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APPROXIMATION BY ARCHIMEDEAN LATTICE CONES

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A *root system* Λ is a partially ordered set having the property that no two incomparable elements λ and μ have a common lower bound. $\Pi(\Lambda, \mathbf{R}_\lambda)$ will denote the direct product of copies of R, the set of real numbers, one for each $\lambda \in \Lambda$. $V(\Lambda, \mathbf{R}_\lambda)$ is the following subgroup: $v \in V = V(\Lambda, \mathbf{R}_\lambda)$ if the support of v has no infinite ascending sequences. We put a lattice order on v by setting $v \geqq 0$ if $v = 0$ or else every maximal component of v is positive in R.

This paper has two main results: we first show that the cone of any finite dimensional vector lattice G can be obtained as the union of an increasing sequence $P_1, P_2 \cdots$ of archimedean vector lattice cones on G such that $(G, P_1) \cong (G, P_2) \cong \cdots$, as vector lattices. Next, generalizing this, we show that for any root system Λ the cone of the ℓ-group $V = V(\Lambda, \mathbf{R}_\lambda)$ can be obtained as the union of a family of archimedean vector ℓ-cones $\{P_\gamma: \gamma \in \Gamma\}$ on V, where $(V, P_\gamma) \cong (V, P_\delta)$, as vector lattices, for all $\gamma, \delta \in \Gamma$.

It is proved in [1], Theorem 2.2, that $V(\Lambda, \mathbf{R}_\lambda)$ is indeed an ℓ-group when Λ is a root system. In an ℓ-group K, $x \in K$ is a *strong order unit* if $x \geqq 0$, and for each $0 < a \in K$ there is an $n = 1, 2, \cdots$ such that $nx \geqq a$. The symbol $\boxplus$ will denote the cardinal sum of ℓ-groups; that is, if $K_i (i \in I)$ are ℓ-groups then $K = \boxplus \{K_i: i \in I\}$ means that K is the direct sum of the K_i, as groups, and $0 \leqq x \in K$ if and only if $0 \leqq x_i \in K_i$, for each $i \in I$. Finally, if r is a real number, $\langle r \rangle$ will denote the smallest integer exceeding r.

Throughout the paper the pair (G, P) will denote an abelian ℓ-group; that is, G is an abelian group, and P is the cone for a lattice-group order on G. An ℓ-group (G, P) is said to be *archimedean* if for any pair $a, b \in P$ there is a positive integer n such that $na \not\leq b$; P is then called an *archimedean ℓ-cone*. We restrict our considerations to abelian groups since archimedean ℓ-groups are necessarily abelian (see [2]).

Let (G, Q) be an ℓ-group; we say that Q *can be approximated by the archimedean ℓ-cone* P if there is a family $\{P_\gamma: \gamma \in \Gamma\}$ of archimedean ℓ-cones on G, such that (i) $(G, P_\gamma) \cong (G, P_\delta)$, for all $\gamma, \delta \in \Gamma$, (ii) $Q = \bigcup \{P_\gamma: \gamma \in \Gamma\}$ and (iii) $P = P_\gamma$, for some $\gamma \in \Gamma$. The ℓ-group (G, Q) is then called a *limit A-group*. If the approximating family is directed by set inclusion (resp. a chain under set inclusion) we call

(G, Q) a *directed* (resp. *linear*) *limit A-group*. If $\Gamma = \{1, 2, \cdots\}$ and $P_n \subseteqq P_{n+1}$ for all $n = 1, 2, \cdots$, we call (G, Q) a *sequential limit A-group*.

(G, Q) is a *vector lattice* if G is a real vector space, and in addition to being an ℓ-cone, P is closed under scalar multiplication by positive real numbers. The vector lattice (G, Q) *can be approximated by the archimedean vector lattice cone* P if there is a family $\{P_\gamma : \gamma \in \Gamma\}$ of archimedean vector ℓ-cones on G, such that (i) $(G, P_\gamma) \cong (G, P_\delta)$, as vector lattices, for all $\gamma, \delta \in \Gamma$, (ii) $Q = \bigcup \{P_\gamma : \gamma \in \Gamma\}$ and (iii) $P = P_\gamma$, for some $\gamma \in \Gamma$. In this case we call (G, Q) a *limit A-space*. By a *directed* (resp. *linear*, resp. *sequential*) *limit A-space* (G, Q) we mean one where the approximating vector ℓ-cones form a directed set (resp. a chain, resp. an increasing sequence.)

It will be useful to denote a limit A-group (G, Q) by (G, Q, P), where $P \cong P_\gamma$, for all $\gamma \in \Gamma$; this way we can keep track of what approximation is being used.

Let (G, Q, P) be a limit A-group (resp. limit A-space); we call it a *strong limit* A-group (resp. *strong limit* A-space) if Q is essential over each P_γ. (Let (G, P) be an ℓ-group, Q be an extension of the cone P. Q is an *essential extention* of P if every ℓ-ideal of (G, Q) is an ℓ-ideal of (G, P). For further discussion on essential extensions see [3]). Suppose the family $\{P_\gamma : \gamma \in \Gamma\}$ has a smallest member (which is once again denoted by P); it follows from a remark in [3] concerning essential extensions, that (G, Q, P) is a strong limit A-group if and only if Q is essential over P.

