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THE LEBESGUE DECOMPOSITION, RADON-NIKODYM DERIVATIVE, CONDITIONAL EXPECTATION, AND MARTINGALE CONVERGENCE FOR LATTICES OF SETS

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In the setting of additive set functions defined on lattices of sets, a Lebesgue decomposition and a Radon-Nikodym derivative are constructed and characterized. In the appropriate case (L_2), the Radon-Nikodym derivative is shown to be the conditional expectation. Finally, a martingale convergence theorem for Radon-Nikodym derivatives is obtained.

The origin of this paper was an interesting colloquium lecture given by H. D. Brunk at the University of California, Riverside, in December, 1968. Brunk's lecture dealt with a Radon-Nikodym derivative for σ-additive set functions defined on a σ-lattice of sets and applications of this Radon-Nikodym derivative to probability. An excellent interpretation of the role of σ-lattices in probability theory can be found in the papers of H. D. Brunk (c.f. [1], where additional references can be found). The purpose of this paper is to extend the underlying mathematical theory to encompass the case of additive set functions defined on lattices of sets.

Perhaps we should remind the reader that both the closed subsets of a metric space, M, and the open subsets of M comprise lattices of subsets of M, so many familiar families of functions are instances of the setting with which this paper deals. For example, the bounded upper semi-continuous functions on the interval $I = [0, 1]$ are the uniform limits of simple (see paragraph two of § 3) functions which are measurable with respect to the lattice of closed subsets of I. If M is a Borel subset of a separable complete metric space, then the analytic subsets of M comprise an important sigma lattice of subsets of M.

Let $\mathfrak{A}$ be an algebra of subsets of a nonempty set Ω (i.e., $\Omega \in \mathfrak{A}$ and if each of E and F is an element of $\mathfrak{A}$, then each of $E \cap F$ and $E^c = \Omega - E$ is an element of $\mathfrak{A}$).

Let $\mathscr{M}$ be a lattice of subsets of $\mathfrak{A}$ (i.e., $\mathscr{M}$ is a subset of $\mathfrak{A}$ such that $\mathscr{M}$ contains each of the empty set φ and Ω and, moreover, $E, F \in \mathscr{M}$ imply $E \cup F$, $E \cap F \in \mathscr{M}$).

Let $\mathscr{F} = \{A \cap B^c; A, B \in \mathscr{M}\}$, and denote by $\mathscr{A}$ the set of finite disjoint unions of elements of $\mathscr{F}$.

Let us examine $\mathscr{A}$ more closely. A finite intersection of elements of $\mathscr{A}$ is an element of $\mathscr{A}$. Moreover, if $E_i = A_i \cap B_i^c$ where

A_i and $B_i \in \mathscr{M}$, then $E_i^c = A_i^c \cup (A_i \cap B_i) \in \mathscr{A}$; thus,

$$E^c = (\cup_i E_i)^c = \cap E_i^c \in \mathscr{A},$$

and $\mathscr{A}$ is closed under complementation. Therefore, $\mathscr{A}$ is the algebra of subsets of Ω that is generated by $\mathscr{M}$.

Notice that if each of E and F is an element of $\mathfrak{A}$ (or $\mathscr{A}$), then $E \cup F \in \mathfrak{A}$ (or $\mathscr{A}$); and if each of E and F is an element of $\mathscr{F}$, then $E \cap F \in \mathscr{F}$.

Let each of λ and μ be a nonnegative additive set function defined on $\mathscr{F}$.

It seems appropriate to consider briefly the implications of the assumption that, say, λ be additive on $\mathscr{F}$. If each of A and B is an element of $\mathscr{M}$, then

$$\lambda(A \cup B) = \lambda(A) + \lambda(A^c \cap B) ,$$

(a) $\lambda(A \cup B) + \lambda(A \cap B) = \lambda(A) + \lambda(B)$, and

(b) $\lambda(\varnothing) = 0$.

Results of B. J. Pettis [6] assert that a real valued function, λ, defined on a lattice, $\mathscr{M}$, has an additive extension to the algebra, $\mathscr{A}$, generated by $\mathscr{M}$ if, and only if, λ satisfies (a) for all $A, B \in \mathscr{M}$ and (b). Moreover, the following elementary example illustrates the fact that the conditions

(a′) $\lambda(A \cup B) = \lambda(A) + \lambda(B)$, $A, B \in \mathscr{M}$, $A \cap B = \varnothing$, and

(b) $\lambda(\varnothing) = 0$

do not imply (a).

Example. $\Omega = \{1, 2, 3\}$,

$\mathscr{M} = \{\varnothing, \{1\}, \{1, 2\}, \{1, 3\}, \Omega\}$,

$\lambda(\varnothing) = \lambda(\{1\}) = 0$,

$\lambda(\{1, 2\}) = \lambda(\{1, 3\}) = 1$, and

$\lambda(\Omega) = 3$.

Recall that the norm, $\|\varphi\|$, of a bounded, real valued, additive set function, φ, defined on an algebra, say, $\mathscr{A}$ of subsets of Ω satisfies

$$\begin{aligned}\|\varphi\| &= \sup_{E \in \mathscr{A}} (|\varphi(E)| + |\varphi(\Omega - E)|) \\ &= \sup_{E \in \mathscr{A}} \varphi(E) - \inf_{F \in \mathscr{A}} \varphi(F) \\ &= \sup \left\{ \sum_{i=1}^{n} |\varphi(E_i)|; \{E_i\}_{i=1}^{n} \text{ a partition of } \Omega,\ E_i \in \mathscr{A} \right\} .\end{aligned}$$

Moreover, $\varphi \geqq 0 \Leftrightarrow \varphi(A) \leqq \varphi(B)$, $A \subset B$. A definition of the integral of a simple function can be gleaned from (13).

The primary purposes of this paper are fourfold. In § 2, we will decompose λ into a part s which is singular with respect to μ and a part t which is absolutely continuous with respect to μ. Section 3 is devoted to constructing and characterizing the Radon-Nikodym derivative $F = \{f_n\}$ of λ with respect to μ. In § 4, it is shown that if λ is the restriction to $\mathscr{F}$ of an element of $L_2(\Omega, \mathfrak{A}, \mu)$, then F is the conditional expectation of λ. Finally, in § 5 we shall establish an appropriate martingale convergence theorem.

2. The Lebesgue decomposition for lattices of sets. Let us begin this section by reviewing a few points concerning Lebesgue-Radon-Nikodym theorems.

When it is appropriate to apply a Lebesgue decomposition theorem to an object u with respect to an object v, u is split, uniquely, into an absolutely continuous part u_a and a singular part u_s. The parts u_a and u_s exhibit antipodal behavior with respect to v; qualitatively, the local behavior of u_a depends on the local behavior of v while u_s acts separately from v. Then one seeks a Radon-Nikodym theorem which applies to u_a: one seeks to represent u_a in terms of v. A Lebesgue-Radon-Nikodym theorem asserts not only that u splits but also that u_a can be represented in an appropriate fashion.

In [4] S. Johansen gives a definition and construction of a Radon-Nikodym derivative of a σ-additive set function with respect to a finite σ-additive measure on a σ-lattice.

