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UPPER AND LOWER BOUNDS FOR EIGENVALUES BY FINITE DIFFERENCES

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Upper and lower bounds for the eigenvalues of elliptic partial differential equations associated with fixed membranes and clamped plates are given in terms of corresponding eigenvalues of their finite difference analogues. The upper bounds are found by interpolating piecewise polynomials through the solutions to the difference equations and substituting into the variational principle associated with the differential equations. The lower bounds are found by averaging the solutions to the differential equations and substituting into the discrete variational principle.

In this paper we are concerned with the following eigenvalue problems:

the vibration of a fixed membrane,

$$\Delta u + \lambda u = 0 \text{ in } R,\ u = 0 \text{ on } \partial R\ ; \tag{1}$$

the vibration of a clamped plate,

$$\Delta^2 v - \Omega v = 0 \text{ in } R,\ v = \frac{\partial v}{\partial n} = 0 \text{ on } \partial R\ ; \tag{2}$$

the buckling of a clamped plate,

$$\Delta^2 w + \Lambda\Delta w = 0 \text{ in } R,\ w = \frac{\partial w}{\partial n} = 0 \text{ on } \partial R\ . \tag{3}$$

Here R is a bounded region of Euclidean n-space with boundary ∂R, Δ is the Laplacian, $\partial/\partial n$ the normal derivative.

Each of these problems has a positive sequence of eigenvalues having no finite accumulation point:

$$0 < \lambda^{(1)} \leqq \lambda^{(2)} \leqq \cdots,\ 0 < \Omega^{(1)} \leqq \Omega^{(2)} \leqq \cdots,\ 0 < \Lambda^{(1)} \leqq \Lambda^{(2)} \leqq \cdots\ .$$

These eigenvalues may be characterized by the following minimax principles:

$$\lambda^{(k)} = \min \max_{a_1,\cdots,a_k} \frac{\sum_{i=1}^{n} \int_R \left[\frac{\partial}{\partial x_i}(a_1u_1 + \cdots + a_ku_k)\right]^2 dx}{\int_R [a_1u_1 + \cdots + a_ku_k]^2}\ , \tag{4}$$

where the minimum is over linearly independent sets of functions

$u_1, \cdots, u_k$ which are continuous, have piecewise continuous first derivatives, and have support in R;

$$(5) \qquad \Omega^{(k)} = \min \max_{a_1,\cdots,a_k} \frac{\int_R [\Delta(a_1v_1 + \cdots + a_kv_k)]^2 dx}{\int_R [a_1v_1 + \cdots + a_kv_k]^2 dx},$$

$$(6) \qquad \Lambda^{(k)} = \min \max_{a_1,\cdots,a_k} \frac{\int_R [\Delta(a_1w_1 + \cdots + a_kw_k)]^2 dx}{\sum_{i=1}^n \int_R \left[\frac{\partial}{\partial x_i}(a_1w_1 + \cdots + a_kw_k)\right]^2 dx},$$

where the minima are over linearly independent sets of functions $v_1, \cdots, v_k$ and $w_1, \cdots, w_k$, respectively, which are continuous, have continuous first derivatives, piecewise continuous second derivatives, and have support in R.

We will obtain explicit upper and lower bounds for these eigenvalues in terms of the corresponding eigenvalues of the finite difference analogues:

$$(7) \qquad \Delta_h U + \lambda_h U = 0 \text{ on } R_h, \ U = 0 \text{ off } R_h ;$$

$$(8) \qquad \Delta_h^2 V - \Omega_h V = 0 \text{ on } R_h, \ V = 0 \text{ off } R_h ;$$

$$(9) \qquad \Delta_h^2 W + \Lambda_h \Delta_h W = 0 \text{ on } R_h, \ W = 0 \text{ off } R_h .$$

Here R_h is a bounded subset of the mesh

$$S_h \equiv \{(i_1h, \cdots, i_nh): i_1, \cdots, i_n \text{ are integers}\}$$

for $h > 0$, and $\Delta_h \equiv \sum_{i=1}^n \partial_i\bar{\partial}_i$ is the $(2n+1)$-point approximation of the Laplacian, where $\partial_i, \bar{\partial}_i$ are forward and backward i-th difference operators:

$$\partial_i U(x_1, \cdots, x_n) = h^{-1}[U(x_1, \cdots, x_i + h, \cdots, x_n) - U(x_1, \cdots, x_i, \cdots, x_n)] ,$$

$$\bar{\partial}_i U(x_1, \cdots, x_n) = h^{-1}[U(x_1, \cdots, x_i, \cdots, x_n) - U(x_1, \cdots, x_i - h, \cdots, x_n)] .$$