PROPOSITION 1. *The cardinal sum of (strong) sequential limit A-groups is a (strong) sequential limit A-group. The same statement holds for (strong) sequential limit A-spaces.*

Proof. Let $(G, Q) = \boxplus (G_i, Q_i)$, $i \in I$. *Suppose each* Q_i *is the limit of the sequence* $\{P_{n,i} : n = 1, 2, \cdots\}$ of archimedean ℓ-cones on G_i, and $(G_i, P_{1,i}) \cong (G_i, P_{2,i}) \cong \cdots$, for all $i \in I$. Fix n, and let P_n be the ℓ-cone of the cardinal sum of the $(G_i, P_{n,i})$. Since each $P_{n,i}$ is archimedean, so is P_n; clearly $P_n \subseteqq P_{n+1}$, for each $n = 1, 2, \cdots$, and $P_n \subseteqq Q$.

So let $y \in Q$ and $i_1, i_2, \cdots, i_k$ be the nonzero components of y. Then each y_{i_m} is in Q_{i_m}, for $m = 1, 2, \cdots, k$, and there exists an $n(m)$ such that $y_{i_m} \in P_{n(m), i_m}$. Let $n = \max\ \{n(m) : m = 1, 2, \cdots, k\}$; then each $y_{i_m} \in P_{n, i_m}$, which implies that $y \in P_n$. This show that $Q = \bigcup_{n=1}^{\infty} P_n$; it is obvious that $(G, P_1) \cong (G, P_2) \cong \cdots$. It follows therefore that (G, Q, P_1) is a sequential limit A-group.

Now suppose Q_i is essential over each $P_{n,i}$, $i \in I$. (This is equivalent to saying that each ℓ-ideal of (G_i, Q_i) is an ℓ-ideal of $(G_i P_{n,i})$.) Let K be an ℓ-ideal of (G, Q); then $K = \boxplus \{K_i\colon\ i \in I\}$, where $K_i = K \cap G_i$. Each K_i is an ℓ-ideal of (G_i, Q_i), and hence an ℓ-ideal of $(G_i, P_{n,i})$. Thus K is an ℓ-ideal of (G, P_n), proving that Q is esssential over P_n, that is, (G, Q, P_1) is a strong sequential limit A-group.

The above proposition can be generalized, in a sense:

PROPOSITION 2. *The cardinal sum of* (strong) *directed limit A-groups is a* (strong) *directed limit A-group. The same statement holds for cardinal products.*

Proof. Let $(G, Q) = \boxplus (G_i, Q_i)$, $i \in I$. Suppose $(G_i, Q_i) = (G_i, Q_i, P_i)$ is a directed limit A-group, and $\{P_{\gamma_i}\colon \gamma_i \in \Gamma^{(i)}\}$ is the approximating family. Let $\Gamma = \pi\{\Gamma^{(i)}\colon i \in I\}$ and consider the family $\{P_\gamma\colon \gamma \in \Gamma\}$ of ℓ-cones defined by: $x \in P_\gamma$ if for each $i \in I$ $x_i \in P_{\gamma_i}(\gamma_i \in \Gamma^{(i)})$. Each P_γ is clearly an archimedean ℓ-cone for G, and $(G, P_\gamma) \cong (G, P_\delta)$, for $\gamma \neq \delta$. The P_γ obviously form a directed system, and finally, if $y \in Q$ then $y_i = 0$ or $y_i \in Q_i$; in either case $y_i \in P_{\delta_i}$, for some $\delta_i \in \Gamma^{(i)}$, and therefore $y \in P_\delta$, where $\delta = (\cdots, \delta_i, \cdots) \in \Gamma$. Thus Q is the join of the P_γ and we're done.

Notice that the above proof works for the cardinal product of directed limit A-groups. If each (G_i, Q_i, P_i) is a strong limit A-group then one uses the technique of the proof of Proposition 1 to show that (G, Q, P) is also a strong limit A-group. We should also point out once more, that a similar version of this theorem holds for directed limit A-spaces.

It is not known whether the cardinal sum (resp. product) of linear limit A-groups is again a linear limit A-group. By Proposition 2 it is certainly a directed limit A-group.