Johansen's results are based on the fact that the Hahn decomposition remains valid in his setting. However, in the case of algebras of sets it is possible to have a bounded and finitely additive set function on a σ-algebra for which no Hahn decomposition exists, and it is possible to have a bounded σ-additive set function on an algebra of sets for which no Hahn decomposition exists. Nevertheless, in dealing with additive set functions on algebras of sets, approximations to Hahn decompositions (ε-decompositions) exist and can be used to obtain a Lebesgue-Radon-Nikodym theorem (cf. [2]).

In this section we shall show that the ε-decomposition approach used in [2] carries over to lattices of sets and permits us to obtain a Lebesgue decomposition. However, in § 3, by a simple example, we illustrate the fact that it is impossible to establish a general Radon-Nikodym theorem for lattices of sets. Our example shows that even in the σ-additive, σ-algebra setting of Johansen's paper the Radon-Nikodym derivative may not represent the absolutely continuous part of λ. Nevertheless, in § 3 we shall refine the elementary construction of this section and obtain a Radon-Nikodym derivative for lattices of sets. In § 4 we show that the Radon-Nikodym derivative of λ represents the best L_2 approximation to λ by $\mathscr{M}$-measurable functions.

Definition (ε-decomposition). Suppose that ν is a finitely additive set function on $\mathscr{F}$ which is bounded above. Let $\varepsilon > 0$. Let $K \in \mathscr{F}$. Let $A \in \mathscr{M}$ such that $\nu(A \cap K) > \sup_{E \in \mathscr{M}} \nu(E \cap K) - \varepsilon$. Then for each $B \in \mathscr{M}$, $\nu(K \cap A \cap B^c) = \nu(K \cap A) - \nu(K \cap A \cap B) > -\varepsilon$ and $\nu(K \cap A^c \cap B) = \nu(K \cap [A \cup B]) - \nu(K \cap A) < \varepsilon$. We shall call $K \cap A$ an ε-positive set for ν in K, $K \cap A^c$ an ε-negative set for ν in K, and $(K \cap A, K \cap A^c)$ an ε-decomposition for ν in K.

In order to obtain a Lebesgue decomposition of λ with respect to μ by splitting off the singular part of λ, we introduce the following simple construction.

For each positive integer n, let $\varepsilon_n = (64)^{-(n+1)}$ and let $([n], [n]^c)$ be an ε_n-decomposition for $\lambda - n\mu$ in Ω.

Let s_n be the restriction of λ to $[n]$ (i.e., $s_n(E) = \lambda(E \cap [n])$, $E \in \mathscr{F}$).

We shall establish two lemmas to show that $\{s_n\}$ is a Cauchy sequence and, hence, $\{s_n\}$ converges to a nonnegative bounded additive function $s = \lambda_s$ on $\mathscr{F}$. The restrictions $t_n = \lambda - s_n$ then converge to a nonnegative bounded additive function $t = \lambda_a$; s and t will be shown to comprise that Lebesgue decomposition of λ with respect to μ.

LEMMA 1. *Let $m > n$, let M denote $[m]$ and let N denote $[n]$. Then $\lambda(M \cap N^c) \to 0$.*

Proof. From the construction and the definitions follow

$$(\lambda - m\mu)(M \cap N^c) > -\varepsilon_m \quad \text{and} \quad (\lambda - n\mu)(M \cap N^c) < \varepsilon_n .$$

Hence

$$m\mu(M \cap N^c) - \varepsilon_m < \lambda(M \cap N^c) < n\mu(M \cap N^c) + \varepsilon_n ,$$

which implies

$$\mu(M \cap N^c) < (\varepsilon_m + \varepsilon_n)/(m - n) \leqq \varepsilon_m + \varepsilon_n ,$$

and, in turn,

$$\lambda(M \cap N^c) < n(\varepsilon_m + \varepsilon_n) + \varepsilon_n = n\varepsilon_m + (n + 1)\varepsilon_n .$$

Next, we will use Lemma 1 and the following pertinent remarks to show that $\lambda(M^c \cap N) \to 0$.

Because λ is nonnegative and additive, it follows that

$$\lambda(G \cap H) \leqq \lambda(H)$$

if G and $H \in \mathscr{F}$.

If $\{K_i\}$ is a sequence of elements of $\mathscr{M}$ and p is a positive integer, then

$$K_p = (K_p \cap K_{p-1}^c) \cup (K_p \cap K_{p-1} \cap K_{p-2}^c) \cup \cdots \cup \left(\left[\bigcap_{p \geqq i > 1} K_i\right] \cap K_1^c\right) \cup T_p ,$$

where

$$T_p = \bigcap_{i \leqq p} K_i .$$

Moreover,

$$K_{p+1}^c \cap K_p = (K_{p+1}^c \cap [K_p \cap K_{p-1}^c]) \cup (K_{p+1}^c \cap K_{p-1} \cap [K_p \cap K_{p-2}^c]) \cup \cdots \cup (K_{p+1}^c \cap T_p) ,$$

and

$$K_{p+1}^c \cap T_p = (K_{p+1} \cap T_p)^c \cap T_p = T_{p+1}^c \cap T_p .$$

LEMMA 2. $\lambda(M^c \cap N) \to 0$.

Proof. Suppose, on the contrary, that there exists $\varepsilon > 0$ and an increasing sequence $\{n_i\}$ of positive integers such that $5n_1\varepsilon_{n_1} < \varepsilon$ and $\lambda(K_{2i}^c \cap K_{2i-1}) \geqq \varepsilon$, where $K_j = [n_j]$. Then

$$\begin{aligned} \lambda(T_{p+1}^c \cap T_p) &= \lambda(K_{p+1}^c \cap K_p) - \lambda(K_{p+1}^c \cap [K_p \cap K_{p-1}^c]) - \cdots \\ &\geqq \lambda(K_{p+1}^c \cap K_p) - \sum_{1 \leqq i < p} \lambda(K_p \cap K_i^c) \\ &> \lambda(K_{p+1}^c \cap K_p) - \sum_{1 \leqq i < p} [n_i\varepsilon_{n_p} + (n_i + 1)\varepsilon_{n_i}] \\ &> \lambda(K_{p+1}^c \cap K_p) - \varepsilon/2 . \end{aligned}$$

Moreover, for each positive integer p,

$$\begin{aligned} \lambda(T_1) &= \sum_{i \leqq p} \lambda(T_i \cap T_{i+1}^c) + \lambda(T_{p+1}) \\ &\geqq \sum_{i \leqq p} \lambda(T_i \cap T_{i+1}^c) . \end{aligned}$$

Hence, choosing k large and $2k \leqq p + 1$, the following contradiction is obtained, and Lemma 2 is thereby established.

$$\begin{aligned} \lambda(T_1) &\geqq \sum_{i \leqq k} \lambda(T_{2i-1} \cap T_{2i}^c) \\ &> \sum_{i \leqq k} (\lambda(K_{2i-1} \cap K_{2i}^c) - \varepsilon/2) \\ &> k\varepsilon/2 \\ &> \sup_{E \in \mathscr{F}} \lambda(E) . \end{aligned}$$

From $\lambda(M \cap N^c) \to 0$, $\lambda(M^c \cap N) \to 0$, and the monotonicity of λ, it follows that $\{s_n\}$ converges uniformly in n on $\mathscr{F}$ to a function s on $\mathscr{F}$ such that

(1) s is a nonnegative additive function on $\mathscr{F}$,

(2) if $\delta > 0$, then there exists $E \in \mathscr{M}$ such that $\mu(E) < \delta$ and $s(\Omega - E) = s(E^c) = \sup_{F \in \mathscr{F}} s(E^c \cap F) < \delta$. Moreover, $t_n = \lambda - s_n$ converges uniformly to a function t on $\mathscr{F}$ such that

(3) t is a nonnegative additive set function on $\mathscr{F}$

(4) $\lambda = s + t$

(5) if $\varepsilon > 0$, then there exists $\delta > 0$ such that if $E \in \mathscr{M}$ and $\mu(E) < \delta$, then $t(E) < \varepsilon$.