Each difference problem has a finite positive sequence of eigenvalues:

$$0 < \lambda_h^{(1)} \leqq \lambda_h^{(2)} \leqq \cdots \leqq \lambda_h^{(\nu)}, 0 < \Omega_h^{(1)} \leqq \Omega_h^{(2)} \leqq \cdots \leqq \Omega_h^{(\nu)}, 0 < \Lambda_h^{(1)} \leqq \Lambda_h^{(2)} \leqq \cdots \leqq \Lambda_h^{(\nu)} ,$$

where ν is the number of points in R_h. These eigenvalues also may be characterized by minimax principles:

$$\lambda_h^{(k)} = \min_{a_1,\cdots,a_k} \max \frac{\sum_{i=1}^{n} h^n \sum_{S_h} [\partial_i(a_1 U_1 + \cdots + a_k U_k)]^2}{h^n \sum_{S_h} [a_1 U_1 + \cdots + a_k U_k]^2}, \tag{10}$$

$$\Omega_h^{(k)} = \min_{a_1,\cdots,a_k} \max \frac{h^n \sum_{S_h} [\varDelta_h(a_1 V_1 + \cdots + a_k V_k)]^2}{h^n \sum_{S_h} [a_1 V_1 + \cdots + a_k V_k]^2}, \tag{11}$$

$$\varLambda_h^{(k)} = \min_{a_1,\cdots,a_k} \max \frac{h^n \sum_{S_h} [\varDelta_h(a_1 W_1 + \cdots + a_k W_k)]^2}{\sum_i h^n \sum_{S_h} [\partial_i(a_1 W_1 + \cdots + a_k W_k)]^2}, \tag{12}$$

where the minima are over linearly independent sets of mesh functions $U_1, \cdots, U_k$ and $V_1, \cdots, V_k$ and $W_1, \cdots, W_k$, respectively, which vanish off R_h.

2. The lower bounds. To obtain lower bounds we take the continuous eigenfunctions of problems (1), (2), (3), and average them over cubes of sides h about mesh points. The resulting mesh functions are then admissible candidates for the minimax principles (10), (11), (12). The technique is due to Weinberger [4], who applied it to problems (1) and (3), among others.

To simplify notation, let $x = (x_1, \cdots, x_n)$, let e_i be the unit vector in the i-th coordinate direction, and let

$$C_h(x) = \{(y_1, \cdots, y_n) : |y_i - x_i| \leqq \tfrac{1}{2}h,\ i = 1, \cdots, n\}$$

be the cube of side h about x.

If u is a continuous and piecewise differentiable function with support in R, then

$$U(x) = h^{-n}\int_{C_h(x)} u(y)dy\ , \qquad x \in S_h\ , \tag{13}$$

is a mesh function which vanishes off R_h, the subset of S_h consisting of points x for which $C_h(x) \cap R$ is not empty. Then,

$$\int_R u^2 dx - h^n \sum_{R_h} U^2 = \sum_{x \in R_h} \int_{C_h(x)} [u(y) - U(x)]^2 dy\ . \tag{14}$$

Now since

$$\int_{C_h(x)} [u(y) - U(x)]dy = 0\ ,$$

each integral on the right of (14) is bounded by the integral of the square of the gradient of u times the reciprocal of the second free membrane eigenvalue for the cube of side h, and

$$\int_R u^2 dx - h^n \sum_{R_h} U^2 \leqq \frac{h^2}{\pi^2} \sum_i \int_R \left[\frac{\partial u}{\partial x_i}\right]^2 dx . \tag{15}$$

We also have, by integration by parts,

$$\partial_i U(x) = h^{-n-1} \int_{C_h(x+e_i h) \cup C_h(x)} \psi(y_i - x_i) \frac{\partial u(y)}{\partial y_i} dy , \tag{16}$$

where

$$\psi(\xi) = \begin{cases} \xi + \frac{1}{2}h , & -\frac{1}{2}h \leqq \xi \leqq \frac{1}{2}h , \\ \frac{3}{2}h - \xi , & \frac{1}{2}h \leqq \xi \leqq \frac{3}{2}h , \\ 0 , & \text{otherwise} . \end{cases}$$