THEOREM 3. *Let (G, Q, P_1) be a strong sequential limit A-space having a strong order unit. Let $K = \mathbf{R} \oplus G$ and $Q' = \{r + g\colon r > 0,$ or else $r = 0$ and $g \in Q\}$. Then $(K, Q', \mathbf{R}^+ \oplus P_1)$ is a strong sequential limit A-space.*

Proof. Let $u \in G$ be a strong order unit relative to Q; without loss of generality we can assume $u \in P_n$ for each $n = 1, 2, \cdots$. Let v be any positive real number and define

$$v^{(n)} = \left(\frac{1}{n}\right)v + \left(\frac{1-n}{n}\right)u\,, \qquad \text{for } n = 1, 2, \cdots.$$

Let $V^{(n)} = \{rv^{(n)}\colon r \in \mathbf{R}\}$; $V^{(n)}$ is a one-dimensional space, and clearly $V^{(n)} \cap G = 0$, so $K = V^{(n)} \oplus G$. Now let $P_n' = \{rv^{(n)} + g\colon 0 \leqq r$ and $g \in P_n\}$; then (K, P_n') is the cardinal sum of $V^{(n)}$, ordered as the reals, and (G, P_n). Since each P_n is archimedean it follows that each P_n' is also. Notice that $V^{(1)} = \mathbf{R}$ and $P_1' = \mathbf{R} \boxplus P_1$. If H is an ℓ-ideal of (K, Q') then either $H = K$ or $H = G$, or else H is a proper ℓ-ideal of (G, Q); in any case H is an ℓ-ideal of (K, P_1'), since Q is essential over P_1. Notice also that $(K, P_n') \cong (K, P_{n+1}')$, for all n.

We must show (1) $P_n' \subsetneqq P_{n+1}' \subsetneqq Q'$ and (2) $Q' = \bigcup_{n=1}^{\infty} P_n'$.

(1) We show first that $P_1' \subsetneqq P_k' \subsetneqq Q'$, for all $k = 1, 2, \cdots$. The first inequality will follow if we can prove that $v \in P_k'$, the second, if $v^{(k)} \in Q'$, because we know that $P_1 \subsetneqq P_k \subsetneqq Q$. That $v^{(k)}$ is in Q' is clear since $(1/n)v > 0$. One can easily show that

$$v = kv^{(k)} + (k-1)u ,$$

proving that $v \in P_k'$.

But now observe that for each $n = 1, 2, \cdots$ we have

$$v^{(n)} - v^{(n+1)} = \frac{1}{n(n+1)}(v + u) \in P_1' \subsetneqq P_{n+1}' ,$$

so $v^{(n)}$ is the sum of two elements in P_{n+1}', and hence $v^{(n)} \in P_{n+1}'$. That is enough to show that $P_n' \subsetneqq P_{n+1}'$.

(2) Let $y \in Q'$; we have the following expressions for y: $y = sv + y_0 = s^{(n)}v^{(n)} + y^{(n)}$, with $s, s^{(n)} \in \mathbf{R}$ and $y_0, y^{(n)} \in G$. This forces certain relations:

$$(1) \qquad s^{(n)} = ns \geqq 0 \qquad \text{(since } y \in Q'\text{)} ,$$

and

$$(2) \qquad \left(\frac{(1-n)}{n}\right)s^{(n)}u + y^{(n)} = y_0 .$$

Thus each $s^{(n)} \geqq 0$; moreover, the above equations give

$$(2') \qquad y^{(n)} = (n-1)su + y_0 .$$

Writing y_0 as the difference of its positive and negative parts relative to Q, we obtain

$$(2'') \qquad y^{(n)} = (n - \ )su + y_0^+ - y_0^- .$$

Observe that since u is a strong order unit of (G, Q), then so is su. Therefore if n is large enough, $(n-1)su > y_0^-$(rel. Q). But since the P_n form a chain we can certainly find an n_0 such that $y_0^+, y_0^- \in P_{n_0}$ and $(n_0 - 1)su > y_0^-$(rel. P_{n_0}). Thus $y_0^{(n)} \in P_{n_0}$; together with the fact that $s_0^{(n)} \geqq 0$ this impies that $y \in P_{n_0}$. This proves the theorem.

COROLLARY 3.1. *Every finite dimensional vector lattice is a strong sequential limit A-space.*

Proof. Note at the outset that every finite dimensional vector lattice has a strong order unit. For if (V, Q) is a t-dimensional vector lattice, we may regard (V, Q) as $V(\Lambda, \mathbf{R}_\lambda)$, where Λ is a root system of t elements, and for each $\lambda \in \Lambda$, $\mathbf{R}_\lambda = \mathbf{R}$. ([1], Theorem 5.11) Then $x = (1, 1, \cdots, 1)$ is a strong order unit.

We proceed by induction on t:

Case I. Λ has a largest element λ_0. Let $\Lambda' = \Lambda\backslash(\lambda_0\}$; then (V, Q) is a direct lexicographic extension of $V(\Lambda', \mathbf{R}_\lambda)$ by $\mathbf{R}$. But $V(\Lambda', \mathbf{R}_\lambda)$ has dimension $t - 1$, so it is a strong sequential limit A-space. By Theorem 3 (V, Q) is also a strong sequential limit A-space.