Proof of (5). Choose n such that $\sup_{E \in \mathscr{F}} |(t - t_n)(E)| < \varepsilon/4$ and $\varepsilon_n < \varepsilon/4$. Then choose $\delta = \varepsilon/2n$. If $E \in \mathscr{M}$ and $\mu(E) < \delta$, then

$$t(E) < \varepsilon/4 + t_n(E) = \varepsilon/4 + \lambda(E \cap [n]^c) < \varepsilon/4 + n\mu(E \cap [n]^c) + \varepsilon_n < \varepsilon/2 + n\mu(E) < \varepsilon .$$

Now that we have established the existence of a Lebesgue decomposition, it remains to establish uniqueness.

Proof of uniqueness. Suppose that y and z are bounded, additive functions on $\mathscr{F}$ such that

(i) $\lambda = y + z$,

(ii) if $\delta > 0$, then there exists $E \in \mathscr{M}$ such that $\mu(E) < \delta$ and $\sup_{B \in \mathscr{F}} |y(E^c \cap B)| < \delta$, and

(iii) if $\varepsilon > 0$, then there exists $\delta > 0$ such that if $E \in \mathscr{M}$ and $\mu(E) < \delta$, then $|z(E)| < \varepsilon$.

Look at $s - y = z - t$. Let $\varepsilon > 0$. Let δ be a positive number less than ε such that if $E \in \mathscr{M}$ and $\mu(E) < 2\delta$ then $t(E) < \varepsilon$ and $|z(E)| < \varepsilon$. Let E and $F \in \mathscr{M}$ such that $\mu(E) < \delta$, $s(E^c) < \delta$, $\mu(F) < \delta$, and $\sup_{B \in \mathscr{F}} |y(F^c \cap B)| < \delta$. Let $K = E \cup F$. Then $\mu(K) < 2\delta$, $s(K^c) < \delta$, and

$$\sup_{B \in \mathscr{F}} |y(K^c \cap B)| = \sup_{B \in \mathscr{F}} y(F^c \cap E^c \cap B) \leqq \sup_{A \in \mathscr{F}} |y(F^c \cap A)| < \delta .$$

Let $A \in \mathscr{M}$. Then

$$\begin{aligned} |(s - y)(A)| &= |(s - y)(A \cap K^c) + (z - t)(A \cap K)| \\ &\leqq s(A \cap K^c) + |y(A \cap K^c)| + |z(A \cap K)| + t(A \cap K) \\ &< 4\varepsilon . \end{aligned}$$

Recall that if ω is an additive function on $\mathscr{F}$, then $A, B \in \mathscr{M}$ imply $\omega(A \cap B^c) = \omega(A) - \omega(A \cap B)$ and, hence,

$$\sup_{C \in \mathscr{F}} |\omega(C)| \leqq 2 \sup_{C \in \mathscr{M}} |\omega(C)| .$$

Thus, $\sup_{C \in \mathscr{F}} |(s - y)(C)| \leqq 8\varepsilon$ which implies that $s = y$ and $t = z$.

3. The Radon-Nikodym derivative. In this section, we shall construct a Radon-Nikodym derivative of λ with respect to μ. In order to describe what we shall construct, it is necessary to introduce the following notation.

A (real valued) function f on Ω is said to be $\mathscr{M}$-measurable if $(f > r) = \{x; f(x) > r\} \in \mathscr{M}$ whenever $r \in R$, the set of real numbers. If f is a $\mathscr{M}$-measurable function on Ω and the range of f is a finite subset of R, then f is said to be a simple $\mathscr{M}$-measurable function.

Suppose that $\mathscr{S}$ is an algebra of subsets of Ω, and that ρ is a nonnegative additive set function on $\mathscr{S}$. Let $L_p(\Omega, \mathscr{S}, \rho)$, $p \geqq 1$, denote the space of functions f on Ω such that if $\varepsilon > 0$, then there exists an $\mathscr{S}$-measurable function g and an $\mathscr{S}$-measurable function h such that

(i) $\int |g|^p d\rho < \infty$,

(ii) $|f - g| \leqq h$, and

(iii) $\int |h|^p d\rho < \varepsilon$.

The spaces $L_p(\Omega, \mathscr{S}, \rho)$ are not, in general, complete unless $\mathscr{S}$ is a σ-algebra of subsets of Ω and ρ is countably additive on $\mathscr{S}$. The completions $V_p(\Omega, \mathscr{S}, \rho)$ of $L_p(\Omega, \mathscr{S}, \rho)$ are spaces of additive set functions on $\mathscr{S}$. These additive set functions can be identified with sequences $\{g_n\}$ of simple $\mathscr{S}$-measurable functions such that

(i) $\int |g_n|^p d\mu < \infty$ and

(ii) $\int |g_m - g_n|^p d\mu \to 0$,

and we shall often identify the elements of the V_p-spaces that we will encounter with appropriate Cauchy sequences of simple functions. Primary sources of information about such L_p and V_p spaces are [3], [4], and [7].

If $\mathscr{M}$ were an algebra of subsets of Ω, then it would follow from [2] that there would exist a sequence $\{f_n\}$ of simple $\mathscr{M}$-measurable functions such that

$$\left\| t - \int f_n d\mu \right\|_\infty \to 0 .$$

The following example shows that even if $\mathscr{M}$ is a σ-lattice of subsets of Ω and each of λ and μ is σ-additive on $\mathscr{F}$, then it may not be possible to uniformly approximate t by integrals of simple $\mathscr{M}$-measurable functions.

EXAMPLE. Let Φ be the union of two one element sets a and b. Let $\mathscr{M}$ be comprised of φ, a, and Ω. Then $\mathscr{M}^c = \{\varphi, b, \Omega\}$ and $\mathscr{F} = \{\varphi, a, b, \Omega\}$. Let λ and μ be defined on $\mathscr{F}$ by $\lambda(\varphi) = \mu(\varphi) = 0$, $\lambda(a) = 2$, $\lambda(b) = 4$, $\lambda(\Omega) = 6$, $\mu(a) = \mu(b) = 1$, $\mu(\Omega) = 2$. A function f

on Ω is $\mathscr{M}$-measurable if, and only if, $f(a) \geqq f(b)$. Hence if f is a $\mathscr{M}$-measurable function on Ω, and F is defined on $\mathscr{F}$ by

$$F(E) = \int_E f d\mu ,$$

then $F(a) \geqq F(b)$. Thus, λ cannot be uniformly approximated by integrating $\mathscr{M}$-measurable functions with respect to μ. But, λ is absolutely continuous with respect to μ.