It follows that

$$\begin{aligned} &\int_R \left[\frac{\partial u}{\partial x_i}\right]^2 dx - h^n \sum_{S_h} [\partial_i U]^2 \\ &= h^{-1} \sum_{x \in S_h} \int_{C_h(x+e_i h) \cup C_h(x)} \psi(y_i - x_i) \left[\frac{\partial u(y)}{\partial y_i} - \partial_i U(x)\right]^2 dy , \\ &\qquad\qquad i = 1, \cdots, n . \end{aligned} \tag{17}$$

Therefore, since the right side is positive,

$$\sum_{i=1}^n h^n \sum_{S_h} [\partial_i U]^2 \leqq \sum_{i=1}^n \int_R \left[\frac{\partial u}{\partial x_i}\right]^2 dx . \tag{18}$$

If the function u is continuous, has continuous first derivatives and piecewise continuous second derivatives, each integral on the right side of (17) is bounded by the integral of the square of the gradient of $\partial u/\partial y_i$ times the reciprocal of the second eigenvalue η_2 of the weighted free membrane problem

$$\begin{cases} \Delta\varphi(y) + \eta\psi(y_i - x_i)\varphi(y) = 0 , & y \in C_h(x + e_i h) \cup C_h(x) , \\ \dfrac{\partial\varphi(y)}{\partial n} = 0 , \qquad , & y \in \partial[C_h(x + e_i h) \cup C_h(x)] . \end{cases} \tag{19}$$

The eigenvalue here is the second one because

$$\int_{C_h(x+e_i h) \cup C_h(x)} \psi(y_i - x_i) \left[\frac{\partial u(y)}{\partial y_i} - \partial_i U(x)\right] dy = 0 .$$

Since $\psi(y_i - x_i) \leqq h$, a lower bound for η_2 is the second eigenvalue of the problem obtained by replacing ψ with h in (19), i.e.,

$$\eta_2 \geqq \frac{1}{4} \pi^2 h^{-3} .$$

Therefore,

$$\sum_{1=i}^{n} h^n \sum_{S_h} [\partial_i U]^2 \geqq \sum_{i=1}^{n} \int_R \left[\frac{\partial u}{\partial x_i}\right]^2 dx - 8\frac{h^2}{\pi^2}\int_R [\Delta u]^2 dx \ . \tag{20}$$

Still assuming u is continuous, has continuous first derivatives, and piecewise continuous second derivatives, we have, by integration by parts,

$$\partial_i \bar{\partial}_i U(x) = h^{-n-2} \int_{C_h(x-e_i h) \cup C_h(x) \cup C_h(x+e_i h)} \tilde{\psi}(y_i - x_i) \frac{\partial^2 u(y)}{\partial y_i^2} dy \ , \quad i = 1, \cdots, n \ ,$$

where

$$\tilde{\psi}(\xi) = \begin{cases} \frac{1}{2}(\xi + \frac{3}{2}h)^2 \ , & -\frac{3}{2}h \leqq \xi \leqq -\frac{1}{2}h \ , \\ \frac{3}{4}h^2 - \xi^2 \ , & -\frac{1}{2}h \leqq \xi \leqq \frac{1}{2}h \ , \\ \frac{1}{2}(\xi - \frac{3}{2}h)^2 \ , & \frac{1}{2}h \leqq \xi \leqq \frac{3}{2}h \ , \\ 0 \ , & \text{otherwise} \ . \end{cases}$$

Then

$$\begin{aligned} &\int_R \left[\frac{\partial^2 u}{\partial x_i^2}\right]^2 dx - h^n \sum_{S_h} [\partial_i \bar{\partial}_i U]^2 \\ &= h^{-2} \sum_{x \in S_h} \int_{C_h(x-e_i h) \cup C_h(x) \cup C_h(x+e_i h)} \tilde{\psi}(y_i - x_i) \left[\frac{\partial^2 u(y)}{\partial y_i^2} - \partial_i \bar{\partial}_i U(x)\right]^2 dy \geqq 0 \ , \\ &\qquad i = 1, \cdots, n \ . \end{aligned} \tag{21}$$

