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RINGS WHOSE HOMOMORPHIC IMAGES ARE q-RINGS

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L. Levy has characterised a commutative noetherian ring in which every proper homomorphic image is self-injective to be a Dedekind domain, or a principal ideal ring with descending chain condition, or a local ring whose maximal ideal M has composition length 2 and satisfies $M^2 = 0$. The object of this paper is to generalise Levy's result to the noncommutative case by studying right noetherian rings in which every proper homomorphic image is a right q-ring. Since every commutative self-injective ring is a q-ring, one can get Levy's result as a special case of Theorems 2.12 and 2.13.

1. **Definitions, notations and preliminaries.** Throughout this paper, every ring has unity. J and $B(R)$ will denote the Jacobson and the prime radical of a ring R respectively. If X is a subset of a ring R, $r(X)$ (resp. $l(X)$) will mean the right (resp. left) annihilator of X in R. A right (left) ideal of a ring R is said to be a right (left) annulet if $A = r(X)(A = l(X))$ for some subset X of R. A ring R is called a duo ring if every one-sided ideal of R is two-sided A ring R is said to be a right (left) q-ring if every right (left) ideal of R is a quasi-injective right (left) R-module. By a q-ring we mean a ring which is both right and left q-ring. It was proved in [6] that a ring R is a right q-ring if and only if R is right self-injective and every large right ideal of R is two-sided. It is clear that a right self-injective duo ring is a right q-ring. In particular, every commutative self-injective ring is a q-ring. It is also shown in [8, Corollary 1.6] that a local right q-ring is a duo ring.

A nonzero ideal A of a ring R is said to be an indecomposable ideal if A is not expressed as the direct sum of two nonzero ideals of R. An indecomposable ideal B of a ring R is called a block if there exists an ideal B' of R such that $R = B \oplus B'$.

A ring R is said to be a principal ideal ring (PIR) if every one-sided ideal of R is cyclic. The notation $PIRD$ will mean a principal ideal ring which is also a duo ring.

2. We start by proving and recording some results which are essential for proving the main theorems.

LEMMA 2.1. *Let R be a right noetherian ring with the property that every proper homomorphic image is right self-injective. Then,*

(i) *every proper homomorphic image is QF,*

(ii) *every nonzero prime ideal is maximal, and*
(iii) *if R is not prime, then $J = B(R)$.*

Proof. (i) Since every proper homomorphic image is right noetherian and right self-injective, then every proper homomorphic image is QF (Cf. [2, Th. (1)])

(ii) Let P be a nonzero prime ideal of R, then R/P is artinian, by (i). Hence R/P is a simple ring, proving (ii). (iii) is an immediate consequence of (ii).

The following two well known results will be stated without proof.

LEMMA 2.2. *If R is a right noetherian ring, then*
(i) *every ideal of R contains a product of prime ideals, and*
(ii) *in case R is prime, every nonzero ideal contains a product of nonzero prime ideals.*

LEMMA 2.3. *If $M_1, M_2, \cdots, M_n$ are maximal ideal of a ring R such that $M_1M_2, \cdots, M_n = 0$, then every prime ideal of R is in the set $\{M_1, M_2, \cdots, M_n\}$.*

Jain [5] proved the following lemma. We are giving here another proof.

LEMMA 2.4. *Let R be a right noetherian ring with the property that every proper homomorphic image is right self-injective. If R is not prime, then R is right artinian.*

Proof. By 2.1 and 2.2,

$$0 = M_1M_2 \cdots M_t$$

where $\{M_i, 1 \leqq i \leqq t\}$ is a set of maximal ideals of R. After renumbering, if necessary, let $M_1, M_2, \cdots, M_n, n \leqq t$ be the distinct ideals in the set $\{M_i, 1 \leqq i \leqq t\}$. Then, by 2.1 and 2.3,

$$J = B(R) = \bigcap_{i=1}^{n} M_i \, .$$

Now, if $J = 0$, we get

$$R \cong \sum_{i=1}^{n} \oplus R/M_i \, .$$

Since each R/M_i is artinian by 2.1, R is also artinian.

On the other hand, if $J \neq 0$, then R/J is artinian. Since $J = B(R)$, J is a nil ideal and hence nilpotent, by Levitzki's Theorem [4,

p. 199]. Hence R is right artinian follows by the method of Hopkins [3].

