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# DIFFERENTIAL MAPPINGS ON A VECTOR SPACE

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**Let $E$ and $F$ be two normed vector spaces with real scalars, and $G$ an open subset of $E$. A mapping $f\colon E \to F$ is said to be differentiable at $x$ in $G$ if there is a bounded linear map $L\colon E \to F$ such that for every $y$ in $G$,**

$$f(y) - f(x) = L(y - x) + R(x, y)\ ,$$

**where $R\colon E \to F$ and**

$$\lim_{\|y-x\| \to 0} \frac{\| R(x, y) \|}{\| y - x \|} = 0\ .$$

**$L$ is called the differential of $f$ at $x$. An extension of this definition is possible is such a way as to include a point $x$ on the boundary of $G$. In such cases $f$ is said to have a differential at $x$ from side $G$. Some properties of side differentials and relationships between the differential of $f$ at $x$ and its side differentials at $x$ are shown in this paper.**

Theorems 1, 2, 3, and 4, listed below without proofs, are known theorems. The balance of the paper will be used to extend this theory.

THEOREM 1. *If $f$ is differentiable at $x$, then for every $y$ in $G$,*

$$L(y) = \lim_{a \to 0} \frac{f(x + ay) - f(x)}{a} \quad [1,\ \text{p. } 32].$$

THEOREM 2. *If $f$ is differentiable at $x$, then the differential is unique* [1, p. 33].

THEOREM 3. *The differential of $f$ is independent of the norms in $E$ and $F$* [1, p. 35].

THEOREM 4. *If $f$ is a real valued functional and differentiable, a necessary condition for $f$ to have an extreme value at $x$ is that $L(x - y)$ vanish for all $y$ in $G$* [2, p. 13].

DEFINITION 1. Let $h$ be a vector in $E$, then

$$|m| = \lim_{a \to 0} \frac{\| f(x + ah) - f(x) \|}{\| ah \|}$$

if it exists, is said to be the absolute-slope of $f$ at $x$ in the $h$ direction.

THEOREM 5. *If $f$ is differentiable at $x$ then the absolute-slope of $f$ at $x$ in the $h$ direction exists and furthermore*

$$|m| = \left\|L\left(\frac{h}{\|h\|}\right)\right\| .$$

*Proof.* Writing $f(y) - f(x) = L(ah) + R(x, x + ah)$ and dividing by $\|ah\|$ we have

$$\frac{f(x + ah) - f(x)}{\|ah\|} = \frac{L(ah)}{\|ah\|} + \frac{R(x, x + ah)}{\|ah\|} ,$$

and

$$\lim_{a\to 0} \frac{\|f(x + ah) - f(x)\|}{\|ah\|} = \lim_{a\to 0} \left\|\frac{a}{|a|} L\left(\frac{h}{\|h\|}\right)\right\| = \left\|L\frac{h}{\|h\|}\right\| .$$

COROLLARY. $|m| \leqq \|L\|$ *for all $h$ in $E$.*

The proof is immediate since $L$ is bounded and

$$\|L\| = \sup_{\|h\|=1} \|L(h)\| .$$

DEFINITION 2. A function $S$ from $E$ to $F$ with the properties
(i) $S(x + y) = S(x) + S(y)$ and
(ii) $S(ax) = aS(x)$ if $a > 0$ will be called almost-linear.

It should be noted that if the domain of $S$ is the whole space, then $S$ is linear.

DEFINITION 3. Suppose that $f$ is differentiable on an open set $G$ and $x$ is on the boundary of $G$. $f$ is said to have a derivative from side $G$ if there exists an almost-linear function $S$ such that:

$$f(y) - f(x) = S(y - x) + R(x, y)$$

for all $y$ such that $(1 - a)x + ay$ is in $G$ whenever $0 < a \leqq 1$ and

$$\lim_{\|y-x\|\to 0} \frac{\|R(x, y)\|}{\|y - x\|} = 0 .$$

THEOREM 6. *If $f$ is differentiable at $x$ from side $G$ then*

$$S(y - x) = \lim_{a\to 0^+} \frac{f((1 - a)x + ay) - f(x)}{a} .$$

The proof is similar to that of Theorem 1.

COROLLARY.

$$S(y) = \lim_{a \to 0^+} \frac{f(x + ay) - f(x)}{a} .$$

THEOREM 7. *If $f$ has a differential at $x$ from side $G$, then this differential is unique.*

The proof is similar to that of Theorem 2.