We also have, for $i \neq j$,

$$\begin{aligned} &\int_R \left[\frac{\partial^2 u}{\partial x_i \partial x_j}\right]^2 dx - h^n \sum_{x \in S_h} [\partial_i \partial_j U]^2 \\ &= h^{-2} \sum_{x \in S_h} \int_{C_h(x) \cup C_h(x+e_i h) \cup C_h(x+e_j h) \cup C_h(x+e_i h+e_j h)} \psi(y_i - x_i) \Psi(y_j - x_j) \\ &\quad \times \left[\frac{\partial^2 u(y)}{\partial y_i \partial y_j} - \partial_i \partial_j U(x)\right]^2 dy \geqq 0 \ , \qquad i, j = 1, \cdots, n \ . \end{aligned} \tag{22}$$

Combining (21) and (22), we have

$$h^n \sum_{S_h} [\Delta_h U]^2 \leqq \int_R [\Delta u]^2 dx \ . \tag{23}$$

Now we obtain the desired lower bounds. Let $u^{(j)}$ be the eigenfunction associated with $\lambda^{(j)}$ in (1). We may assume

$$\int_R u^{(i)} u^{(j)} dx = \delta(i, j) \ ,$$

where $\delta(i, j)$ is the Kronecker delta. Let

$$U_j(x) = h^{-n} \int_{C_h(x)} u^{(j)}(y) dy \ , \qquad x \in R_h \ .$$

We employ (15) and (18) with $u = a_1 u^{(1)} + \cdots + a_k u^{(k)}$, $U = a_1 U_1 + \cdots + a_k U_k$ in (10) and see that

$$\lambda_h^{(k)} \leqq \frac{\lambda^{(k)}}{1 - \dfrac{h^2}{\pi^2}\lambda^{(k)}} ,$$

or, what is the same thing,

$$\frac{\lambda_h^{(k)}}{1 + \dfrac{h^2}{\pi^2}\lambda_h^{(k)}} \leqq \lambda^{(k)} . \tag{24}$$

Next, let $v^{(j)}$ be the eigenfunction associated with $\Omega^{(j)}$ in (2), also such that

$$\int_R v^{(i)} v^{(j)} dx = \delta(i, j) .$$

Let

$$V_j(x) = h^{-n} \int_{C_h(x)} v^{(j)}(y) dy , \qquad x \in R_h .$$

Employing (15) and (23) with $u = a_1 v^{(1)} + \cdots + a_k v^{(k)}$, $U = a_1 V_1 + \cdots + a_k V_k$ in (11), we see that

$$\Omega_h^{(k)} \leqq \frac{\Omega^{(k)}}{1 - \dfrac{h^2}{\pi^2}\Omega^{(k)}} ,$$

or, equivalently,

$$\frac{\Omega_h^{(k)}}{1 + \dfrac{h^2}{\pi^2}\Omega_h^{(k)}} \leqq \Omega^{(k)} . \tag{25}$$

(Inequalities (24) and (25) correspond to (2.25) and (8.10) of [4].) Next, let $w^{(j)}$ be the eigenfunction associated with $\Lambda^{(j)}$ in (3), such that

$$\sum_{i=1}^{n} \int_R \frac{\partial w^{(j)}}{\partial x_i} \frac{\partial w^{(l)}}{\partial x_i} dx = \delta(j, l) .$$

Let

$$W_j(x) = h^{-n} \int_{C_h(x)} w^{(j)}(y) dy , \qquad x \in R_h .$$

Employing (20) and (23) with $u = a_1 w^{(1)} + \cdots + a_k w^{(k)}$, $U = a_1 W_1 + \cdots + a_k W_k$ in (12), we see that

$$\Lambda_h^{(k)} \leqq \frac{\Lambda^{(k)}}{1 - 8\frac{h^2}{\pi^2}\Lambda^{(k)}},$$

or,

$$\frac{\Lambda_h^{(k)}}{1 + 8\frac{h^2}{\pi^2}\Lambda_h^{(k)}} \leqq \Lambda^{(k)} . \tag{26}$$

This inequality is new.