SUBLEMMA 2.5. *If M is a maximal ideal of a QF-ring R, then $r(M)$ is a minimal ideal in R.*

LEMMA 2.6. *Let R be a ring with the property that every proper homomorphic image is QF. If M_1 and M_2 are distinct maximal ideals of R such that $M_1M_2 \neq 0$ and $M_2M_1 \neq 0$, then*

$$M_1M_2 = M_1 \cap M_2 = M_2M_1 .$$

Proof. Let $S = R/M_1M_2$, $A = M_1/M_1M_2$ and $B = M_2/M_1M_2$. Because B is a maximal ideal in S and $AB = 0$, $r(A) = B$. Hence, B is a minimal ideal by 2.5. This implies that there exists no ideal of R between M_2 and M_1M_2. Thus $M_1 \cap M_2 = M_1M_2$ Similarly $M_1 \cap M_2 = M_2M_1$. This completes the proof.

LEMMA 2.7. *Let $\{A_i, 1 \leqq i \leqq n\}$ be a set of pairwise comaximal ideals of a ring R such that $A_iA_j = A_jA_i$, $1 \leqq i, j \leqq n$. Then, for arbitrary positive integers $\alpha_1, \alpha_2, \cdots, \alpha_n$.*

(i) *$(A_1^{\alpha_1}A_2^{\alpha_2} \cdots A_{n-1}^{\alpha_{n-1}})$ and $A_n^{\alpha_n}$ are comaximal ideals.*

(ii) *$A_1^{\alpha_1}A_2^{\alpha_2} \cdots A_n^{\alpha_n} = A_1^{\alpha_1} \cap A_2^{\alpha_2} \cap \cdots \cap A_n^{\alpha_n}$.*

Proof. Similar to that given in [9, Th. 31] for commutative rings.

Next we prove the following

PROPOSITION 2.8. *Let R be a right artinian block such that every proper homomorphic image of R is right self-injective. Then,*

(i) *R contains at most two maximal ideals M_1, and M_2,*

(ii) *if $M_1 \neq M_2$, then $M_1M_2 = 0$ or $M_2M_1 = 0$ (not both); moreover if $M_1M_2 = 0$, then R has exactly three proper ideals M_1, M_2 and M_2M_1,*

(iii) *if R is right self-injective, then R has a unique maximal ideal.*

Proof. If R is prime, then it is simple and we have nothing to prove. So assume that R is not prime.

(i) If possible, let R contain more than two distinct maximal ideals. Then, in view of 2.3, the product of any two maximal ideals of R is not zero. Hence, by 2.6, $MN = NM$ for every pair of maximal ideals M and N of R. Let $\{M_1, M_2, \cdots, M_k\}$ be a minimal set of maximal ideals such that $M_1M_2\cdots M_k = 0$. Collecting the powers of the M_i's together, we get

$$M_1^{\alpha_1}M_2^{\alpha_2}\cdots M_n^{\alpha_n} = 0$$

where $\{M_1, M_2, \cdots, M_n\}$ is a set of distinct maximal ideals (renumbering may take place) and $\alpha_1 + \alpha_2 + \cdots + \alpha_n = k$. It is clear that $n > 1$; otherwise M_1 would be the unique maximal ideal of R. Let

$$A = M_1^{\alpha_1}M_2^{\alpha_2}\cdots M_{n-1}^{\alpha_{n-1}} \quad\text{and}\quad B = M_n^{\alpha_n}\ .$$

Then, by Lemma 2.7, $R = A + B$. But since $AB = 0 = BA$, $R = A \oplus B$, which contradicts the fact that R is a block. Hence R contains at most two maximal ideals, say M_1 and M_2.

(ii) Suppose that $M_1M_2 \neq 0$ and $M_2M_1 \neq 0$. Then, by Lemma 2.6 $M_1M_2 = M_2M_1$. But this would imply, as in (i), that $R = M_1^{\alpha_1} \oplus M_2^{\alpha_2}$, for some positive integers α_1 and α_2, which is a contradiction. Thus if $M_1 \neq M_2$, then either $M_1M_2 = 0$ or $M_2M_1 = 0$.