Case II. Λ has no largest element. Then Λ can be written as the union of two nonempty, disjoint subsets Λ_1 and Λ_2 having the property that λ is incomparable to μ, for all $\lambda \in \Lambda_1$ and $\mu \in \Lambda_2$. It follows that $(V, Q) = V(\Lambda_1, \mathbf{R}_\lambda) \boxplus V(\Lambda_2, \mathbf{R}_\lambda)$, and both these summands have dimension less than t; thus they both are strong sequential limit A-spaces, and by Proposition 1 so is (V, Q).

Let Λ be a root system, $\Pi = \Pi(\Lambda, \mathbf{R}_\lambda)$, $V = V(\Lambda, \mathbf{R}_\lambda)$ and $P = V \cap \Pi^+$, where $\Pi^+ = \{x: x_\lambda \geqq 0$, for all $\lambda \in \Lambda\}$. The following discussion will establish that V is a limit A-space. (Of course we consider V as a vector lattice relative to the cone $V^+ = \{v$: all the maximal nonzero components of v are positive$\}$.) Notice that (V, P) is an ℓ-subgroup of Π. For each $x \in P$ let $s(x)$ denote the support of x, $m(x)$ the set of maximal nonzero components of x. Choose a family $\{n_\lambda: \lambda \in m(x)\}$ of positive integers, and define a map $\theta_{x,\{n_\lambda\}}$ on Π by:

$$(y\theta_{x,\{n_\lambda\}})_\lambda = \begin{cases} y_\lambda & \text{if } \lambda \notin s(x) \text{ or } \lambda \in m(x); \\ y_\lambda - n_{\lambda(x)}^{\langle x\lambda\rangle} y_{\lambda(x)} & \text{if } \lambda \in s(x) \setminus m(x) \text{ and } \lambda \text{ has no successor in } s(x); \\ y_\lambda - n_{\lambda(x)}^{\langle x\lambda\rangle} y_{\lambda-1} & \text{if } \lambda \in s(x) \setminus m(x) \text{ and } \lambda-1 \text{ is the seccessor of } \lambda \text{ in } s(x). \end{cases}$$

(Note: $\lambda(x)$ is the maximal component of x that exceeds λ.) This map has an inverse $\theta^{-1}_{x,\{n_\lambda^-\}}$:

$$(a\theta_{x,\{n_\lambda\}}{}^{-1})_\lambda = \begin{cases} y_\lambda & \text{if } \lambda \notin s(x) \text{ or } \lambda \in m(x); \\ n_{\lambda(x)}^{\langle x\lambda\rangle} y_{\lambda(x)} + y_\lambda & \text{if } \lambda \in s(x) \setminus m(x) \text{ and } \lambda \text{ has no successor in } s(x); \\ n_{\lambda(x)}^{\langle x\lambda_2\rangle+\cdots+\langle x\lambda_k\rangle} y_{\lambda_1} + \cdots + n_{\lambda(x)}^{\langle x\lambda_k\rangle} y_{\lambda_{k-1}} + y_{\lambda_k=\lambda} & \\ \quad \text{if } \lambda \in s(x)\setminus m(x) \text{ and } \lambda_{i-1} \text{ is the successor of } \lambda_i; \text{ also } \lambda_1 = \lambda(x): & \\ n_{\lambda(x)}^{\langle x\lambda_1\rangle+\cdots+\langle x\lambda_k\rangle} y_{\lambda(x)} + n_{\lambda(x)}^{\langle x\lambda_2\rangle+\cdots+\langle x\lambda_k\rangle} y_{\lambda_1} + \cdots + y_{\lambda_k=\lambda} & \\ \quad \text{if } \lambda \in s(x)\setminus m(x) \text{ and } \lambda_{i-1} \text{ is the successor of } \lambda_i; \lambda_1 \text{ has no successor.} & \end{cases}$$

Clearly then $\theta_{x,\{n_\lambda\}}$ is a vector space isomorphism of Π onto itself. Let $P_{x,\{n_\lambda\}} = P\theta_{x,\{n_\lambda\}}$; we claim first that, restricted to V, each $\theta_{x,\{n_\lambda\}}$ is an isomorphism of V onto itself. This is due to the fact that for all $y \in \Pi$

$$s(y) \subseteqq s(x) \cup s(y\theta_{x,\{n_\lambda\}}) \quad \text{and} \quad s(y\theta_{x,\{n_\lambda\}}) \subseteqq s(y) \cup s(x) .$$

A quick look at the definition of $\theta^{-1}_{x,\{n_\lambda\}}$ readily shows that $P\theta_{x,\{n_\lambda\}} \subseteqq P$, that is: $P \subseteqq P_{x,\{n_\lambda\}}$. Thus $P_{x,\{n_\lambda\}}$ is an archimedean vector lattice order on V, and $(V, P) \cong (V, P_{x,\{n_\lambda\}})$, for all $x \in P$ and $\{n_\lambda\colon \lambda \in m(x)\}$.