3. The upper bounds. To obtain upper bounds we take the mesh eigenfunctions of problems (7), (8), (9) and interpolate to obtain admissible candidates for the minimax problems (4), (5), (6).

Pólya [3] has applied this technique to problem (1) using piecewise linear interpolation. Specifically, he considered the mesh domain R_h consisting of points x in S_h such that $C_{2h}(x) \subset R$. Each mesh square with vertices at points of S_h he divided into two triangles by a diagonal through two vertices. Given a mesh function U which vanishes off R_h, he interpolated a function u, linear on each triangle and agreeing with U at the vertices. He then proved the estimates

$$\int_R u^2 dx \geqq h^2 \sum_{x \in R_h} U^2 - \tfrac{1}{4}h^2 \sum_{i=1}^{2} h^2 \sum_{x \in R_h} [\partial_i U]^2 ,$$

$$\sum_{i=1}^{2} \int_R \left[\frac{\partial u}{\partial v_i}\right]^2 dx = \sum_{i=1}^{2} h^2 \sum_{x \in S_h} [\partial_i U]^2 ,$$

from which it follows that, for $n = 2$,

$$\lambda^{(k)} \leqq \frac{\lambda_h^{(k)}}{1 - \frac{1}{4}h^2\lambda_h^{(k)}} . \tag{27}$$

Weinberger [4] indicates how this may be extended to higher dimensions.

For the problems (2) and (3), however, piecewise linear functions are not smooth enough to be admissible in (5) and (6). We must interpolate with functions which are cubic polynomials in each space variable in each mesh cube, and such that the function is continuous with continuous first derivatives across the sides of the cube.

Let us first consider the one-dimensional case ($n = 1$). Given a mesh function U, we uniquely define the interpolating function, $P_h U$, by requiring that for $x \in S_h$

$$P_h U(x) = U(x), \frac{d}{dx}[P_h U(x)] = \tfrac{1}{2}[\partial U(x) + \bar{\partial} U(x)] .$$

By linearity,

$$P_h U(x) = \sum_{y \in S_h} U(y) P_h \delta(x, y) ,$$

so it suffices to define

$$k_h(x-y) \equiv P_h \delta(x, y)$$
$$= \begin{cases} 1 - \frac{5}{2}\left|\frac{x-y}{h}\right|^2 + \frac{3}{2}\left|\frac{x-y}{h}\right|^3 & , \ |x-y| \leqq h , \\ 2 - 4\left|\frac{x-y}{h}\right| + \frac{5}{2}\left|\frac{x-y}{h}\right|^2 - \frac{1}{2}\left|\frac{x-y}{h}\right|^3 , & h \leqq |x-y| \leqq 2h , \\ 0 & , \ 2h \leqq |x-y| . \end{cases}$$

For general n, then, we define

$$P_h U(x) = P_{h,x_1} P_{h,x_2} \cdots P_{h,x_n} U(x_1, \cdots, x_n)$$
$$= \sum_{y \in R_h} U(y) \prod_{i=1}^{n} k_h(x_i - y_i) .$$

Let us assume R_h consists of point x of R_h such that $C_{4h}(x) \subset R$. Then, for U vanishing off R_h, $P_h U$ will vanish off R. We now wish to estimate

$$\int_R [P_h U]^2 dx .$$

Let us again first do the case $n = 1$. We have

$$\int_{-\infty}^{\infty} [P_h U]^2 dz = \sum_{x, y \in S_h} U(x) U(y) \int_{-\infty}^{+\infty} k_h(x-z) k_h(y-z) dz$$
$$= \sum_{x, y \in S_h} U(x) U(y) \int_{-\infty}^{+\infty} k_h(z) k_h(z + x - y) dz$$
$$= \sum_{x \in S_h} U(x) \Big\{ U(x) \int_{-\infty}^{+\infty} [k_h(z)]^2 dz + [U(x-h) + U(x+h)]$$
$$\times \int_{-\infty}^{+\infty} k_h(z) k_h(z+h) dz + [U(x-2h) + U(x+2h)]$$
$$\times \int_{-\infty}^{+\infty} k_h(z) k_h(z+2h) dz + [U(x-3h) + U(x+3h)]$$
$$\times \int_{-\infty}^{+\infty} k(z) k(z+3h) dz \Big\}$$
$$= h \sum_{x \in S_h} U(x) \Big\{ \frac{57}{70} U(x) + \frac{71}{560} [U(x-h) + U(x+h)]$$
$$- \frac{1}{28} [U(x-2h) + U(x+2h)]$$
$$+ \frac{1}{560} [U(x-3h) + U(x+3h)] \Big\}$$
$$= h \sum_{x \in S_h} U(x) \Big\{ I - \frac{1}{40} h^4 \partial^2 \bar{\partial}^2 + \frac{1}{560} h^6 \partial^3 \bar{\partial}^3 \Big\} U(x) .$$