Let $M_1M_2 = 0$. Then, it is clear that $M_2M_1 \neq 0$; otherwise $R = M_1 \oplus M_2$, a contradiction. Let X be a proper ideal of R such that $X \not\subset M_1$. Then $R = X + M_1$, which implies that $M_2 \subset X + M_1M_2$, whence $M_2 = X$. Similarly if Y is a proper ideal of R such that $Y \not\subset M_2$, then $Y = M_1$. Therefore, if A is a proper ideal of R such that $A \neq M_1$ and $A \neq M_2$, then $A \subset M_1 \cap M_2 = M_2M_1$. Now, consider the QF-ring R/A and let $X = M_1/A$ and $Y = M_2/A$. Then X and Y are maximal ideals in R/A, and $Y = r(X)$. Consequently, Y is a minimal ideal in R/A and there exists no ideal of R between A and M_2. This implies that $A = M_2M_1$, proving (ii).

(iii) Suppose that $M_1 \neq M_2$ and let $M_1M_2 = 0$. Since R is QF, M_2 is a minimal ideal of R. Then $M_1 \cap M_2 = 0$; otherwise $M_2 \subset M_1$ which implies that $M_2 = M_1$. But $M_1 \cap M_2 = 0$ implies $R = M_1 \oplus M_2$, which is not the case, since R is a block. Therefore $M_1 = M_2$, completing the proof.

LEMMA 2.9. *Let R be a right q-ring. If M is a maximal ideal of R such that $M^n = 0$ and $M^{n-1} \neq 0$, $n > 1$, then R is a local duo ring and M^{n-1} is the unique minimal* (*right*) *ideal of R.*

Proof. By 2.3, M is the unique maximal ideal of R. Then, since $M \neq 0$, $l(M) \subset M$, which implies that M is a large right ideal of R. Thus, the fact that R is a right q-ring implies that every right ideal containing M is two-sided [6, Th. 2.3]. Hence M is a maximal right ideal, and therefore is the unique maximal right ideal of R. Then R is a local ring having M as its Jacobson radical. Now, R is a duo ring follows by [8, Corollary 1.6].

Since $M^n = 0$ and $M^{n-1} \neq 0$, $0 \neq M^{n-1} \subset l(M) = \operatorname{soc} R$. But since R_R is indecomposable and injective, R_R is uniform which implies that M^{n-1} is the unique minimal (right) ideal of R, completing the proof.

LEMMA 2.10. *Let R be a ring with the property that every proper homomorphic image is a right q-ring. If M is a maximal ideal of R such that $M^n \neq 0$, $n > 2$, then the only right (or left) ideals of R between M and M^n are powers of M.*

Proof. For every integer i, $2 \leqq i \leqq n$, let $S_i = R/M^i$ and $N_i = M/M^i$. If $(N_i)^{i-1} = 0$, then $M^{i-1} = M^i$. On the other hand, if $(N_i)^{i-1} \neq 0$, then by 2.9, $(N_i)^{i-1}$ is a minimal right (left) ideal in S_i. In either case, there exists no right (or left) ideal of R between M^{i-1} and M^i, $2 \leqq i \leqq n$.

Let A be a (right) ideal of R such that $M \supset A \supset M^n$. If $A \neq M$, then we assert that $A \subset M^2$. For, if $A \not\subset M^2$, then $M = A + M^2$, whence

$$M^{i-1} \subset A + M^i \qquad i = 2, 3, \cdots, n .$$

But this would imply that $A = M$, a contradiction. This proves our assertion.

Again, if $A \neq M^2$, then as before $A \subset M^3$. Proceeding in this way and assuming that $A \neq M^i$, $1 \leqq i \leqq n-2$, we get $M^{n-1} \supset A \supset M^n$. But this implies that $A = M^{n-1}$ or $A = M^n$. This proves that the only right (or left) ideals between M and M^n are powers of M.

PROPOSITION 2.11. *Let R be a ring with the property that every proper homomorphic image is a right q-ring. If M is a maximal ideal of R such that $M^n = 0$, $n > 2$, then R is a PIRD with descending chain condition.*

Proof. Let $S = R/M^2$ and $N = M/M^2$. Then S is a right q-ring and N is a maximal ideal in S satisfying $N^2 = 0$. Hence, by Lemma 2.9, N is the only right (left) ideal of S. This implies that M is a maximal right ideal of R and there exists no right (or left) ideal of R between M and M^2.

