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ON NETS OF CONTRACTIVE MAPS IN UNIFORM SPACES

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R. B. Fraser and S. Nadler have recently proved the following theorem: If X is a locally compact metric space, if $f_n \to f_0$ pointwise, where each f_n, $n = 0, 1, 2, \cdots$ is a contractive map with fixed point a_n, then $f_n \to f_0$ uniformly on compacta, and $a_n \to a_0$. Their method of proof actually showed more. In fact it implied that if a_0 was a fixed point of f_0, and if U is a compact neighborhood of a_0, then there exists a natural number $N(U)$ such that if $n \geqq N(U)$ then f_n had a fixed point $a_n \in U$, and $a_n \to a_0$. In 1963, W. J. Kammerer and R. H. Kasriel proved a theorem giving conditions for existence and uniqueness of fixed points of a general type contractive map on a uniform space. Edelstein in 1965, was able to considerably strengthen their results and achieved a significant extension of the Banach fixed point theorem. In this paper we show that the theorem of Fraser and Nadler may be extended with minor alteration to include locally compact uniform spaces. It was evident in the context of uniform spaces that the covergent sequences of their theorem should be replaced by covergent nets. Our method of proof is similar to their proof and used Edelstein's fixed point theorem.

1. Some preliminary results. In what follows let $(X, \mathfrak{U})$ be a uniform space and let f be a mapping of X into itself. Let $\mathfrak{B}$ be a symmetric base for the uniformity $\mathfrak{U}$.

DEFINITION A. f is said to be a $\mathfrak{B}$-contraction if for each $U \in \mathfrak{B}$ and $(x, y) \in U$, $x \neq y$, a $W \in \mathfrak{B}$ exists such that $(f(x), f(y)) \in W \subset \operatorname{int} U$, and $(x, y) \notin W$.

DEFINITION B. f is said to be $\mathfrak{B}$-contractive if for each $U \in \mathfrak{B}$ and $(x, y) \in U$, $x \neq y$, a $W \in \mathfrak{B}$ exists such that $(f(x), f(y)) \in W \subset U$ and $(x, y) \notin W$.

DEFINITION C. $\mathfrak{B}$ is said to be ample if whenever $(x, y) \in U \in \mathfrak{B}$ there is $V \in \mathfrak{B}$ such that $(x, y) \in V \subset \bar{V} \subset U$.

REMARK. Except for minor changes the above terminology agrees with that used in the papers of Rhodes [8], Brown and Comfort [1], Kamerer and Kasriel [5], Edelstein [3], and Knill [7]. In addition, throughout we will use notation as standard in Kelley [6].

NOTATION. If $\mathfrak{B}$ is a base for the uniformity $\mathfrak{U}$ we will write $\bar{\mathfrak{B}} = \{\bar{B}: B \in \mathfrak{B}\}$. Note that if $\mathfrak{B}$ is ample then $\bar{\mathfrak{B}}$ is a base.

LEMMA 1.1. *Let $\mathfrak{B}$ be an ample base for $\mathfrak{U}$, let f be a $\bar{\mathfrak{B}}$-contraction, and let $a_0 \in X$, be such that $\bar{U}[a_0]$ is compact for some $U \in \mathfrak{B}$. Then there is $V \in \mathfrak{U}$ such that*

$$\bar{V}[f(a_0)] \subset \operatorname{int} U[f(a_0)]$$

and $f(x) \in \bar{V}[f(a_0)]$ if $x \in \bar{U}[a_0]$.