Then, for general n, we have

$$
\begin{aligned}
\int_R [P_h U]^2 dz &= \sum_{x,y \in S_h} U(x)U(y) \prod_{i=1}^{n} \int_{-\infty}^{+\infty} k_h(x_i - z_i) k_h(y_i - z_i) dz_i \\
&= h^n \sum_{x \in S_h} U(x) \prod_{i=1}^{n} \left[I - \frac{1}{40} h^4 \partial_i^2 \bar{\partial}_i^2 + \frac{1}{560} h^6 \partial_i^3 \bar{\partial}_i^3 \right] U(x) .
\end{aligned} \tag{28}
$$

Similarly, we have

$$
\begin{aligned}
\sum_{i=1}^{n} \int_R \left[\frac{\partial}{\partial z_i} P_h U \right]^2 dz &= \sum_{i=1}^{n} \sum_{x,y \in S_h} U(x)U(y) \int_{-\infty}^{+\infty} k_h'(x_i - z_i) k_h'(y_i - z_i) dz \\
&\quad \times \prod_{\substack{j=1 \\ j \neq i}}^{n} \int_{-\infty}^{+\infty} k_h(x_j - z_j) k_h(y_j - z_j) dz \\
&= -\sum_{i=1}^{n} h^n \sum_{x \in S_h} U(x) \left[\partial_i \bar{\partial}_i - \frac{1}{12} h^2 \partial_i^2 \bar{\partial}_i^2 - \frac{1}{120} h^4 \partial_i^3 \bar{\partial}_i^3 \right] \\
&\quad \times \prod_{\substack{j=1 \\ j \neq i}}^{n} \left[I - \frac{1}{40} h^4 \partial_j^2 \bar{\partial}_j^2 + \frac{1}{560} h^6 \partial_j^3 \bar{\partial}_j^3 \right] U(x) .
\end{aligned} \tag{29}
$$

Also,

$$
\begin{aligned}
\int_R [\Delta P_h U]^2 dz &= \sum_{x,y \in S_h} U(x)U(y) \Bigg[\sum_{i=1}^{n} \int_{-\infty}^{+\infty} k_h''(x_i - z_i) k''(y_i - z_i) dz \\
&\quad \times \prod_{\substack{j=1 \\ j \neq i}}^{n} \int_{-\infty}^{+\infty} k_h(x_j - z_j) k_h(y_j - z_j) dz_j \\
&\quad + \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \int_{-\infty}^{+\infty} k_h'(x_i - z_i) k_h'(y_i - z_i) dz_i \\
&\quad \times \int_{-\infty}^{+\infty} k_h'(x_j - z_j) k_h'(y_j - z_j) dz_j \\
&\quad \times \prod_{\substack{l=1 \\ l \neq i,j}}^{n} \int_{-\infty}^{+\infty} k_h(x_l - z_l) k_h(y_l - z_l) dz_l \Bigg] \\
&= h^n \sum_{x \in S_h} U(x) \Bigg\{ \sum_{i=1}^{n} \left[\partial_i^2 \bar{\partial}_i^2 - \frac{1}{2} h^2 \partial_i^3 \bar{\partial}_i^3 \right] \\
&\quad \times \prod_{\substack{j=1 \\ j \neq i}}^{n} \left[I - \frac{1}{40} h^4 \partial_j^2 \bar{\partial}_j^2 + \frac{1}{560} h^6 \partial_j^3 \bar{\partial}_j^3 \right] \\
&\quad + \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \left[\partial_i \bar{\partial}_i - \frac{1}{12} h^2 \partial_i^2 \bar{\partial}_i^2 - \frac{1}{120} h^4 \partial_i^3 \bar{\partial}_i^3 \right] \\
&\quad \times \left[\partial_j \bar{\partial}_j - \frac{1}{12} h^2 \partial_j^2 \bar{\partial}_j^2 - \frac{1}{120} h^4 \partial_j^3 \bar{\partial}_j^3 \right] \\
&\quad \times \prod_{\substack{l=1 \\ l \neq i,j}}^{n} \left[I - \frac{1}{40} h^4 \partial_l^2 \bar{\partial}_l^2 + \frac{1}{560} h^6 \partial_l^3 \bar{\partial}_l^3 \right] \Bigg\} U(x) .
\end{aligned} \tag{30}
$$