Now, $M^n = 0$ implies that M is the unique maximal right ideal of R. Hence R is a local ring having M as its Jacobson radical. Let x be an element of M such that $x \notin M^2$. Then,

$$M = xR + M^2 = Rx + M^2 .$$

This implies that

$$M^{i-1} \subset xR + M^i, \qquad i = 3, 4, \cdots, n .$$

And since $M^n = 0$, $M^2 \subset xR$. Hence $M = xR = Rx$.

We proceed to show that every principal right ideal of R is of the form $x^\alpha R$ for some positive integer α. Let $0 \neq y \in R$. If y is not a unit, then $y \in M = xR$. This implies that $y = xy_1$ for some $y_1 \in R$. Again if y_1 is not a unit, then $y_1 = xy_2$ for some $y_2 \in R$, and hence $y = x^2 y_2$. Proceeding in this way, noting that $x^n = 0$, we get $y = x^\alpha y_\alpha$ for some positive integer $\alpha < n$, where y_α is a unit in R. Hence,

$$yR = x^\alpha y_\alpha R = x^\alpha R ,$$

as desired. Moreover, since $xR = Rx$,

$$yR = x^\alpha R = Rx^\alpha .$$

Now, let A be a right ideal of R. Since $A = \sum_{z \in A} zR$, it follows that $A = x^\beta R = Rx^\beta$ for some positive integer $\beta \leqq n$. Thus, the only right (left) ideals of R are

$$R \supset xR \supset \cdots \supset x^{n-1}R \supset x^n R = 0 .$$

This completes the proof.

The following theorem generalizes Levy's result for the class of non-prime rings.

THEOREM 2.12. *Let R be a nonprime right noetherian ring. Then, every proper homomorphic image of R is a right q-ring if and only if*

(1) *$R = S \oplus T$, where S is semisimple artinian and T is a PIRD with descending chain condition, or*

(2) *R/J is artinian and every nonzero ideal of R contains J, or*

(3) *R is a local ring whose maximal right ideal M satisfies $M^2 = 0$, and every proper homomorphic image of R contains at most one proper right (left) ideal.*

Proof. Suppose that all the proper homomorphic images of R are right q-rings. Then, by Lemma 2.4, R is right artinian. Let

$$R = R_1 \oplus R_2 \oplus \cdots \oplus R_k .$$

where $\{R_i, 1 \leqq i \leqq k\}$ are the blocks of R.

First assume that $k > 1$. Then, every homomorphic image of the ring R_i is isomorphic to a proper homomorphic image of R. Then every homomorphic image of R_i is a QF-ring, and by Proposition 2.8, each R_i has a unique maximal ideal M_i. If R_i is prime, then it is simple artinian. Suppose that R_i is not prime, then by 2.1 and 2.2,

$M_i^{n_i} = 0$ for some positive integer $n_i > 1$. Now, either by 2.9 (in case $n_i = 2$) or by 2.11, R_i is a *PIRD* with descending chain condition. Hence R is of type (1).

Next, let $k = 1$. Then in view of Prop. 2.8, we have to consider two cases:

Case (i), R contains a unique maximal ideal M. Then, as before, $M^n = 0$ for some positive integer $n > 1$.

If $M^2 \neq 0$, then by 2.11, R is a local *PIRD* with descending chain condition, and hence of type (1).

If $M^2 = 0$, we consider the following subcases:

(a) M is a minimal ideal. Then M is the only proper ideal of R, and R is of type (2).

(b) M contains a nonzero ideal A. Let $S = R/A$ and $N = M/A$. Then N is a maximal ideal in the right q-ring S satisfying $N^2 = 0$. Consequently, N is the only proper right (left) ideal of S. This also proves that M is a maximal right ideal of R, and hence the unique maximal right ideal. Therefore R is a local ring. Thus R is of type (3).

Case (ii), R contains two distinct maximal ideals M_1 and M_2 such that $M_1M_2 = 0$. Then by Proposition 2.8, M_1, M_2 and M_2M_1 are the only proper ideals of R. Then it is clear that

$$J = M_1 \cap M_2 = M_2M_1 .$$

Therefore, R is of type (2).

