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A NOTE ON THE MINIMALITY OF CERTAIN BITRANSFORMATION GROUPS

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Let (T, X) be a transformation group with compact Hausdorff space X and topological group T. Let (X, G) be a transformation group with G a compact topological group. Then the triple (T, X, G) is a bitransformation group if $(tx)g = t(xg)$ for all $t \in T, x \in X, g \in G$ and the action of G on X is strongly effective, (that is $xg = x$ if and only if $g =$ the identity element e of G). A bitransformation group (T, X, G), induces canonically the transformation group $(T, X/G)$ where X/G is the orbit space of (X, G). Let (T, X, G) be a bitransformation group. Suppose $(T, X/G)$ is a minimal transformation group whereas (T, X) is not a minimal transformation group then what is the possible structure of (T, X, G)?

In this note, it is proved that the fundamental group of X must be of certain form when G is a circle group. Use this result together with some results of Malcev, a necessary and sufficient condition is found for the minimality of certain nilflows.

THEOREM 1. *Let (T, X, G) be a bitransformation group with circle group G. If $(T, X/G)$ is a minimal transformation group and (T, X) is not minimal, then there exists a finite group H of G such that X is a covering space of X/H and X/H admits a section over X/G.*

Proof. Let M be a minimal set in (T, X). Let $H = \{g \in G: gM = M\}$. Then H is a proper closed subgroup of G. Thus H is a finite group. The natural projection $p: X/H \rightarrow X/G$ is a principal bundle map with fiber G/H. Then $p \mid M/H: M/H \rightarrow X/G$ is a homeomorphism. Thus p admits a global cross section.

COROLLARY. *Besides all the notation of Theorem 1, assume that X is path connected. Then $\pi(X)$ is a isomorphic with a subgroup of $\pi(X/G) \cdot Z$, where Z is the integer group and the dot denotes semidirect product.*

From now on, we shall assume that N is a simply connected nilpotent analytic group. A subgroup H of N is a uniform subgroup if the homogeneous space N/H is compact. Let Γ be a discrete uniform subgroup of N. Then Γ is torsion-free and finitely generated [2]. For each discrete uniform subgroup Γ of N, there is a subset

$D = \{d_1, \cdots, d_m\}$ of Γ with the following properties:

(1) there exist m one-parameter groups $d_i(t)$ such that $N = \{d_1(t_1)d_2(t_2) \cdots d_m(t_m): t_1, \cdots, t_m \in R, \text{ reals}\}$.

(2) $\Gamma = \{d_1(n_1)d_2(n_2) \cdots d_m(n_m): n_1, n_2, \cdots, n_m \in Z, \text{ integers}\}$.

(3) If $N_i = \{d_i(t_i) \cdots d_m(t_m): t_i, t_{i+1}, \cdots, t_m$ any real numbers, then N_i is a closed subgroup of N and N_i is normal in N_{i-1}. D is called a canonical basis of Γ.

Let F be a nilpotent group and $F = F^0 \supset F^1 \supset F^2 \supset \cdots \supset F^p \supset F^{p+1} = (e)$ be the descending central series. We recall that $F^i = [F, F^{i-1}]$, where $[F, F^{i-1}]$ is the subgroup of F generated by $\{[a, b] = aba^{-1}b^{-1}$: $a \in F, b \in F^{i-1}$. Let $N = N^0 \supset N^1 \supset \cdots \supset N^m \supset N^{m+1} = (e)$ be the descending central series. Then $\Gamma^p \subset N^p \cap \Gamma \subset N^p$ we shall prove that.

LEMMA 1. *Γ^p is uniform in N^p and $\Gamma \cap N^p/\Gamma^p$ is finite.*

Proof. Let V be the vector subspace of N^p spanned by Γ^p. Let $D = \{d_1, \cdots, d_l, \cdots, d_k, \cdots, d_m\}$ be a canonical basis of D such that $\{d_l, \cdots, d_m\}$, $\{d_k, \cdots, d_m\}$ are canonical basis for N^{p-1} and N^p respectively. Then $\{d_i d_j(t) d_i^{-1} d_j(t)^{-1}: t \in R\}$ is an one-parameter group containing $d_i d_j d_i^{-1} d_j^{-1}$ if $l \leqq j$. Hence $\{d_i d_j(t) d_i^{-1} d_j(t)\} \subseteqq V$. For each fixed $t_0 \in R$, $\{d_i(t) d_j(t_0) d_i(t)^{-1} d_j(t_0)^{-1}: t \in R\}$ is an one parameter group containing $d_i d_j(t_0) d_i d_j(t_0)^{-1} \in V$ if $l \leqq j$. This implies that $d_i(s) d_j(t) d_i(s)^{-1} d_j(t)^{-1} \in V$ for any $s, t \in R$. Thus $N^p = [N, N^{p-1}] \subseteqq V$ and $N^p = V$. Hence Γ^p is uniform in N^p and $\Gamma \cap N^p/\Gamma^p$ is finite.

In order to state our next result, we recall the definition of coset transformation group. Let T be a topological group and G/H a coset space. Let $\mathcal{O}$ be a continuous homomorphism from T into G. Then $(T, G/H)$ is a coset transformation group (relative to $\mathcal{O}$) if $tgH = \mathcal{O}(t)gH$ for $t \in T$, $g \in G$.