Now if $y \in V^+$ then consider $x = |y|_p$; of course $s(x) = s(y)$ and $m(x) = m(y)$. We proceed by induction on the maximal chains of $s(x)$. Let μ be a fixed maximal component of x; of course $(y\theta^{-1}_{x,\{n_\lambda\}})_\lambda = y_\lambda$ for all $\lambda \geqq \mu$ and every choice of integers $\{n_\lambda\colon \lambda \in m(x)\}$. So assume $\lambda < \mu$ and $\lambda \in s(x)$; if λ has no successor in $s(x)$, let n_μ be the smallest positive integer $\geqq 2$ such that $n_\mu x_\mu \geqq 2$. If $y_\lambda > 0$ then $n_\mu^{\langle x\lambda\rangle} y_\mu + y_\lambda \geqq 1$, since $x_\mu = y_\mu$. If $y_\lambda < 0$ then $y_\lambda = -x_\lambda$; now if $x_\lambda > 1$ we get $n_\mu^{\langle x\lambda\rangle-1} \geqq x_\lambda$, for all $n_\mu \geqq 2$. This implies that $n_\mu^{\langle x\lambda\rangle} y_\mu \geqq 2x_\lambda \geqq x_\lambda + 1$. If $0 > y_\lambda \geqq -1$ then $n_\mu^{\langle x\lambda\rangle} y_\mu = n_\mu y_\mu \geqq 2 = 1 + 1 \geqq x_\lambda + 1$. Hence in any of the above cases $n_\mu^{\langle x\lambda\rangle} y_\mu + y_\mu \geqq 1$, for large enough n_μ. Notice that n_μ is independent of λ.

If λ does have a successor in $s(x)$ there are two cases for $(y\theta^{-1}_{x,\{n_\lambda\}})_\lambda$.

Case I. $(y\theta^{-1}_{x,\{n_\lambda\}})_\lambda = n_\mu^{\langle x\lambda_2\rangle+\cdots+\langle x\lambda_k\rangle} y_{\lambda_1} + \cdots + n_\mu^{\langle x\lambda_k\rangle} y_{\lambda_{k-1}} + y_{\lambda_k}$, where $\lambda_k = \lambda$, λ_{i-1} is the successor of λ_i in $s(x)$ and $\lambda_1 = \mu$.
Thus

$$(y\theta^{-1}_{x,\{n_\lambda\}})_\lambda = n_\mu^{\langle x\lambda_k\rangle}[n_\mu^{\langle x\lambda_2\rangle+\cdots+\langle x\lambda_{k-1}\rangle} y_\mu + \cdots + y_{\lambda_{k-1}}] + y_{\lambda_k} ,$$

and by induction the sum in the square brackets is $\geqq 1$; so

$$(y\theta_{x}^{-1},_{\{n_\lambda\}})_\lambda \geqq n_\mu^{\langle x_{\lambda_k}\rangle} + y_{\lambda_k} \geqq 1 \; .$$

(The last inequality holds since for any real number r, $n^{\langle |r| \rangle} \geqq r + 1$, for all $n \geqq 2$.)

Case II.

$$(y\theta_{x}^{-1},_{\{n_\lambda\}})_\lambda = n_\mu^{\langle x_{\lambda_1}\rangle + \langle x_{\lambda_2}\rangle + \cdots + \langle x_{\lambda_k}\rangle} y_\mu + n_\mu^{\langle x_{\lambda_2}\rangle + \cdots + \langle x_{\lambda_k}\rangle} y_{\lambda_1} + \cdots + n_\mu^{\langle x_{\lambda_k}\rangle} y_{\lambda_{k-1}} + y_{\lambda_k} \; ,$$

where $\lambda_k = \lambda$, λ_{i-1} is the successor of λ_i in $s(x)$ and λ_1 has no successor in $s(x)$. Again

$$(y\theta_{x}^{-1},_{\{n_\lambda\}})_\lambda = n_\mu^{\langle x_{\lambda_k}\rangle}[n_\mu^{\langle x_{\lambda_1}\rangle + \cdots + \langle x_{\lambda_{k-1}}\rangle} y_\mu + \cdots + y_{\lambda_{k-1}}] + y_{\lambda_k} \; ,$$

and again by induction the bracketed sum is $\geqq 1$; so

$$(y\theta_{x}^{-1},_{\{n_\lambda\}})_\lambda \geqq n^{\langle x_{\lambda_k}\rangle} + y_{\lambda_k} \geqq 1 \; .$$

Out of all of this we get that if $\lambda < \mu$ and $\lambda \in s(x)$ then there is an n_μ(independent of λ) such that $(y\theta_{x}^{-1},_{(n_\lambda)})_\lambda \geqq 1$. This works for every $\mu \in m(x) = m(y)$, and so we can find integers $\{n_\lambda\colon \lambda \in m(x)\}$ such that $y\theta_{x}^{-1},_{\{n_\lambda\}} \in P$. (Remark: if $\lambda < \mu$ in the above arguments, but $x_\lambda = y_\lambda = 0$, then there is no problem; any θ^{-1} will fix this component.) Putting it differently: we've discovered an x in P and integers $\{n_\lambda\colon \lambda \in m(x)\}$ such that $y \in P_{x,\{n_\lambda\}}$; hence

$$V^+ \subseteqq \bigcup \{P_{x,\{n_\lambda\}}\colon x \in P,\ \{n_\lambda\colon \lambda \in m(x)\}\} \; .$$