The Radon-Nikodym derivative of λ with respect to μ will be a L_1-Cauchy sequence $\{f_n\}$ of simple $\mathscr{M}$-measurable functions with properties that will be discussed. The Radon-Nikodym derivative $F = \{f_n\}$ of λ is then absolutely continuous with respect to μ, and it is reasonable to ask whether F is the Radon-Nikodym derivative of the absolutely continuous part t of λ. The answer to this latter question is yes (see (14) and the definition of t).

The construction of the sequence $\{f_n\}$ is fairly straightforward; but, a direct proof that it is a Cauchy sequence appears to involve a dreadful computation which we shall avoid by complicated but conceptually reasonable means.

To simplify the notation, we often denote $(x/y)\mu$ by $x/y\mu$ when each of x and y is a real number. For example, $n\mu = n2^n/2^n\mu$.

CONSTRUCTION. Let n be a positive integer. We wish to refine the construction that we used to establish a Lebesgue decomposition. Recall that $\varepsilon_n = (64)^{-(n+1)}$ and that $N_{n2^n} = N = [n]$ is an ε_n-positive set for $(\lambda - n\mu) = (\lambda - n2^n/2^n\mu)$ in Ω. For $n2^n > i \geqq 1$, let $N_i \in \mathscr{M}$ such that $N_i \cap (\bigcap_{n2^n \geqq j > i} N_j^c)$ is an ε_n-positive set for $(\lambda - i/2^n\mu)$ in $\bigcap_{n2^n \geqq j > i} N_j^c$. Notice that we can assume that

$$\varphi = N_{n2^n+1} \subset N_{n2^n} \subset \cdots \subset N_i \subset \cdots \subset N_1 \subset N_0 = \Omega ;$$

let us call such a sequence an ε_n-decomposition sequence for n.

It will greatly simplify the typography to introduce the following notation.

(6) $$\text{Let } L_i = N_{i+1}^c \cap N_i .$$

The following are immediate consequences of the construction.

(7) $$(\lambda - i/2^n\mu)(L_i \cap B^c) > -\varepsilon_n , \qquad B \in \mathscr{M},$$

(8) $$-\varepsilon_n + i/2^n\mu(L_i \cap B^c) < \lambda(L_i \cap B^c) , \qquad B \in \mathscr{M},$$

(9) $$(\lambda - i/2^n\mu)(N_i^c \cap A) < \varepsilon_n , \qquad A \in \mathscr{M},$$

(10) $$\lambda(N_i^c \cap A) < i/2^n\mu(N_i^c \cap A) + \varepsilon_n , \qquad A \in \mathscr{M}.$$

The following consequence of (9) will be applied several times in the paper; since $(\lambda - i/2^n\mu)(N_{i+1}^c \cap [N_i \cap A]) = (\lambda - i/2^n\mu)(L_i \cap A)$,

$$(\lambda - i/2^n\mu)(L_i \cap A) < 2^{-n}\mu(L_i \cap A) + \varepsilon_n . \tag{11}$$

Let f_n be the simple $\mathscr{M}$-measurable function defined by

$$f_n = 2^{-n} \sum_{1 \leq i \leq a_n} \chi_{N_i} = \sum_{i \leq a_n} i/2^n \chi_{L_i}, \text{ where } a_n = n2^n . \tag{12}$$

Let F_n be the nonnegative additive function defined by

$$\begin{aligned} F_n(E) &= \int_E f_n d\mu \\ &= \sum_{1 \leq i \leq a_n} i/2^n \mu(L_i \cap E) \\ &= 2^{-n} \sum_{1 \leq i \leq a_n} \mu(N_i \cap E) . \end{aligned} \tag{13}$$

Recall that $(\lambda - n\mu)(N \cap A^c) > -\varepsilon_n$, $A \in \mathscr{M}$, and that

$$(\lambda - m\mu)(M \cap A^c) > -\varepsilon_m , \qquad A \in \mathscr{M} .$$

Hence $n\mu(N \cap M^c) < \lambda(N \cap M^c) + \varepsilon_n$. Moreover, $m\mu(M) < \lambda(M) + \varepsilon_m$ which implies that $n\mu(M) < n/m(\lambda(M) + \varepsilon_m)$. Thus,

$$\begin{aligned} n\mu(N) &= n(\mu(N \cap M^c) + \mu(N \cap M)) \\ &< \lambda(N \cap M^c) + \varepsilon_n + n/m(\lambda(\Omega) + 1) . \end{aligned}$$

Hence, choosing m to be large and applying Lemma 2, we obtain

$$n\mu(N) \to 0 . \tag{14}$$

Recall that $t_n(E) = \lambda(E \cap N^c) = \sum_{0 \leq i < a_n} \lambda(L_i \cap E)$; hence

$$t_n(E) - F_n(E) = \sum_{0 \leq i < a_n} (\lambda - i/2^n\mu)(L_i \cap E) - n\mu(N \cap E) . \tag{15}$$

Then (7) implies

$$t_n(B^c) - F_n(B^c) > -a_n\varepsilon_n - n\mu(N \cap E) , \qquad B \in \mathscr{M} . \tag{16}$$

Moreover, applying (11) to (15) yields

$$t_n(A) - F_n(A) < a_n\varepsilon_n + 2^{-n}\mu(N^c \cap A) , \qquad A \in \mathscr{M} . \tag{17}$$

We wish to show that F_n converges uniformly to a nonnegative additive function F on $\mathscr{F}$. We know from (14), (16), and (17) that F_n converges to t uniformly on $\mathscr{M} \cap \mathscr{M}^c$. Hence if $\mathscr{M}$ were an algebra of subsets of Ω, then it would follow that $F_n \to t$ on $\mathscr{M} = \mathscr{F}$. In general, we know that F_n need not converge to t. We shall establish the uniform convergence of F_n on $\mathscr{F}$ by showing that F_n is almost increasing on $\mathscr{F}$. (What we mean by the term "almost increasing" will become clear in due course.)

Let $m > n$, and let $\varphi = M_{a_m+1} \subset M = M_{a_m} \subset \cdots \subset M_1 \subset M_0 = \Omega$ be an ε_m-decomposition sequence for m.

Since $N_i = \bigcup_{i \leqq j \leqq a_n} L_j$, it follows from (8) that for $A \in \mathscr{M}$,

$$\begin{aligned} -a_n\varepsilon_n + i/2^n\mu(N_i \cap A^c) &= -a_n\varepsilon_n + \sum_{i \leqq j \leqq a_n} i/2^n\mu(L_j \cap A^c) \\ &\leqq -a_n\varepsilon_n + \sum_{i \leqq j \leqq a_n} j/2^n\mu(L_j \cap A^c) \\ &< \sum_{i \leqq j \leqq a_n} \lambda(L_j \cap A^c) . \end{aligned}$$

Hence,

$$i/2^n\mu(N_i \cap A^c) < a_n\varepsilon_n + \lambda(N_i \cap A^c) , \qquad A \in \mathscr{M} . \tag{18}$$

If $A^c = M_j^c$, then (10) implies that $\lambda(N_i \cap M_j^c) < \varepsilon_m + j/2^m\mu(N_i \cap M_j^c)$. Thus,

$$(i/2^n - j/2^m)\mu(N_i \cap M_j^c) < a_n\varepsilon_n + \varepsilon_m . \tag{19}$$

Let $p = 2^{m-n}$, and let $K_j = M_{jp}$, $j = 1, \cdots, m2^n$. Then (see (13))

$$\begin{aligned} F_m(E) = 2^{-m} \sum_{i \leqq a_m} \mu(M_i \cap E) &= 2^{-m}[(\mu(M_1 \cap E) + \cdots + \mu(M_p \cap E)) \\ &\quad + (\mu(M_{p+1} \cap E) + \cdots + \mu(M_{2p} \cap E)) + \cdots] \\ &\geqq 2^{-n}[\mu(K_1 \cap E) + \mu(K_2 \cap E) + \cdots + \mu(K_{m2^n} \cap E)] \end{aligned}$$

(recall that $K_1 \supset K_2 \supset \cdots$).