Proof. Let $x \in \bar{U}[a_0]$ be arbitrary. Then there is $W_x \subset \operatorname{int} U$, $W_x \in \mathfrak{B}$, such that $f(x) \in \operatorname{int} W_x[f(a_0)]$. Since the map $g: (a_0, x) \mapsto f(x)$ is continuous and $\mathfrak{B}$ is ample, there is $U_x \subset \bar{U}_x \subset W_x$, $U_x \in \mathfrak{B}$, and $V_x \in \mathfrak{U}$, such that $x \in V_x[a_0]$, $f(x) \in U_x[f(a_0)]$ and if $y \in V_x[a_0]$ then $f(y) \in U_x[f(a_0)]$. Observe that the neighborhoods $\{V_x[a_0]: x \in \bar{U}[a_0]\}$ are an open cover of $\bar{U}[a_0]$ and so there is a finite subcover $V_1[a_0], \cdots, V_k[a_0]$ of $\bar{U}[a_0]$. Furthermore,

$$\begin{aligned} f(\bar{U}[a_0]) &\subset \bigcup_{i=1}^{k} f(V_i[a_0]) \subset \bigcup_{i=1}^{k} U_i[f(a_0)] \\ &\subset \bigcup_{i=1}^{k} \bar{U}_i[f(a_0)] \subset \bigcup_{i=1}^{k} W_i[f(a_0)] \subset \operatorname{int} U[f(a_0)] . \end{aligned}$$

Since $V = \bigcup_{i=1}^{n} U_i$ satisfies $V \in \mathfrak{U}$,

$$\bar{V}[f(a_0)] = \bigcup_{i=1}^{k} U_i[f(a_0)] \subset \operatorname{int} U[f(a_0)] ,$$

and

$$f(x) \in \bar{V}[f(a_0)] , \quad \text{if} \quad x \in \bar{U}[a_0] ,$$

the lemma is proved.

REMARK. It should be noted that the proof went through by means of a standard compactness argument because a contraction f maps a point of $\bar{U}[a_0]$ into the interior of $\bar{U}[f(a_0)]$. Thus any contractive map g on $\bar{U}[a_0]$ that sends $x \in \bar{U}[a_0]$ into $\operatorname{int} \bar{U}[g(a_0)]$, for each $x \in \bar{U}[a_0]$, could be substituted for f in the lemma. We shall be applying this fact in the future in the special case where $g(a_0) = a_0$.

DEFINITION D. A uniform space $(X, \mathfrak{U})$ is U-chainable, if for each pair $x, y \in X$, a finite sequence $x = x_0, x_1, \cdots, x_n = y$ exists such that $(x_{i-1}, x_i) \in U \in \mathfrak{U}$ for $i = 1, 2, \cdots, n$. Such a sequence is called a U-chain.

COROLLARY 1.2. *If $U \in \mathfrak{U}$ is symmetric then $\bar{U}[x]$ is U-chainable, for each $x \in X$.*

Proof. If $y, r \in \bar{U}[x]$, there is $z \in U[y] \cap U[x]$ and $s \in U[r] \cap U[x]$, so that $(y, z) \in U$, $(z, x) \in U$, $(x, s) \in U$, and $(s, r) \in U$.

DEFINITION E. Let $\xi \in X$ then $X^f(\xi) = \{y: y$ is a cluster point of the sequence $\{f^n(\xi)\}_{n=1}^{\infty}\}$. Here $f^n[x] = f(f^{n-1}(x))$.

THEOREM 1.3. (*M. Edelstein* [3]). *Let $\mathfrak{B}$ be an open ample base for $(X, \mathfrak{U})$, a U-chainable uniform space for some $U \in \mathfrak{B}$, and let $f: X \to X$ be B-contractive. If $X^f(\xi) \neq \emptyset$, for some ξ, if $x \in X^f(\xi)$, and if $X^f(y) \neq \emptyset$ for each $y \in U[x]$, then f has the unique fixed point x.*

REMARK. The theorem above is a slightly weaker form of the theorem of M. Edelstein. The condition $X^f(\xi) \neq \emptyset$, for some ξ is needed in the theorem as may be seen by the following example.

Let $G = R \times Z$ where R denotes the real line with the usual topology, and Z denotes the integers with the discrete topology. This G is locally compact, σ-compact, locally connected, and even metrizable. Let $f: G \to G$ be defined by $f(x, k) = (\frac{1}{2}x, k+1)$. Then f is a contraction on the base $\mathfrak{B} = \{U^*: (\xi, y) \in U^*$ if and only if $y \in \xi(U \times \{0\})$, where U is a basic open neighborhood of $0 \in R\}$. However $X^f(\xi) = \emptyset$ for every $\xi \in G$, and clearly f has no fixed point.