THEOREM 8. *Suppose that $x \in G$, an open set. Let $\{F_i\}$ be the collection of all open subsets of $G$ such that $x$ is on the boundary of $F_i$. Then $f$ is differentiable at $x$ if and only if $f$ has a differential from side $F_i$ for each $i$ and all of these side differentials are equal.*

*Proof.* The only if proof is trivial. For the converse, we need only show that the almost-linear function $S$ is a linear function. Pick $F_1$ and $F_2$ so that $F_1 \cup F_2 \cup \{x\}$ is balanced with respect to $x$. For each $y$ in $F_2$, we can write $f(y) - f(x) = S(x - y) + R(x, y)$, where $S$ is almost-linear and

$$\lim_{||y-x|| \to 0} \frac{||R(x, y)||}{||y - x||} = 0 .$$

Since the differential from side $F_1$ is the same as the differential from side $F_2$, we have for each $z$ in $F_2$, $f(z) - f(x) = S(z - x) + P(x, z)$, where

$$\lim_{||z-x|| \to 0} \frac{||P(x, z)||}{||z - x||} = 0 .$$

If $z$ is chosen so that $z - x = x - y$, then

$$\begin{aligned} S(x - y) = S(z - x) &= \lim_{a \to 0^+} \frac{f((1 - a)x + az) - f(x)}{a} \\ &= \lim_{a \to 0^+} \frac{f(x - ay + 2ax - ay) - f(x)}{a} \\ &= \lim_{a \to 0^+} \frac{f((1 + a)x - ay) - f(x)}{a} \\ &= \lim_{a \to 0^-} \frac{f((1 - a)x + ay) - f(x)}{a} \\ &= -\lim_{a \to 0^-} \frac{f((1 - a)x + ay) - f(x)}{a} \\ &= -S(y - x) . \end{aligned}$$

Hence $S$ is linear.

THEOREM 9. *Suppose that $G$ and $H$ are open sets with a non-*

*empty intersection and that $x$ is a boundary point of $G$, $H$ and $G \cap H$. If $f$ has a differential at $x$ from the $G$ and $H$ sides, then $f$ has a differential from the $G \cup H$ side.*

*Proof.* Since $f$ has a differential from the $G$ side we may write $f(y) - f(x) = S(y - x) + R(x, y)$ for all $y$ such that $(1 - a)x + ay$ is in $G$ whenever $0 < a \leqq 1$, where $S$ is almost-linear and

$$\lim_{||x-y||\to 0} \frac{||R(x, y)||}{||x - y||} = 0 .$$

Also we may write $f(y) - f(x) = T(y - x) + P(x, y)$ for all $y$ such that $(1 - a)x + ay$ is in $H$ whenever $0 < a \leqq 1$, where $T$ is almost-linear and

$$\lim_{||x-y||\to 0} \frac{||P(x, y)||}{||x - y||} = 0 .$$

By Theorem 6, $S(y - x)$ and $T(y - x)$ must agree for all $y$ such that $(1 - a)x + ay$ is in $G \cap H$ whenever $0 < a \leqq 1$. Hence $T$ and $S$ are extensions of each other and are unique by Theorem 7. Let

$$V(y - x) = \begin{cases} S(y - x), y \text{ in } G \\ T(y - x), y \text{ in } H . \end{cases}$$

Then

$$f(y) - f(x) = V(y - x) + \begin{cases} R(x, y), y \text{ in } G \\ P(x, y), y \text{ in } H - G \end{cases}$$

for all $y$ such that $(1 - a)x + ay$ is in $G \cup H$. Therefore $f$ has a differential at $x$ from the $G \cup H$ side.

THEOREM 10. *Let $U$ be an open set containing $x$. Then $f$ has a differential at $x$ if there exist in $U$ a finite number of open sets $\{G_i\}_{i=1}^n$ such that $G_i \cap G_{i+1} \neq \varnothing$ (Take $G_{n+1} = G_1$), $x$ is on the boundary of each $G_i$ and $G_i \cap G_{i+1}$, $U = \{x\} \cup \bigcup_{i=1}^n G_i$ and $f$ has a differential from the $G_i$ side for each $i$.*

This is a corollary to Theorems 7 and 9.

## REFERENCES

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