In (19), let $j = (i-1)p$. Then $(i/2^n - j/2^m)) = 2^{-n}$, and (19) becomes

$$\mu(N_i \cap K_{i-1}^c) < 2^n(a_n\varepsilon_n + \varepsilon_m) . \tag{20}$$

From (13) it follows that

$$\begin{aligned} F_n(E) &= 2^{-n} \sum_{i \leqq a_n} \mu(N_i \cap E) \\ &= 2^{-n} \sum_{i \leqq a_n} [\mu(N_i \cap K_{i-1}^c \cap E) + \mu(N_i \cap K_{i-1} \cap E)] \\ &\leqq 2^{-n} \sum_{i \leqq a_n} \mu(N_i \cap K_{i-1}^c \cap E) + 2^{-n} \sum_{i \leqq a_n} \mu(K_{i-1} \cap E) . \end{aligned}$$

Hence using (20) and the paragraph between (19) and (20), we obtain

$$\begin{aligned} &F_n(E) < \delta_n + F_m(E), \text{ where} \\ &\delta_n = 2^{-n}a_n2^n(a_n\varepsilon_n + \varepsilon_m) = a_n(a_n\varepsilon + \varepsilon_m) . \end{aligned} \tag{21}$$

Moreover, an inspection of the argument which produced (21) shows that (21) is valid if $\{E_i\}$ is a finite collection of pairwise disjoint elements of $\mathscr{F}$, $E = \bigcup_i E_i$, and $F_n(E)$ is defined to be $\sum_i F_n(E_i)$.

Inequality (21) says that F_n is almost increasing. Since the F_n's are nonnegative and $\lim\sup_n F_n(E) \leqq \lim_n F_n(\Omega) = t(\Omega)$, the F_n's are uniformly bounded. Hence $F(E) = \lim_n F_n(E)$ exists for each finite union E of pairwise disjoint elements of $\mathscr{F}$.

We have extended the F_n's to the set $\mathscr{A}$ of finite disjoint unions of elements of $\mathscr{F}$. Recall that $\mathscr{A}$ is the algebra of subjects of Ω which is generated by $\mathscr{M}$ or $\mathscr{F}$, and a nonnegative additive function on $\mathscr{F}$ has a unique additive extension to $\mathscr{A}$. Moreover, we have shown that the extensions converge (almost increasingly) pointwise on $\mathscr{A}$ to a nonnegative additive function F.

Because the F_n's are almost increasing to F on $\mathscr{A}$ and $\mathscr{A}$ is an algebra of subsets of Ω, it is easy to see that the F_n's coverge uniformly to F on $\mathscr{A}$. But, $\mathscr{F} \subset \mathscr{A}$, and the sequence $\{F_n\}$ converges to F in L_1-norm.

In summary, the sequence $\{f_n\}$ is a Cauchy sequence in $L_1(\Omega, \mathscr{A}, \mu)$ and the integrals F_n of f_n converge to F uniformly.

By the Radon-Nikodym derivative of λ with respect to μ, we shall mean the object $\{f_n\} = F$.

We shall conclude this section with a characterization of the Radon-Nikodym $\{f_n\}$ that is analogous to that given by S. Johansen ([4, Th. 4]) in the case where $\mathscr{M}$ is a σ-lattice in $\mathfrak{A}$ and each of λ and μ is countably additive on $\mathscr{F}$. From (10):

$$\lambda(N_i^c \cap B) < i/2^n \mu(N_i^c \cap B) + \varepsilon_n \,, \qquad B \in \mathscr{M} \,,$$

and (18):

$$\lambda(A^c \cap N_i) > i/2^n \mu(A^c \cap N_i) - a_n\varepsilon_n \,, \qquad A \in \mathscr{M} \,,$$

we obtain the following proposition which will be shown to characterize the Radon-Nikodym derivative.

(22) Let $M > 0$ and $\varepsilon > 0$. Then there exists a positive integer k such that if $n > k$, $a, b \in R$, $b < M$, and $A, B \in \mathscr{M}$, then

(i) $\lambda([f_n \leqq b] \cap A) < b\mu([f_n \leqq b] \cap A) + \varepsilon$, and
(ii) $\lambda(B^c \cap [f_n > a]) > a\mu(B^c \cap [f_n > a]) - \varepsilon$.

THEOREM (*characterization of the Radon-Nikodym derivative*). *Suppose that* $\{g_n\}$ *is a sequence of simple* $\mathscr{M}$*-measurable functions such that* $\{g_n\}$ *is an* L_1*-Cauchy sequence* $\left(i.e., \int_\Omega |g_m - g_n| d\mu \to 0\right)$. *If* $\{g_n\}$ *can play the role of* $\{f_n\}$ *in* (22), *then* $\{g_n\} \equiv \{f_n\}$.

Proof. Suppose that $\{g_n\}$ can play the role of $\{f_n\}$ in (22). Let $\delta > 0$. Let ε be a positive number less than one; ε will be specified later. Notice that for sufficiently large n, (22-(ii)) implies the inequality $\mu([f_n > a]) < (\varepsilon + \lambda(\Omega))/a$ for f_n and the analogous inequality for g_n. Suppose that $a = (i + 1)\delta$ and $b = i\delta$. Then for sufficiently large n, $a\mu([f_n \leqq b] \cap [g_n > a]) - \varepsilon < b\mu([f_n \leqq b] \cap [g_n > a]) + \varepsilon$ or $\mu([f_n \leqq b] \cap [g_n > a]) < 2\varepsilon/\delta$; symmetrically, $\mu([g_n \leqq b] \cap [f_n > a]) < 2\varepsilon/\delta$. Choose a positive integer m such that $(1 + \lambda(\Omega))/m\delta < \delta/2$. Then, using consecutive terms of the sequence $0 < \delta < 2\delta < \cdots < m\delta$ for b and a, we obtain $\mu([|f_n - g_n| > 2\delta) < (1 + \lambda(\Omega))/m\delta + 2m(2\varepsilon/\delta)$ which can be made $< \delta$ by choosing ε to be sufficiently small. Hence the sequence $\{f_n - g_n\}$ converges to zero in μ-measure. However, the sequence $\{f_n - g_n\}$ is an element of $V_1(\Omega, \mathscr{A}, \mu)$ and, hence, the integrals

$$\int (f_n - g_n) d\mu$$

define a weakly convergent sequence of bounded and finitely additive set functions on $\mathscr{A}$. It then follows from the proof of Theorem 2.1 in [2] that $\int_\Omega |f_n - g_n| d\mu \to 0$ and, hence, $\{f_n\} \equiv \{g_n\}$.

4. Conditional expectation. Suppose that μ is a nonnegative additive set function on $\mathfrak{A}$ and that we have been looking at its restrictions to $\mathscr{M}$, $\mathscr{F}$, and $\mathscr{A}$.