To show the reverse containment we show a little bit more. The maps $\theta_{x,\{n_\lambda\}}$ all take V^+ into itself. For if $a \in V^+$ and $\mu \in m(a)$ then $(a\theta_{x,\{n_\lambda\}})_\mu = a_\mu$. And if $\lambda > \mu$ then $(a\theta_{x,\{n_\lambda\}})_\lambda = a_\lambda = 0$; thus $m(a) \subseteqq m(a\theta_{x,\{n_\lambda\}})$. One shows in a similar fashion that $m(a\theta_{x,\{n_\lambda\}}) \subseteqq m(a)$, and hence equality holds. This clearly shows that $V^+\theta_{x,\{n_\lambda\}} = V^+$ and therefore $P_{x,\{n_\lambda\}} \subseteqq V^+$, for all $x \in P$ and $\{n_\lambda\colon \lambda \in m(x)\}$.

In addition V^+ is essential over P, in view of Proposition 2.5 in [3]. We've thus proved the following theorem:

THEOREM 4. *If Λ is any root system, then $V = V(\Lambda, \mathbf{R}_\lambda)$ is a strong limit A-space.*

Again let Λ be a root system, and $F = F(\Lambda, \mathbf{R}_\lambda) = \{v \in V\colon s(v)$ is contained in the union of finitely many maximal chains;$\}$ F is then an ℓ-subgroup of V. In the above construction we can throw out quite a few of the $P_{x,\{n_\lambda\}}$; in this case we take for each $x \in Q = P \cap F$ and $n = 1, 2, \cdots$, mappings $\theta_{x,\{n_\lambda\}}$ where each $n_\lambda = n$. We abbreviate the notation to $\theta_{x,n}$ and $P_{x,n}$ respectively. (We mention in passing that (F, Q) is an ℓ-subgroup of (V, P).) For each $a \in Q$ and each positive integer n, we denote by $Q_{a,n}$ the cone $P_{a,n} \cap F = (P \cap F)\theta_{a,n} = Q\theta_{a,n}$. Notice that since $s(b) \subseteqq s(a) \cup s(b\theta_{a,n})$ and $s(b\theta_{a,n}) \subseteqq s(a) \cup s(b)$ it follows that $F\theta_{a,n} = F$. This means that $Q_{a,n}$ is an ℓ-cone for F and $(F, Q) \cong (F, Q_{a,n})$.

If $y \in F^+ = F \cap V^+$ then $x = |y|_P \in F$; pick n_0 to be the smallest integer $\geqq 2$ such that $n_0 x_{\mu_j} \geqq 2$, for all $j = 1, \cdots, k$, with $m(x) = m(y) = \{\mu_1, \cdots, \mu_k\}$. With this notation, we can follow the technique of the proof of Theorem 4 and show that $y \in Q_{x,n_0}$. We get therefore that $F^+ = \bigcup \{Q_{x,n}: x \in Q, n = 1, 2, \cdots\}$, and we've proved the following:

THEOREM 5. *If Λ is a root system, then $F = F(\Lambda, \mathbf{R}_\lambda)$ is a strong limit A-space.*

REMARK. Once again in view of 2.5 in [3] we can conclude that F^+ is essential over Q.

Now let Λ be a root system having finitely many maximal chains and no infinite ascending sequences; note that in this case $V = \Pi$. Let $m(\Lambda)$ denote the set of maximal components of Λ. For each $x \in P$ define $\Psi_{x,n}$ on Π by

$$(y\Psi_{x,n})_\lambda = \begin{cases} y_\lambda & \text{if } \lambda \in m(\Lambda); \\ y_\lambda - n^{\langle x_\lambda \rangle} y_{\lambda^*} & \text{if } \lambda \notin m(\Lambda) \text{ and } \lambda \text{ has no successor in } \Lambda; \\ y_\lambda - n^{\langle x_\lambda \rangle} y_{\lambda-1} & \text{if } \lambda \notin m(\Lambda) \text{ and } \lambda - 1 \text{ is its successor in } \Lambda. \end{cases}$$

(Note: λ^* denotes the maximal entry of Λ exceeding λ.) As before $\Psi_{x,n}$ is a vector space isomorphism on V, and $Q_{x,n} = P\Psi_{x,n} \supseteqq P$, for all $x \in P$ and $n = 1, 2, \cdots$. Once again $(V, P) \cong (V, Q_{x,n})$; and if $y \in V^+$ and $x = |y|_P$ we pick n_0 to be the smallest integer $\geqq 2$ such that $n_0 x_{\mu_j} \geqq 2$, for all maximal components $\mu_1, \mu_2, \cdots, \mu_k$ of x. Then as in the proof of Theorem 4, with the various cases, one shows that for all $\lambda < \mu_j$ $(j = 1, \cdots, k)$ we get $(y\Psi^{-1}_{x,n_0})_\lambda \geqq 1$. (We have to assume here that $x_{\mu_j} \geqq 1$, for each j, but this can be done without loss of generality.) Therefore $V^+ = \bigcup \{Q_{x,n}: x \in P, n = 1, 2, \cdots\}$.