Conversely, suppose that R is of type (1), (2) or (3). Then it is obvious that any proper homomorphic image of R is of the form $A \oplus B$, where A is semisimple artinian and B is a *PIRD* with descending chain condition. Since a *PIR* with descending chain condition is *QF* [1, Th. 4.1], B is a q-ring. Then it is clear that $A \oplus B$ is a q-ring, completing the proof.

Now, we consider the prime case.

THEOREM 2.13. *Let R be a prime right noetherian ring with the property that every proper homomorphic image is a right q-ring. Then,*

(i) *every ideal of R is a product of prime ideals, and*

(ii) *for every nonzero prime ideal of R, R/P is a division ring.*

Proof. If R is a simple ring, the result holds trivially. So, we assume that R is not simple. Let P be a nonzero prime ideal of R. Then, P is a maximal ideal by Lemma 2.1. Also $P^2 \neq 0$, since R is prime. Consider the ring R/P^2. By Lemma 2.9, P/P^2 is the only proper right ideal in R/P^2. Hence P is a maximal right ideal in R, proving (ii).

Let M and N be maximal ideals of R. Since R is prime, $MN \neq 0$ and $NM \neq 0$. Hence $MN = NM$, by Lemma 2.6. This proves that all maximal ideals of R commute. Now, let A be a nonzero ideal of R. Then, A contains a product of nonzero prime ideals, by 2.2. Let $\{M_1, M_2, \cdots, M_S\}$ be a minimal set of prime ideals such that $A \supset M_1 M_2 \cdots M_S$. Since each M_i is a maximal ideal, we can collect the powers of M_i's together. Renumbering if necessary, we get $A \supset M_1^{\alpha_1} M_2^{\alpha_2} \cdots M_n^{\alpha_n}$, where $\alpha_1 + \alpha_2 + \cdots + \alpha_n = s$, and $\{M_1, M_2, \cdots, M_n\}$ is a set of distinct maximal ideals.

If possible, suppose that $A \not\subset M_1$. Then $R = A + M_1$ implies

$$M_1^{\alpha_1 - 1} M_2^{\alpha_2} \cdots M_n^{\alpha_n} \not\subset A + M_1^{\alpha_1} M_2^{\alpha_2} \cdots M_n^{\alpha_n} = A .$$

But this contradicts the minimality of the set $\{M_1, M_2, \cdots, M_S\}$. Hence $A \subset M_1$. Further if $A \not\subset M_1^2$, then by 2.10 $M_1 = A + M_1^2$, which implies

$$M_1^{\alpha_1 - 1} M_2^{\alpha_2} \cdots M_n^{\alpha_n} \subset A .$$

Again, a contradiction, thus, $A \subset M_1^2$. Proceeding in this way, using Lemma 2.10, we get $A \subset M_1^{\alpha_1}$. Similarly, $A \subset M_i^{\alpha_i}$, $1 \leqq i \leqq n$. Therefore,

$$M_1^{\alpha_1} \cap M_2^{\alpha_2} \cap \cdots \cap M_n^{\alpha_n} \supset A \supset M_1^{\alpha_1} M_2^{\alpha_2} \cdots M_n^{\alpha_n} .$$

But since,

$$M_1^{\alpha_1} \cap M_2^{\alpha_2} \cap \cdots \cap M_n^{\alpha_n} = M_1^{\alpha_1} M_2^{\alpha_2} \cdots M_n^{\alpha_n}$$

by Lemma 2.7, we get

$$A = M_1^{\alpha_1} M_2^{\alpha_2} \cdots M_n^{\alpha_n} .$$

Proving (i), and completing the proof.

REMARK. For the nonprime case, Levy obtained only two types of rings which are analogous to (1) and (3) of Theorem 2.12. It is not difficult to see that a commutative ring of type (2) is of type (1). Hence, Theorem 2.12 generalises Levy's result only for the class of nonprime rings. On the other hand, Theorem 2.13 shows that a prime right noetherian ring whose proper homomorphic images are right q-rings, is a ring which may be called a 'noncommutative Dedekind ring'.

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