We note that W. J. Krammerer and R. K. Kasriel [5] proved a slightly weaker form of 1.3. They showed that if $\mathfrak{B}$ is an ample base for $(X, \mathfrak{U})$, a U-chainable uniform space $(U \in \mathfrak{B})$, and if some iterate of X under f is compact, where f is $\mathfrak{B}$-contractive, then f has a unique fixed point.

COROLLARY 1.4. *Let $\mathfrak{B}$ be an ample open base of the uniformly locally compact uniform space $(X, \mathfrak{U})$. Let $U \in \mathfrak{B}$ be such that $\bar{U}[x]$ is compact for all $x \in X$. Suppose that*

(i) *f is $\mathfrak{B}$-contractive*

(ii) *there exists an $x \in X$ such that f maps $U[x]$ into itself. Then the following occur*

(i) *f has exactly one fixed point y in $\bar{U}[x]$.*

(ii) *the filter base $f^n[\bar{U}[x]]$ converges to y*

(iii) *for each $\bar{V} \subset \bar{\mathfrak{B}}$ there exists k such that $f^k(\bar{U}[x]) \subset \operatorname{int} \bar{V}[f^k(x)]$. Furthermore, if X is U-chainable, then f has a unique fixed point.*

Proof. We may observe that f is continuous so that $f(\bar{U}[x]) \subset \bar{U}[x]$. Also $\bar{U}$ is U-chainable by 1.2. Let G be the restriction of f

to $\bar{U}[x]$. Then $g: \bar{U}[x] \to \bar{U}[x]$ and $\bar{U}[x]$ is compact. Thus g and $\bar{U}$ satisfy the conditions of the theorem of Kamerer and Kasriel so that $\bar{U}[x]$ has exactly one fixed point of g, say y. This proves (i).

According to the theorem of Kammerer and Kasriel [5], $\{y\} = \bigcup \{X^g(\xi): \xi \in \bar{U}[x]\}$. Let now $z \in \bar{U}[x]$, and let $V \in \mathfrak{B}$. Since $f^n(z) \to y$ there is an integer $N(z)$ such that $n \geqq N(z)$ implies that $f^n(z) \in V[y]$. Since f is continuous there is $W_z \in \mathfrak{B}$ such that $f^n(W_z[z]) \subset V[y]$. Clearly the collection $\{W_z[z]: z \in \bar{U}[x]\}$ covers $\bar{U}[x]$ and so a finite subcollection $W_1[z_1], \cdots, W_k[z_k]$ also covers $\bar{U}[x]$. But then if $n \geqq N(z_1) + \cdots + N(z_k)$, it follows that $f^n(\bar{U}[x]) \subset V[y]$, proving (ii).

Let $V \in \mathfrak{B}$ and let $W \in \mathfrak{B}$ be such that $W \circ W \subset \operatorname{int} \bar{V}$, and $W \subset V$. By part (ii) there is k such that

$$f^k(\bar{U}[x]) \subset W[y] .$$

Let $z \in \bar{U}[x]$, then

$$(f^k(z), y) \in W \text{ and } (y, f^k(x)) \in W$$

so that $f^k(z) \in W \circ W[f^k(x)] \subset \operatorname{int} \bar{V}[f^k(x)]$. But then

$$f^k(\bar{U}[x]) \subset \operatorname{int} \bar{V}[f^k(x)]$$

proving (iii).

The final statement of this corollary is evident from Edelstein's theorem.

REMARK. Condition (ii) was suggested by theorem due to Knill [7]; p. 453.

2. The main results. The above theorem of M. Edelstein and the resultant corollary allows us to choose the setting for a generalization of the theorem of R. B. Fraser and S. Nadler [4] on contractive maps with fixed points. It is evident that in a uniform space where sequential convergence does not suffice to describe topological properties it is natural to replace sequential convergence of functions by convergence of nets of functions. Thus we use the following.

