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A GENERAL THEOREM FOR BILINEAR GENERATING FUNCTIONS

S. SARAN

The following theorem was proved by Chatterjea for ultraspherical polynomials:

$$If \quad F(x, t) = \sum_{m=0}^{\infty} a_m t^m P_m^\lambda(x)$$

then

$$\bar{\rho}^{2\lambda} F\left(\frac{x-t}{\rho}, \frac{ty}{\rho}\right) = \sum_{r=0}^{\infty} t^r b_r(y) P_r^\lambda(x)$$

where

$$b_r(y) = \sum_{m=0}^{\infty} (v_m) a_m y^m \text{ and } \rho = (1 - 2xt + t^2)^{1/2} .$$

The object of this paper is to show that a general theorem for any polynomial satisfying certain conditions can be given so as to include the above case, and may be applicable in obtaining new generating functions for other polynomials also.

THEOREM. **If**

$$f_n(x) = \mu(n) G(x) D^n\{g(x)\} \tag{1.1}$$

where $g(x)$ **and** $G(x)$ **are independent of** n, **and**

$$F(x, t) = \sum_{m=0}^{\infty} a_m t^m f_m(x) \tag{1.2}$$

then

$$\frac{G(x) F(x-t, ty)}{G(x-t)} = \sum_{r=0}^{\infty} \frac{(-t)^r}{\mu(r) r!} b_r(y) f_r(x)$$

where

$$b_r(y) = \sum_{m=0}^{r} (-r)_m \mu(m) a_m y^m .$$

2. Proof of the theorem. Writing ty for t in (1.2) and using (1.1) we get

$$[G(x)]^{-1} F(x, ty) = \sum_{m=0}^{\infty} a_m t^m y^m \mu(m) D^m[g(x)] .$$

Applying the operator $\bar{e}^{tD}$ where $D \equiv d/dx$ on both sides, we get, since

$$\bar{e}^{tD} f(x) = f(x - t)$$

$$\begin{aligned}\frac{F(x-t,ty)}{G(x-t)} &= \sum_{m=0}^{\infty} a_m t^m y^m \mu(m) \bar{e}^{tD} D^m[g(x)] \\ &= \sum_{m=0}^{\infty} a_m t^m y^m \mu(m) \sum_{r=0}^{\infty} \frac{(-tD)^r}{r!} D^m[g(x)] \\ &= \sum_{m=0}^{\infty} a_m y^m \mu(m) \sum_{r=0}^{\infty} \frac{(-)^r t^{m+r}}{r!} D^{m+r}[g(x)] \\ &= \sum_{r=0}^{\infty} \frac{(-)^r t^r}{r!} D^r[g(x)] \sum_{r=0}^{m} (-r)_m a_m \mu(m) y^m .\end{aligned}$$

This on using (1.1) proves the theorem.

3. Some applications of theorem. (i) First we consider the generating function given by Brafman

$$(1-2xt)^{-C}{}_2F_0\left(\tfrac{1}{2}C, \tfrac{1}{2}C+\tfrac{1}{2}, -: \frac{-4t^2}{(1-2xt)^2}\right) \cong \sum_{n=0}^{\infty} \frac{(c)_n H_n(x) t^n}{n!} .$$

If we take $(a)_m = (c)_m/m!$, $f_m(x) = H_m(x)$, $G(x) = \exp(x^2)$, $\mu(n) = (-)^n$ from Rodrigues formula [5] we obtain

$$\begin{aligned}&[1+2yt(x-t)]^{-c} \exp(2xt - t^2)_2F_0\left(\tfrac{1}{2}c, \tfrac{1}{2}c+1, -: \frac{-4y^2t^2}{(1+2xyt-2yt^2)^2}\right) \\ &\cong \sum_{n=0}^{\infty} \frac{{}_2F_0(-n, c, -: y) H_n(x) t^n}{n!} ,\end{aligned}$$

a relation proved by Brafman [1] and Rainville [5] in different ways.

(ii) We now consider the generating function given by Carlitz [2]

$$[1+\tfrac{1}{2}(x+1)t]^{\alpha}[1+\tfrac{1}{2}(x-1)t]^{\beta} = \sum_{n=0}^{\infty} P_n^{\alpha-n,\beta-n}(x) t^n . \tag{3.1}$$

Writing tz for t, multiplying both sides by $\bar{e}^z z^{c-1}$ and integrating w.r.t. z we get

$$\begin{aligned}&\sum_{n=0}^{\infty} (c)_n P_n^{\alpha-n,\beta-n}(x) t^n \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-\alpha)_m(-\beta)_n(c)_{m+n}}{m!\,n!} [\tfrac{1}{2}(x+1)t]^m [\tfrac{1}{2}(x-1)t]^n .\end{aligned}$$