We have already remarked that the completions V_p of $L_p = L_p(\Omega, \mathfrak{A}, \mu)$ are spaces of additive set functions on $\mathfrak{A}$ and that the Radon-Nikodym derivative $F = \{f_n\}$ of λ with respect to μ is an element of $V_1(\Omega, \mathscr{A}, \mu)$ which extends to V_1.

Notice that since $\mu(\Omega) < \infty$, $V_2 \subset V_1$.

Suppose that λ is the restriction to $\mathscr{F}$ of an element $H = \{h_m\}$ of V_2 $\left(\text{i.e., } H(E) = \lim_n \int_E h_n d\mu\right)$. Then the following theorem provides a rather satisfying extension of results of H. D. Brunk and others (cf. [1], [4], the references in [1] and [4], $\cdots$). Among other things, our theorem characterizes F as the best V_2 approximation to H that can be obtained via a L-Cauchy sequence of simple $\mathscr{M}$-measurable functions.

THEOREM.

(i) $F \in V_2$,

(ii) $\int HK\, d\mu \leqq \int FK\, d\mu$, $K = \{K_m\}$ *$\mathscr{M}$-measurable*, $K \in V_2$,

(iii) $\int HK\, d\mu \geqq \int FK\, d\mu$, $K = \{k_m\}$ *$\mathscr{M}^c$-measurable*, $K \in V_2$,

(iv) $\int HF\, d\mu = \int F^2 d\mu$,

(v) $\int (H - F)^2 d\mu \leqq \int (H - K)^2 d\mu$, $K = \{k_m\}$ *$\mathscr{M}$-measurable*, $K \in V_2$.

Moreover, the conditions of the theorem characterize F among the L_2-Cauchy sequences of simple $\mathscr{M}$-measurable functions.

Proof of (i). In order to establish (i), it suffices to show that

$$\int_\Omega |f_m - f_n|^2 d\mu \to 0$$

because then there exists $G \in V_2(\Omega, \mathscr{A}, \mu)$ such that $G = \{f_n\}$ and

$$\|F - G\|_1 \leqq \|F - F_n\|_1 + \|F_n - G\|_1 \leqq \|F - F_n\|_1 + (\mu(\Omega))^{1/2}\|F_n - G\|_2 .$$

We can assume, without loss of generality, that $\mu(\Omega) = 1$. We shall first show that $\{f_n\}$ is bounded in L_2. To this end, let $\rho \in V_1$ be defined by

$$\rho(E) = \lim_n \int_E h_n^2 d\mu ,$$

and let $\{u_n\}$ be the Radon-Nikodym derivative of ρ with respect to μ as constructed in § 3.

For the sequence $\{f_n\}$, (22) can be refined to read

$$\text{(23)}\quad \begin{aligned} &\text{(i)}\quad \lambda([f_n \leqq b] \cap A) < (b + 2^{-n})\mu([f_n \leqq b] \cap A) + \varepsilon_n , \qquad A \in \mathscr{M} , \\ &\text{(ii)}\quad \lambda([f_n > a] \cap B^c) > (a - 2^{-n})\mu([f_n > a] \cap B^c) - a_n\varepsilon_n , \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad B \in \mathscr{M} . \end{aligned}$$

Moreover, the inequalities in (23) also hold for $\{u_n\}$ with respect to ρ. These versions of (23) will be utilized in our proof that $\{f_n\}$ is bounded in L_2.

Let $a^2 > b$, let $A = (f_n^2 > a^2)$, and let $B^c = (u_n \leqq b)$. From (23) and Hölder's inequality we obtain for $C = A \cap B^c = (f_n^2 > a^2 > b \geqq u_n)$ the following chain:

$$\begin{aligned} [(a - 2^{-n})\mu(C) - a_n\varepsilon_n]^2 < [\lambda(C)]^2 &\approx \left(\int_C h_m d\mu\right)^2 \\ &\leqq \left(\int_C h_m^2 d\mu\right)\mu(C) \approx \rho(C)\mu(C) \\ &< [(b + 2^{-n})\mu(C) + \varepsilon_n]\mu(C) . \end{aligned}$$

Rearranging the first and last terms of the chain leads to

$$[(a - 2^{-n})^2 - (b + 2^{-n})][\mu(C)]^2 < [2a_n\varepsilon_n(a - 2^{-n}) + \varepsilon_n]\mu(C) + (a_n\varepsilon_n)^2 .$$

But, $A = \varphi$ if $a^2 \geqq n^2$; and if $a^2 < n^2$, then

$$a^2 - 2a2^{-n} + 4^{-n} - b - 2^{-n} > a^2 - b - (2n + 1)2^{-n}$$

and $(a^2 - b)[\mu(C)]^2 < \xi_n$ where

$$\xi_n < (2a_n\varepsilon_n n + \varepsilon_n)\mu(\Omega) + (a_n\varepsilon_n)^2 + (2n + 1)2^{-n}[\mu(\Omega)]^2 .$$

Then, since $(f_n^2 > u_n + 2) \subset \bigcup_{0 \leqq k < n^2} (f_n^2 > k + 1 \geqq k \geqq u_n)$,

$$\mu([f_n^2 > u_n + 2]) < n^2(\xi_n)^{1/2}$$

and, hence,

$$\int f_n^2 d\mu \leqq \int u_n d\mu + 2 + n^4(\xi_n)^{1/2} \leqq P ,$$

where P is independent of n. Therefore, $\{f_n\}$ is bounded in L_2.

Perhaps we should digress briefly and comment on the last term, $(2n + 1)2^{-n}[\mu(\Omega)]^2$, of the inequality that determines ξ_n. Firstly, $2n + 1$ comes from $\|f_n\|_\infty$, 2^{-n} comes from the mesh of f_n, and $\mu(\Omega)$ was taken to be one so this component appears only for emphasis; and secondly, the term $n^4(\xi_n)^{1/2}$ appears in bounding the f_n's in L_2. This is the only argument in which the ratio of $\|f_n\|_\infty$ to the mesh of f_n has to be controlled: all the other arguments can be pushed through by adjusting ε_n.

We know that $\{f_n\}$ converges in L_1, and we have just shown that $\{f_n\}$ is bounded in L_2. Hence $\{f_n\}$ converges weakly in $V_2(\Omega, \mathscr{A}, \mu)$: Suppose that $G \in V_2(\Omega, \mathscr{A}, \mu)$ and $\varepsilon > 0$. Then there exists a simple $\mathscr{A}$-measurable function $k = \sum_{j \leqq v} c_j \chi_{E_j}$ such that if $K = \int k$, then $\|G - K\|_2 < \varepsilon$ and

$$\begin{aligned} |(G, F_m - F_n)| &\leqq |(G - K, F_m - F_n)| + |(K, F_m - F_n)| \\ &\leqq \|G - K\|_2 \cdot (2P^{1/2}) + \left|\sum_{j \leqq v} c_j \int_{E_j} (f_m - f_n) d\mu\right| \\ &< 2\varepsilon P^{1/2} + \|k\|_\infty \|f_m - f_n\|_1 . \end{aligned}$$

Denote the weak limit of $\{f_n\}$ in $V_2(\Omega, \mathscr{A}, \mu)$ by G.

