d_1 + d_2 + 2 - n)$, then $P \in I$.*

COROLLARY 4. *If P is a λ-term of degree n in determinants, and*

(a) *Q is a power product in $y, y_1, \cdots, y_{n-1}, z, z_1, \cdots, z_{n-1}$, and the degree of Q is large enough, then $P \cdot Q \in I$.*

(b) *Q is a power product in y_i and z_j, with $i, j \geqq n$, then $PQ \in I$ if and only if $P \in I$.*

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