Similarly, writing t/p for t, multiplying both sides by $e^p p^{-D}/2\pi i$ and evaluating over the contour C given by Hankel [4] we get

$$\begin{aligned}&F^{(1)}(c, D; -\alpha, -\beta: \tfrac{1}{2}(x+1)t, \tfrac{1}{2}(x-1)t) \\ &= \sum_{n=0}^{\infty} \frac{(c)_n}{(D)_n} P_n^{\alpha-n,\beta-n}(x) t^n ,\end{aligned} \tag{3.2}$$

where $F^{(1)}$ is Appell's hypergeometric function of the first kind [5] defined by

$$F^{(1)}(a, b; c, d: x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n}(c)_m(d)_n}{(b)_{m+n} m!\, n!} x^m y^n \qquad |x| < 1, |y| < 1 .$$

Assuming that $f_n^{\alpha-n,\beta-n}(x) = (1/(1-x^2))p_n^{\alpha-n,\beta-n}(x)$ (3.2) can be put in the form

$$\text{(3.3)} \quad \begin{aligned} &F^{(1)}\left(c, D; -\alpha, -\beta: \frac{-t}{2(x-1)}, \frac{-t}{2(x+1)}\right) \\ &= \sum_{n=0}^{\infty} \frac{(c)_n}{(D)_n} f_n^{\alpha-n,\beta-n}(x) t^n . \end{aligned}$$

The differential formula for $f_n^{\alpha-n,\beta-n}(x)$ is [5]

$$\text{(3.4)} \quad f_n^{\alpha-n,\beta-n}(x) = \frac{(-)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta} D^n[(1-x)^\alpha(1+x)^\beta] .$$

Thus comparing (3.3) and (3.4) with (1.1) and (1.2) we get

$$a_m = \frac{(c)_m}{(D)_m}, \ \mu(n) = \frac{(-)^n}{2^n n!}, \ G(x) = (1-x)^\alpha(1+x)^\beta$$

and therefore our theorem gives, after replacing t by $t(1-x^2)$,

$$\text{(3.5)} \quad \begin{aligned} &[1 + \tfrac{1}{2}t(x+1)]^\alpha[1 + \tfrac{1}{2}t(x-1)]^\beta \\ &\quad \times F^{(1)}\left(C, D; -\alpha, -\beta: \frac{ty(x+1)}{2+t(x+1)}, \frac{ty(x-1)}{2+t(x-1)}\right) \\ &= \sum_{n=0}^{\infty} {}_2F_1(-n, c; D: -y) p_n^{\alpha-n,\beta-n}(x) t^n . \end{aligned}$$

Since $p_n^{\alpha-n,\beta-n}(x)$ can be reduced to $L_n^{\alpha-n}(x)$ by replacing x by $1 - 2x/\beta$ and letting β tending to infinity, the above result can be put in the following interesting form

$$\text{(3.6)} \quad \begin{aligned} &(1+t)^\alpha \exp(xt)\varphi_1\left(C, -\alpha, D; -xty, \frac{ty}{t+1}\right) \\ &= \sum_{n=0}^{\infty} {}_2F_1(-n, C; D; -y) L_n^{\alpha-n}(x) \end{aligned}$$

where ϕ_1 is a confluent form of $F^{(1)}$ given by

$$\phi_1(\alpha, \beta, \gamma; x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_n}{(\gamma)_{m+n} m! n!} x^m y^n , \qquad |x| < 1 .$$

This again reduces to the following result due to Srivastava [6]

$$\text{(3.7)} \quad \begin{aligned} &(1+t)^\alpha \exp(-yt)\phi_3\left(-\alpha, D: \frac{yt}{t+1}, xyt\right) \\ &= \sum_{n=0}^{\infty} \binom{D-1+n}{n}^{-1} L_n^{D-1}(y) L_n^{\alpha-n}(x) t^n \end{aligned}$$

where ϕ_3 is a confluent form of ϕ_1. This is obtained by replacing y by $-(y/C)$ in (3.4) and letting C tend to infinity.

4. Generalization of (3.2). The result (3.1) is capable of generalization by the repeated application of the method indicated in proving (3.2). One can easily write it as

$$\begin{aligned} &F^{r,1}_{0,0}\left(\begin{vmatrix} a \\ b \end{vmatrix}\begin{matrix} r \\ s \end{matrix}; -\alpha, -\beta \colon \tfrac{1}{2}(x+1)t, \tfrac{1}{2}(x-1)t\right) \\ &= \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_r)_n}{(b_1)_n \cdots (b_s)_n} p_n^{\alpha-n,\beta-n}(x)t^n \end{aligned} \tag{4.1}$$

in the notation of Srivastava and Saran [7] for Kampé de Fériet function.

Thus (3.5) can be generalized into

$$\begin{aligned} &[1 + \tfrac{1}{2}t(x+1)]^{\alpha}[1 + \tfrac{1}{2}t(x-1)]_x^{\beta} \\ &\quad \times F^{r,1}_{s,0}\left(\begin{vmatrix} a \\ b \end{vmatrix}\begin{matrix} r \\ s \end{matrix}; -\alpha, -\beta \colon \frac{yt(x+1)}{2+t(x+1)}, \frac{yt(x-1)}{2+t(x-1)}\right) \\ &= \sum_{n=0}^{\infty} {}_{r+1}F_s(-n, a_r; b_s \colon -y) p_n^{\alpha-n,\beta-n}(x)t^n \, . \end{aligned} \tag{4.2}$$

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PUNJABI UNIVERSITY
PATIALA, (INDIA)