We know that $\|G\|_2 \leqq \liminf_n \|f_n\|_2$. Suppose for the moment that $\|f_n\|_2 \to \|G\|_2$ (i.e., $\|G\|_2 \geqq \limsup_n \|f_n\|_2$). Then

$$\|G - F_n\|_2^2 = (G - F_n, G - F_n) = (G, G) - 2(G, F_n) + (F_n, F_n) \to 0$$

(i.e., $\{f_n\}$ converges to G in $V_2(\Omega, \mathscr{A}, \mu)$) and, as we remarked before, $G = F$. Hence in order to complete a proof of (i), it suffices to show below that $\|G\|_2 \geqq \limsup_n \|f_n\|_2$.

Because $\{f_n\}$ is almost increasing, using (20), we have

$$\begin{aligned} \mu([f_n > f_m + 2 \cdot 2^{-n}]) &\leqq \sum_{0 \leqq i < a_n} \mu([f_n > (i + 1)/2^n > i/2^n \geqq f_m]) \\ &= \sum \mu(N_{i+2} \cap K_{i+1}^c) \\ &< a_n 2^n (a_n\varepsilon_n + \varepsilon_m) = \eta_n . \end{aligned}$$

Hence,

$$\int_E f_n^2 d\mu < \int_E (f_m + 2^{-(n-1)})^2 d\mu + n^2\eta_n$$
$$< \int_E f_m^2 d\mu + 2^{-n}\int_E f_m d\mu + 4^{-(n-1)}\mu(E) + n^2\eta_n ,$$

and $\{f_n^2\}$ is almost increasing. Thus, $\|F_n\|_2$ is almost increasing. Moreover, it follows from S. Leader's work ([5]) that

$$\begin{aligned}(G, F_n) &= \sum_{i \leqq a_n} \frac{G(L_i)F_n(L_i)}{\mu(L_i)} = \sum_{i \leqq a_n} i/2^n G(L_i) \\ &= \lim_m \sum_{i \leqq a_n} i/2^n F_m(L_i)(F_m \xrightarrow{\omega} G) \\ &\geqq \lim_m \sum_{i \leqq a_n} i/2^n(F_n(L_i) - \delta_n) \text{ (see (21))} \\ &= \sum_{i \leqq a_n} i/2^n \int_{L_i} f_n d\mu - a_n(a_n + 1)2^{n+1}\delta_n \\ &> \int_\Omega f_n^2 d\mu - n^2 2^n \delta_n .\end{aligned}$$

Hence, $\|F_n\|_2^2 - n^2 2^n\delta_n \leqq \|F_n\|_2\|G\|_2$ and, finally, $\lim \|F_n\|_2 = \|G\|_2$.

Proof of (ii). Firstly, notice that

$$\int HK\, d\mu = \lim_m \int Hk_m d\mu = \lim_m \int k_m d\lambda$$

and that

$$\int FK\, d\mu = \lim_m \int Fk_m d\mu = \lim_m \left(\lim_n \int k_m f_n d\mu\right) .$$

But, fixing m, $k_m = \sum_{j \leqq e_m} b_j \chi_{B_j}$, $b_j > 0$, $B_j \in \mathscr{M}$. Hence

$$\int k_m d\lambda = \sum_j b_j \lambda(B_j)$$

and

$$\int k_m f_n d\mu = \sum_j b_j \int_{B_j} f_n d\mu = \sum_j b_j \sum_{i \leqq a_n} i/2^n \mu(L_i \cap B_j) .$$

Thus,

$$\begin{aligned}&\int k_m\, d\lambda - \int k_m f_n d\mu \\ &= \sum_j b_j \sum_{i \leqq a_n} (\lambda - i/2^n\mu)(L_i \cap B_j) \\ &\leqq \sum_j b_j \sum_{i \leqq a_n} [2^{-n}\mu(L_i \cap B_j) + \varepsilon_n] \text{ (see (11))} \\ &= \textstyle\sum_j b_j[2^{-n}\mu(B_j) + a_n\varepsilon_n] \to 0 \text{ as } n \to \infty,\end{aligned}$$

which implies that

$$\int k_m d\lambda \leqq \lim_n \int k_m f_n d\mu .$$

Hence (ii) is established.

Because the proof of (iii) is analogous to that of (ii) we will omit the details.

Proof of (iv). It follows from (ii) that

$$\int HF\,d\mu \leqq \int F^2 d\mu$$

so it will suffice to show that

$$\int HF\,d\mu = \lim_n \int f_n d\lambda \geqq \lim_n \int f_n^2 d\mu = \int F^2 d\mu .$$

But,

$$\int f_n d\lambda = \sum_{i \leqq a_n} i/2^n \lambda(L_i)$$

and

$$\int f_n^2 d\mu = \sum_{i \leqq a_n} (i/2^n)(i/2^n)\mu(L_i) .$$

Hence

$$\begin{aligned}\int f_n d\lambda - \int f_n^2 d\mu &= \sum_{i \leqq a_n} (i/2^n)[(\lambda - (i/2^n)\mu)(L_i)] \\ &> na_n(-\varepsilon_n) \to 0 \text{ (see (7)).}\end{aligned}$$

Proof of (v).

$$\begin{aligned}\int (H-K)^2 d\mu &= \int (H-F)^2 d\mu + \int (F-K)^2 d\mu \\ &\quad + 2\left(\int (H-F)(F-K)d\mu\right) ;\end{aligned} \tag{24}$$

and

$$\begin{aligned}\int (H-F)(F-K)d\mu &= \int HF\,d\mu - \int F^2 d\mu \\ &\quad - \int HK\,d\mu + \int FK\,d\mu \geqq 0 .\end{aligned}$$

Moreover, (24) also shows that if K can play the role of F in the theorem, then $\int (F-K)^2 = 0$ and, hence, the conditions of the theorem characterize the Radon-Nikodym derivative of H. An excellent interpretation of these results can be found in the papers of H. D. Brunk.

5. A martingale convergence theorem. In this section, we shall establish which features of a martingale convergence theorem carry over to the setting of additive set functions defined on lattices of sets.

Suppose that $\{\mathscr{M}_n\}$ is an increasing sequence of lattices of subsets of Ω, and $\mathscr{M} = \bigcup_n \mathscr{M}_n$. Then the algebras, $\mathscr{A}_n$, of subsets of Ω that are generated by these lattices increase to the algebra, $\mathscr{A}$, generated by $\mathscr{M}$.

Suppose that λ and μ are nonnegative, additive set functions defined on $\mathscr{A}$. Denote by λ_n and μ_n the restrictions of λ and μ to $\mathscr{A}_n$. Denote by $F = \{f_n\}$ the Radon-Nikodym derivative of λ with respect to μ, and denote by $G_k = \{g_{k,n}\}_{n=1}^{\infty}$ the Radon-Nikodym derivative of λ_k with respect to μ_k.

Because the sequences $\{g_{k,n}\}_n$ are Cauchy sequences in $L_1(\Omega, \mathscr{A}_k, \mu_k)$ they are also Cauchy sequences in $L_1(\Omega, \mathscr{A}, \mu)$

$$\left(\text{i.e., } \int_\Omega |g_{k,m} - g_{k,n}|\,d\mu = \int_\Omega |g_{k,m} - g_{k,n}|\,d\mu_k\right).$$

Hence the equation

$$H_k(E) = \lim_n \int_E g_{k,n}\,d\mu\,, \qquad E \in \mathscr{A},$$

defines an additive extension of G_k to $\mathscr{A}$. Notice that $\{H_k\}$ is determined by $\{\mathscr{M}_k\}$, λ, and μ.