But in this case we can say more: the system $\{Q_{x,n}: x \in P, n = 1, 2, \cdots\}$ is directed. To prove this we show that if $m \leqq n$ are positive integers then $Q_{x,m} \subseteqq Q_{x,n}$; and if $0 \leqq x \leqq y$ (rel. p) then $Q_{x,n} \subseteqq Q_{y,n}$. First suppose $m \leqq n$; let $a \in P$ and consider $a\Psi_{x,m}\Psi^{-1}_{x,n}$: given $\lambda \in \Lambda$

there are four cases to consider.

(1) $\lambda \in m(\Lambda)$; then $(a\Psi_{x,m}\Psi_{x,n}^{-1})_\lambda = a_\lambda \geqq 0$.

(2) $\lambda \notin m(\Lambda)$ and λ has no successor in Λ; then

$$\begin{aligned}(a\Psi_{x,m}\Psi_{x,n}^{-1})_\lambda &= n^{\langle x_\lambda\rangle}(a\Psi_{x,m})_{\lambda^*} + (a\Psi_{x,m})_\lambda \\ &= n^{\langle x_\lambda\rangle}a_{\lambda^*} + a_\lambda - m^{\langle x_\lambda\rangle}a_{\lambda^*} \\ &= a_\lambda + (n^{\langle x_\lambda\rangle} - m^{\langle x_\lambda\rangle})a_{\lambda^*} \geqq 0 .\end{aligned}$$

(3) $\lambda \notin m(\Lambda)$ and λ_{i-1} is the successor of λ_i, where $\lambda_k = \lambda$ and $\lambda_1 \in m(\Lambda)$. Then

$$\begin{aligned}&(a\Psi_{x,m}\Psi_{x,n}^{-1})_\lambda \\ &= n^{\langle x_{\lambda_2}\rangle+\cdots+\langle x_{\lambda_k}\rangle}(a\Psi_{x,m})_{\lambda_1} + \cdots + n^{\langle x_{\lambda_k}\rangle}(a\Psi_{x,m})_{\lambda_{k-1}} + (a\Psi_{x,m})_{\lambda_k} \\ &= n^{\langle x_{\lambda_2}\rangle+\cdots+\langle x_{\lambda_k}\rangle}a_{\lambda_1} + \cdots + n^{\langle x_{\lambda_k}\rangle}(a_{\lambda_{k-1}} - m^{\langle x_{\lambda_{k-1}}\rangle}a_{\lambda_{k-2}}) + a_{\lambda_k} - m^{\langle x_{\lambda_k}\rangle}a_{\lambda_{k-1}} \\ &= n^{\langle x_{\lambda_3}\rangle+\cdots+\langle x_{\lambda_k}\rangle}(n^{\langle x_{\lambda_2}\rangle} - m^{\langle x_{\lambda_2}\rangle})a_{\lambda_1} + \cdots + (n^{\langle x_{\lambda_k}\rangle} - m^{\langle x_{\lambda_k}\rangle})a_{\lambda_{k-1}} + a_{\lambda_k} \geqq 0 .\end{aligned}$$

(4) $\lambda \notin m(\Lambda)$ and λ_{i-1} is the successor of λ_i, $\lambda_k = \lambda$ and λ_1 has no successor. As in (3) one shows that $(a\Psi_{x,m}\Psi_{x,n}^{-1})_\lambda \geqq 0$. This proves that $P\Psi_{x,m}\Psi_{x,n}^{-1} \subseteqq P$, or $Q_{x,m} \subseteqq Q_{x,n}$.

Next, suppose $0 \leqq x \leqq y$ (rel. p) and n is a positive integer. Consider $(a\Psi_{x,n}\Psi_{y,n}^{-1})_\lambda$ with $a \in P$; once again there are four cases.

(1) $\lambda \in m(\Lambda)$; then $(a\Psi_{x,n}\Psi_{y,n}^{-1})_\lambda = a_\lambda \geqq 0$.

(2) $\lambda \notin m(\Lambda)$ and λ has no successor in A; then one can check that $(a\Psi_{x,n}\Psi_{y,n}^{-1})_\lambda = a_\lambda + (n^{\langle y_\lambda\rangle} - n^{\langle x_\lambda\rangle})a_{\lambda^*} \geqq 0$, since $\langle y_\lambda\rangle \geqq \langle x_\lambda\rangle$.