THEOREM 2.1. *Let $(X, \mathfrak{U})$ be uniformly locally compact and let $\mathfrak{B}$ be an ample base. If $\{f_\alpha: \alpha \in D\}$ is a net of $\mathfrak{B}$-contractive maps, of X into itself, converging pointwise to f_0, a $\mathfrak{B}$-contractive map, then f_α converges uniformly to f_0 on each compact set $K \subset G$.*

Proof. Let $U \in \mathfrak{B}$ be arbitrary, and let $V \in \mathfrak{B}$ be such that $V \circ V \circ V \subset U$, $V \subset U$ (see Kelley [6], p. 180. Th. 8). Since $f_\alpha \to f_0$ pointwise there exists for each x, an $\alpha(x) \in D$, such that $\alpha \geqq \alpha(x)$

implies that $(f_0(x), f_\alpha(x)) \in V$.

Clearly if K is compact the collection $\{V[x]: x \in K\}$ is a cover of K and hence is reducible to a finite subcover $V[x_1], \cdots, V[x_k]$ of K (note that the collection $\{\text{int } V[x]: x \in K\}$ is an open cover of K). For each i, let $\alpha_i = \alpha(x_i) \in D$, be such that $(f_0(x_i), f_\alpha(x_i)) \in V$ if $\alpha \geqq \alpha_i$.

Let $x \in K$ be arbitrary. Then there is $x_j \in \{x_1, \cdots, x_k\}$ such that $(x_j, x) \in V$ and therefore $(f_\alpha(x_j), f_\alpha(x)) \in V$ for all α, since the f_α are $\mathfrak{B}$-contractive. Thus if $\alpha_0 \geqq (\alpha_1, \cdots, \alpha_n)$ we have for all $\alpha \geqq \alpha_0$ that $(f_0(x), f_\alpha(x)) \in V \circ V \circ V \subset U$. [Since $(f_0(x), f_0(x_j)) \in V$, $(f_0(x_j), f_\alpha(x_j)) \in V$, and $(f_\alpha(x_j), f_\alpha(x)) \in V$.]

REMARK. It is evident from the proof of 2.1 that the theorem is true if the $f_\alpha(f_0)$ are weakly contractive [5] or nonexpansive [3].

THEOREM 2.2. *Let $\mathfrak{B}$ be an ample open base of the uniformly locally compact uniform space $(X, \mathfrak{U})$. Let $U \in \mathfrak{B}$ be such that $\bar{U}[x]$ is compact for all $x \in X$. Let $\{f_\alpha: X \to X \mid \alpha \in D\}$ be a net of $\bar{\mathfrak{B}}$-contractions, and let $f_\alpha \to f$ pointwise, where f is a $\bar{\mathfrak{B}}$-contraction. If f has a fixed point y_0, then*

(i) *there is $\alpha_0 \in D$ such that if $\alpha \geqq \alpha_0$ then f_α has exactly one fixed point in $\bar{U}[y_0]$, say y_α.*

(ii) $y_\alpha \to y_0$.

Furthermore if X is U-chainable then y is a unique fixed point of f and y_α, $\alpha \geqq \alpha_0$, are unique fixed points of f_α.

Proof. We observe that $\bar{U}[y_0]$ is a compact neighborhood of y_0. Since f is a $\bar{\mathfrak{B}}$-contraction there is $V \in \mathfrak{U}$ such that $\bar{V}[y_0] \subset U[y_0]$ and $f(x) \in \bar{V}[y_0]$ if $x \in \bar{U}[y_0]$ (1.1). By Theorem 33 in Kelley [6], there is $W \in \mathfrak{U}$ such that $W[\bar{V}[y_0]] \subset U[y_0]$.

By Theorem 2.1, $f_\alpha \to f$ uniformly on $U[y_0]$, and so there is α_0 such that $\alpha \geqq \alpha_0$ implies that $f_\alpha(x) \in W[f(x)]$ for all $x \in U[y_0]$. But then if $x \in U[y_0]$, and $\alpha \geqq \alpha_0$ we have

$$f_\alpha(x) \in W \circ \bar{V}[y_0] \subset U[y_0]$$

since $f(x) \in \bar{V}[y_0]$ and $f_\alpha(x) \in W[f(x)]$. This shows that f_α, $\alpha \geqq \alpha_0$ maps $U[y_0]$ into itself. By Corollary 1.4, $\bar{U}[y_0]$ contains exactly one fixed point of f_α, if $\alpha \geqq \alpha_0$, proving (i).