PROPOSITION 1. *Let $(T, N/\Gamma)$ be a coset transformation group where N is a simply connected nilpotent analytic group and Γ is discrete uniform subgroup of N. Assume that* $\dim N^q/N^{qH} = 1$ *for $q \geqq 1$. Then $(T, N/\Gamma)$ is minimal if and only if $(T, N/\Gamma N')$ is minimal.*

Proof. We shall prove this theorem by induction based the length of nilpotency of Γ. When Γ is abelian, there is nothing to prove. Assume $(T, N/\Gamma N')$ is minimal. By induction hypothesis $(T, N/N^p/\Gamma N^p/N^p)$ is minimal. Thus $(T, N/\Gamma N^p)$ is minimal. Let $H^q = \{d_q(t_q) \cdots d_m(t_m)$: $t_q, \cdots t_m \in R\}$. Suppose $(T, N/\Gamma H^q)$ is minimal and $(T, N/\Gamma H^{q+1})$ is not minimal. Then $(T, N/\Gamma H^{q+1}, \Gamma H^q/\Gamma H^{q+1})$ is a bitransformation group. By Corollary 1, $\Gamma/\Gamma \cap H^{q+1}$ is isomorphic with a subgroup of $\Gamma/\Gamma \cap H^q \times Z$. Then the image of $d_q(\Gamma \cap H^{q+1})$ under this isomorphism must be of the form (x, z) for some nonzero integer[1]. Thus $(x, z)^\alpha \notin (\Gamma/\Gamma \cap H^q)$ if α is a nonzero integer. On the other hand, $[(\Gamma/\Gamma \cap H^q \times Z, (\Gamma/\Gamma \cap H^q) \times Z] \subset \Gamma/\Gamma \cap H^q$. This fact together with Lemma 1, we have the contradiction. Thus $(T, N/\Gamma H^{q+1})$ is minimal. By finite induction, $(T, N/\Gamma)$ is minimal.

THEOREM 2. *Let $(T, N/H)$ be a coset transformation with nilpotent analytic group N and closed uniform subgroup H such that* $\dim (N/\Gamma H_0)^q/(N/\Gamma H_0)^{q+1} = 1$. *Then $(T, N/H)$ is minimal if and only if $(T, N/H[N, N])$ is minimal.*

Proof. Let H_0 be the identity component of H. Then H_0 is a normal subgroup of N and N/H_0 is simply connected. Let π be the canonical projection from $N \to N/H_0$. Then $\pi^{-1}(\pi(\Gamma)[N/H_0, N/H_0]) = H[N, N]$. Hence $H[N, N]$ is closed uniform subgroup of N. If $(T, N/H[N, N])$ is minimal, then $(T, N/H_0/H/H_0)$ is minimal by Proposition 1. But $(T, N/H)$ is isomorphic with $(T, N/H_0/H/H_0)$. Hence $(T, N/H)$ is minimal.

EXAMPLES. ([1, p. 52]) consider the group G of all real matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

and let D be the uniform discrete subgroup of matrices

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

for all integers a, b, c. Then $M = G/D$ is a nilmanifold. Consider a one-parameter subgroup $\varphi(t)$ of G given by

$$\operatorname{expt}\begin{pmatrix} 0 & \alpha & \gamma \\ 0 & 0 & \beta \\ 0 & 0 & 0 \end{pmatrix} = \begin{vmatrix} 1 & 2t & \lambda t + \dfrac{1}{2}\alpha\beta t^2 \\ 0 & 1 & \beta t \\ 0 & 0 & 1 \end{vmatrix} .$$

Take a point $Q \in M$ given by the coset

$$\begin{pmatrix} 1 & x_0 & z_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{pmatrix} D$$

[1] Since Γ is nilpotent, the semi-direct product here is actually a direct product.

the orbit $\varphi_t^*(t)$ in M is

$$\begin{pmatrix} 1 & t + x_0 & \gamma t + \frac{\alpha\beta}{2} t^2 + z_0 + \alpha + y_0 \\ 0 & 1 & \beta t + y_0 \\ 0 & 0 & 1 \end{pmatrix} D .$$

Then $D[G, G]$ is the set of all the matrices

$$\begin{pmatrix} 1 & a & z \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

for all integers a, b and real number z. And $(\varphi(t), G/D[G, G])$ is isomorphic with the continuous flow on two-dimensional torus with the direction ratio (α, β).

By Theorem 2, $(\varphi(t), M)$ is minimal if and only if $(\varphi(t), G/D[G, G])$. The latter is minimal if and only if α and β are rationally independent. This answers the question in [1, p. 53].

Added in proof. After this note went in print, we have the proof of the following statement. Let G be a simply connected solvable analytic group and Γ be a nilpotent uniform subgroup of G. Then $(T, G/P)$ is minimal if and only if $(T, G/\Gamma N)$ is minimal, here N denotes the analytic subgroup of G which contains $[\Gamma, \Gamma]$ as a uniform subgroup. The proof uses a stronger form of Lemma 1 (replacing the circle group by torus groups) and the nilpotency of Γ. The detail will appear later.

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