The desired inequalities are obtained from (28), (29), (30) by using the summation by parts formula

$$\sum_{S_h} U\bar{\partial}_i V = -\sum_{S_h} V\partial_i U ,$$

for functions with compact support. We consider the case $n = 2$. From (28) we have

$$\begin{aligned}\int_R [P_h U]^2 dx = h^2 \sum_{S_h} \Big\{ & U^2 - \frac{1}{40} h^4([\partial_1^2 U]^2 + [\partial_2^2 U]^2) \\ & - \frac{1}{560} h^6([\partial_1^3 U]^2 + [\partial_2^3 U]^2) + \frac{1}{1600} h^8[\partial_1^2\partial_2^2 U]^2 \\ & + \frac{1}{22400} h^{10}([\partial_1^2\partial_2^3 U]^2 + [\partial_1^3\partial_2 U]^2) + \frac{1}{313600} h^{12}[\partial_1^3\partial_2^3 U]^2 \Big\} \\ \geqq h^2 \sum_{S_h} \Big\{ & U^2 - \frac{1}{40} h^4([\partial_1^2 U]^2 + 2[\partial_1\partial_2 U]^2 + [\partial_2 U]^2) \\ & - \frac{1}{560} h^6([\partial_1^3 U]^2 + 3[\partial_1^2\partial_2 U]^2 + 3[\partial_1\partial_2^2 U]^2 + [\partial_2^3 U]^2) \Big\} .\end{aligned}$$

Therefore,

$$\int_R [P_h U]^2 dx \geqq h^2 \sum_{S_h} U\Big[U - \frac{1}{40} h^2 \Delta_h^2 U + \frac{1}{560} h^6 \Delta_h^3 U \Big] . \tag{31}$$

Similarly, from (29), we have

$$\sum_{i=1}^{2} \int_R \Big[\frac{\partial}{\partial x_i} P_h U\Big]^2 dx \geqq h^2 \sum_{S_h} U\Big[-\Delta_h U + \frac{1}{120} h^4 \Delta_h^3 U - \frac{1}{2240} h^6 \Delta_h^4 U \Big] , \tag{32}$$

$$\begin{aligned}\sum_{i=1}^{2} \int_R \Big[\frac{\partial}{\partial x_i} P_h U(x)\Big]^2 dx \leqq h^2 \sum_{S_h} U\Big[& -\Delta_h U + \frac{1}{12} h^2 \Delta_h^2 U \\ & - \frac{1}{168000} h^8 \Delta_h^5 U + \frac{1}{1334000} h^{10} \Delta_h^6 U \Big] ,\end{aligned} \tag{33}$$

and from (30), we have

$$\int_R [\Delta P_h U]^2 dx \leqq h^2 \sum_{S_h} U[\Delta_h^2 U - \tfrac{1}{2} h^2 \Delta_h^3 U] . \tag{34}$$

Now we obtain the upper bounds. Let $U_h^{(j)}$ be the eigenfunction associated with $\lambda_h^{(j)}$ in (7) such that

$$h^n \sum_{S_h} U_h^{(i)} U_h^{(j)} = \delta(i, j) .$$

Let $u_j = P_h U_h^{(j)}$. We use (31) and (33) in (4) with $U = a_1 U_h^{(1)} + \cdots + a_k U_h^{(k)}$ to see that, for $n = 2$,

$$\lambda^{(k)} \leqq \frac{\lambda_h^{(k)} + \frac{1}{12} h^2 \lambda_h^{(k)2} + \frac{1}{168000} h^8 \lambda_h^{(k)5} + \frac{1}{1334000} h^{10} \lambda_h^{(k)6}}{1 - \frac{1}{40} h^4 \lambda_h^{(k)2} - \frac{1}{560} h^6 \lambda_h^{(k)3}} \tag{35}$$