Now we have enough notation to state our martingale convergence theorem succinctly.

THEOREM. *Suppose that λ is absolutely continuous with respect to μ. Then the sequence $\{H_k\}$ converges to F in $V_1(\Omega, \mathscr{A}, \mu)$.*

Before establishing this theorem, let us give a simple example to illustrate the fact that the requirement that λ be absolutely continuous with respect to μ is not superfluous.

EXAMPLE. Suppose that Ω is the set of positive integers, and $\mathscr{A}_n$ is the algebra of subsets of Ω comprised of the subsets of the first n positive integers and their complements in Ω, $n \geqq 1$. Let λ be the additive function on $\mathscr{A}$ which assigns zero to a finite set and one to the complement of a finite set, and let μ be defined on the elements E of $\mathscr{A}$ by $\mu(E) = \sum_{x \in E} 2^{-x}$. Then λ is singular with respect to μ; but, for each positive integer n, λ_n is absolutely continuous with respect to μ_n. Suppose that $\mathscr{M}_n = \mathscr{A}_n$, $n \geqq 1$. Then

$$G_n = \int 2^n \chi_{[n+1, n+2, \dots]}\,d\mu\,,$$

and the sequence $\{H_n\}$ is not Cauchy.

The example shows that it is possible to have an increasing sequence $\{\mathscr{A}_n)$ of algebras such that λ is not absolutely continuous with respect to μ even though all the λ_n's are absolutely continuous with respect to the corresponding μ_n's. However, λ and μ are determined by the sequences $\{\lambda_n\}$ and $\{\mu_n\}$. Moreover, given sequences $\{\lambda_n\}$ and $\{\mu_n\}$ such that each λ_n is absolutely continuous with respect to μ_n, then λ is absolutely continuous with respect to μ if, and only if, the λ_n's are uniformly absolutely continuous with respect to the μ_n's (i.e., for each $\varepsilon > 0$ there is $\delta > 0$ such that if $E \in \mathscr{A}_n$ and $\mu_n(E) < \delta$, then $\lambda_n(E) < \varepsilon$).

Proceeding to a proof of the theorem, suppose there is a positive number, ε, and a subsequence $\{H_{k_n}\}$ satisfying $\|H_{k_n} - F\| \geqq 3\varepsilon$, $n = 1, 2, \cdots$. Relabeling if necessary, we can suppose that $k_n = n$. Since $F = \{f_i\}$ and the f_i's are simple $\mathscr{A}$-measurable functions, there exists n_i such that f_i is $\mathscr{A}_{n_i}$-measurable. The sequence $\{\mathscr{A}_n\}$ is increasing, so we can take $n_{i+1} > n_i$ and look at the corresponding sequence $\{H_{n_i}\}_i$. Relabeling again, we can thus suppose that f_i is $\mathscr{A}_i$-measurable and $\|H_i - F\| \geqq 3\varepsilon$. Because of the manner in which we defined the sequences $\{f_i\}$ and $\{g_{j,i}\}_i$, we can assume that $g_{j,i} = f_i$ for $i \leqq j$. Referring back to the construction, we have $(\lambda - n\mu)(M^c \cap N) > -\varepsilon_n$ which implies $\mu(M^c \cap N) < (\lambda(\Omega) + \varepsilon_n)/n$. Hence, from the convergence of $\mu(M^c \cap N)$ to zero, follows the convergence of $\lambda(M^c \cap N)$ to zero. Since the λ_k's are uniformly absolutely continuous with respect to the μ_k's, the corresponding values $\lambda_k(M^c_{(k)} \cap N_{(k)})$ converge to zero uniformly in k. (The definitions of $M_{(k)}$ and $N_{(k)}$ are gleaned by putting $\mathscr{M}_k$ into the construction.) Checking the paragraph that produced (14) and then checking (15)-(17) permits us to claim that

$$\lim_n G_{k,n}(\Omega) = \lambda_k(\Omega) = \lambda(\Omega)$$

uniformly in k. Thus, from (21) and the remark that follows (21) we can conclude that $\lim_n G_{k,n} = G_k$ uniformly in k. Since

$$F_k = \int f_k d\mu = \int g_{kk} d\mu,\ \lim_k F_k = F\ ,$$

and

$$\lim_k \left(\left\|\int g_{k,k} d\mu - H_k\right\|\right) = 0\ ,$$

we have $\lim_k (\|H_k - F\|) = 0$.

COROLLARY 1. *Suppose that $\mathscr{M}_n$ is an increasing sequence of lattices of subsets of Ω. For each positive integer n, suppose that* *each of* λ_n *and* μ_n *is a nonnegative additive function defined on* $\mathscr{A}_n$ *such that* $\lambda_{n+1}|\mathscr{A}_n = \lambda_n$ *and* $\mu_{n+1}|\mathscr{A}_n = \mu_n$. *Finally, suppose that the* λ_k*'s are uniformly absolutely continuous with respect to the* μ_k*'s. Then the sequence* $\{H_k\}$ *converges in norm.*

We have restricted our attention to nonnegative functions in this paper because we wished to keep the setting simple enough to make our presentation reasonably easy to follow.

The following corollary for an increasing sequence $\{\mathscr{A}_n\}$ of algebras of subsets of Ω will be established by using the construction given in [2].

COROLLARY 2. *Suppose that* $\{\mathscr{A}_n\}$ *is an increasing sequence of algebras of subsets of* Ω, λ *and* μ *are bounded, additive set functions defined on* $\mathscr{A} = \bigcup_n \mathscr{A}_n$, *with* μ *nonnegative and* λ *is absolutely continuous with respect to* μ. *For each positive integer* n, *take* $\mathscr{M}_n = \mathscr{A}_n$. *Then* $\{H_k\}$ *converges to* λ *in norm.*

Proof. Refer to the construction given in [2]. Adopt the notation of the martingale convergence theorem and repeat the relabelings described in the proof of the martingale convergence theorem. Recall that $F = \lambda$ and $G_k = \lambda_k$, $k \geqq 1$, because $\mathscr{M}_k = \mathscr{A}_k$. Hence

$$\varepsilon \leqq ||H_k - F|| \leqq \left\|H_k - \int g_{k,n} d\mu\right\| + \left\|\int (g_{k,n} - f_k)\mu\right\| + \left\|\int f_k d\mu - \lambda\right\| .$$

But, the first and third terms of the right side of this latter inequality can be made smaller than ε, and the following observations show that the second term can also be made smaller than ε.

$$\begin{aligned}\left\|\int (g_{k,n} - f_k) d\mu\right\| &= \int |g_{k,n} - f_k| d\mu = \int |g_{k,n} - f_k| d\mu_k \\ &= \left\|\int (g_{k,n} - f_k) d\mu_k\right\| \\ &\leqq ||G_{k,n} - G_k|| + ||G_k - \lambda_k|| + ||\lambda_k - f_k d\mu_k|| \\ &\leqq ||G_{k,n} - G_k|| + 0 + \left\|\lambda - \int f_k d\mu\right\| .\end{aligned}$$

The version of Corollary 1 that is appropriate for algebras will not be transcribed.

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