Since we could repeat the above argument for each $V \in \mathfrak{B}$, $V \subset U$, it follows that $y_\alpha \to y_0$, proving (ii). The remainder of the proof is clear.

REMARK. The index α_0 above depends on W which in turn depends on U. This is the reason we may conclude $y_\alpha \to y_0$. Furthermore a straightforward application of 1.4 yields the fact that the fixed point y_α is an isolated fixed point of f_α.

THEOREM 2.3. *Let $\mathfrak{B}$ be an ample open base of the uniformly locally compact uniform space $(X, \mathfrak{U})$. Let $U \in \mathfrak{B}$ be such that $\bar{U}[x]$ is compact for all $x \in X$, and suppose X is U-chainable. Let $\{f_\alpha: X \rightarrow X \mid \alpha \in D\}$ be a net of $\mathfrak{B}$-contractive maps, and let $f_\alpha \rightarrow f$ pointwise, where f is a $\mathfrak{B}$-contractive map.*

If f has fixed point y_0, then

(i) *there is $\alpha_0 \in D$ such that if $\alpha \geqq \alpha_0$ then f_α has exactly one fixed point $y_\alpha \in X$*

(ii) $y_\alpha \rightarrow y_0$.

Proof. We observe that since f is $\mathfrak{B}$-contractive, there is by 1.4 and 1.1 an integer k and a $V \in \mathfrak{U}$, such that $\bar{V}[y_0] \subset U[y_0]$ and $f^k(x) \in V[y_0]$ if $x \in \bar{U}[y_0]$. By Theorem 33 in Kelley [6], there is $W \in \mathfrak{U}$ such that $W[\bar{V}[y_0]] \subset U[y_0]$.

By Theorem 2.1, (since g^k is a $\mathfrak{B}$-contractive map if g is) $f_\alpha^k \rightarrow f^k$ uniformly on $\bar{U}[y_0]$, and so there is α_0 such that $\alpha \geqq \alpha_0$ implies that $f_\alpha^k(x) \in W[f^k(x)]$ for all $x \in \bar{U}[y_0]$. But then if $x \in U[y_0]$, and $\alpha \geqq \alpha_0$ we have

$$f_\alpha^k(x) \in W[\bar{V}[y_0]] \subset U[y_0]$$

since $f^k(x) \in \bar{V}[y_0]$ and $f_\alpha^k(x) \in W[f(x)]$. This shows that f_α^k, $\alpha \geqq \alpha_0$ maps $U[y_0]$ into itself. For each $j = 1, \cdots, k-1$, let n_j be the smallest integer such that $f_\alpha^j(y_0) \in U^{n_j}[y_0]$. Let $f_\alpha^0 = id$, and $Y_\alpha = \bigcup_{j=0}^{k-1} \bar{U}^n[f_\alpha^j(y_0)]$, where $n = \sup\{n_1, \cdots, n_{k-1}\}$. Then Y_α is compact (each $\bar{U}^{n_j}[x] = \bar{U} \circ \bar{U}^{n_j - 1}[x]$ is compact for all $x \in X$), and U-chainable, and $f_\alpha: Y_\alpha \rightarrow Y_\alpha$. Thus the theorem of Kammerer and Kasriel [5] implies that f_α has a unique fixed point in Y_α. But then by 1.3 and the fact that $X^{f_\alpha}(\xi) \cap \bar{U}[y_0] \neq \varnothing$ for $\xi \in \bar{U}[y_0]$ (Since $f_\alpha^k: \bar{U}[y_0] \rightarrow \bar{U}[y_0]$) it follows that $\bar{U}[y_0]$ contains the fixed point y_α. The remainder of the proof follows as in 2.2.

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