$$= \lambda_h^{(k)} + \frac{1}{12}h^2\lambda^{(k)2} + 0(h^4\lambda_h^{(k)3}) ,$$

which, for h sufficiently small, is a better bound than (27). Let $V_h^{(j)}$ be the eigenfunction associated with $\Omega_h^{(j)}$ in (8) such that

$$h^n \sum_{S_h} V_h^{(i)} V_h^{(j)} = \delta(i, j) .$$

Let $v_j = P_h V_h^{(j)}$. Use (31) and (34) in (5) with $U = a_1 V_h^{(1)} + \cdots + a_k V_h^{(k)}$ to see that, for $n = 2$,

$$\Omega^{(k)} \leqq \frac{\Omega_h^{(k)} + \frac{1}{2}h^2\Omega_h^{(k)3/2}}{1 - \frac{1}{40}h^4\Omega_h^{(k)} - \frac{1}{560}h^6\Omega_h^{(k)3/2}} \tag{36}$$

(where the Schwarz inequality was employed).

Finally, let $W_h^{(j)}$ be the eigenfunction associated with $\Lambda_h^{(j)}$ in (9) such that

$$\sum_{i=1}^{n} h^n \sum_{S_h} \partial_i W_h^{(j)} \partial_i W_h^{(l)} = \delta(j, l) .$$

Let $w_j = P_h W_h^{(j)}$. Use (32) and (34) in (6) with $U = a_1 W_h^{(1)} + \cdots + a_k W_h^{(k)}$ to see that, for $n = 2$,

$$\Lambda^{(k)} \leqq \frac{\Lambda_h^{(k)} + \frac{1}{2}h^2\Lambda_h^{(k)2}}{1 - \frac{1}{120}h^4\Lambda_h^{(k)2} - \frac{1}{2240}h^6\Lambda_h^{(k)3}} . \tag{37}$$

Explicit upper bounds for higher dimensions may be obtained in the same fashion from (28), (29), and (30). It is clear that, in general,

$$\lambda^{(k)} \leqq \lambda_h^{(k)} + 0(h^2\lambda_h^{(k)2}) , \tag{38}$$

$$\Omega^{(k)} \leqq \Omega_h^{(k)} + 0(h^2\Omega_h^{(k)3/2}) , \tag{39}$$

$$\Lambda^{(k)} \leqq \Lambda_h^{(k)} + 0(h^2\Lambda_h^{(k)2}) . \tag{40}$$

4. **Conclusion.** We notice that the lower bounds (24), (25), (26) are in terms of difference problems on an R_h such that

$$R \subset \bigcup_{x \in R_h} C_h(x) ,$$

while the upper bounds (38), (39), (40) are in terms of difference problems on an R_h such that

$$\bigcup_{x \in R_h} C_{4h}(x) \subset R .$$

However, the problems (1), (2), (3) depend continuously on the domain

R in such a way that if R, R' are domains whose boundaries are within $0(h)$, then, for each k, the eigenvalues $\lambda^{(k)}, \Omega^{(k)}, \Lambda^{(k)}$ for R are within $0(h)$ of the eigenvalues $\lambda'^{(k)}, \Omega'^{(k)}, \Lambda'^{(k)}$ for R', respectively. With this consideration, we can combine the bounds (24) and (38), (25) and (39), (26) and (40), to say that if R_h is such that $\bigcup_{x \in R_h} C_h(x)$ has boundary within $O(h)$ of the boundary of R, then

$$|\lambda^{(k)} - \lambda_h^{(k)}| = O(h) \ , \tag{41}$$

$$|\Omega^{(k)} - \Omega_h^{(k)}| = O(h) \ , \tag{42}$$

$$|\Lambda^{(k)} - \Lambda_h^{(k)}| = O(h) \ . \tag{43}$$

Estimates like (41), (42), (43) can be used in proving convergence of more accurate finite difference schemes which may be regarded as perturbations of the schemes (7), (8), (9). See the paper [2] for details.

Upper and lower bounds for eigenvalues of free membranes by similar techniques may be found in [1]. Further references may be found in [1], [2] and [4].

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