(3) $\lambda \notin m(\Lambda)$ and λ_{i-1} is the successor of λ_i, $\lambda_k = \lambda$ and λ_1 is a maximal component of Λ. One easily verifies that

$$\begin{aligned}(a\Psi_{x,n}\Psi_{y,n}^{-1})_\lambda &= n^{\langle y_{\lambda_3}\rangle+\cdots+\langle y_{\lambda_k}\rangle}(n^{\langle y_{\lambda_2}\rangle} - n^{\langle x_{\lambda_2}\rangle})a_{\lambda_1} + \cdots \\ &+ (n^{\langle y_{\lambda_k}\rangle} - n^{\langle x_{\lambda_k}\rangle})a_{\lambda_{k-1}} + a_{\lambda_k} \geqq 0 .\end{aligned}$$

(4) $\lambda \notin m(\Lambda)$ and λ_{i-1} is the successor of λ_i, where $\lambda_k = \lambda$ but λ_1 has no successor in Λ. One checks as in the other cases that $(a\Psi_{x,n}\Psi_{y,n}^{-1}) \geqq 0$. Thus $P\Psi_{y,n}\Psi_{x,n}^{-1} \subseteqq P$, that is $Q_{x,n} \subseteqq Q_{y,n}$.

So if $Q_{a,m}$ and $Q_{b,n}$ are given, with $a, b \in P$, then we may assume $m \leqq n$ and so $Q_{a,m} \cup Q_{b,n} \subseteqq Q_{a \vee_p b, n}$; this proves that the system of the $Q_{x,n}$ is directed. Hence:

THEOREM 6. *If Λ is a root system having finitely many roots and no infinite ascending sequences, then $V = V(\Lambda, \mathbf{R}_\lambda) = \Pi(\Lambda, \mathbf{R}_\lambda)$ and V is a strong directed limit A-space.*

As an easy corollary of Theorem 4 we prove the following:

PROPOSITION 7. *Let Λ be a root system, and D be an ℓ-subgroup of $V = V(\Lambda, \mathbf{R}_\lambda)$ having the property that*

(a) *D is an ℓ-subgroup of (V, P); $P = \{x \in V\colon x_\lambda \geqq 0,$ all $\lambda \in \Lambda\}$.*

(b) *And if $a, b \in D$, $c \in V$ and $s(c) \subseteqq s(a) \cup s(b)$, this implies that $c \in D$.*

Then $(D, D \cap V^+)$ is a limit A-group.

Proof. Condition (a) guarantees, of course, that $(D, D \cap P)$ is an ℓ-group. Condition (b) says that for each $x \in D \cap P$ and each family $\{n_\lambda\colon \lambda \in m(x)\}$ the isomorphism $\theta_{x,\{n_\lambda\}}$ takes D onto D. Thus

$$(D, D \cap P) \cong (D, D \cap P_{x,\{n_\lambda\}})$$

and

$$D = \bigcup \{D \cap P_{x,\{n_\lambda\}}\} .$$

This completes the proof.

In particular $\Sigma = \Sigma(\Lambda, \mathbf{R}_\lambda) = \{x \in V\colon s(x) \text{ is finite}\}$ satisfies (a) and (b) in Proposition 7, and so $(\Sigma, \Sigma \cap V^+, \Sigma \cap P)$ is a limit A-space.

In closing we point out that it is unknown whether the construction of Theorem 4 or 5 yields a directed system. Even if this should not be the case, some subsystem might be directed and still fill out V^+. A case in point is $\Sigma = \Sigma(\Lambda, \mathbf{R}_\lambda)$; one can show (the proof being long, but in the spirit of that of Theorems 4 and 5) that Σ is a directed limit A-space, by taking an appropriate subsystem of the $\{P_{x,\{n_\lambda\}}\}$.

Suppose we have an 1-group (G, Q); if we knew under what conditions G admitted an archimedean ℓ-order P, of which Q was a very essential extension, we could perhaps make a construction on P along the lines of the construction of Theorem 4. It is doubtful that the construction of Theorem 4 applies to too many ℓ-subgroups of V. The reason being that the archimedean ℓ-cones $P_{x,\{n_\lambda\}}$ are of a very special type, namely they have a basis.

A question which has some interest on its own: what groups G admit archimedean lattice orders? They must of course be abelian and torsion free, and if G is divisible then G does certainly admit such a cone. There is no guarantee however, that an archimedean ℓ-cone on the divisible closure G^* of G will even induce an ℓ-cone on G.

In view of Corollary 3.1 one can ask of course: what ℓ-groups are (strong) sequential (or linear) limit A-groups. Let us give one example to show that 3.1 does not give all the strong sequential limit A-spaces. This is also an example of a strong sequential limit A-space with infinite descending chains of ℓ-ideals; one can give examples of strong sequential limit A-spaces which have infinite ascending chains of ℓ-ideals. It is even possible to find strong sequential limit A-spaces with descending chains (or ascending chains) of arbitrary length.

Let $G = \mathbf{R} \boxplus \mathbf{R} \boxplus \mathbf{R} \boxplus \cdots = \{$all finitely nonzero real sequences$\}$. Let Q be the lexicographic total order by ordering from the left; let $P = G^+$. Let θ_n be a map defined by

$$x\theta_n = (x_1, x_2 - nx_1, \cdots, x_n - nx_{n-1}, x_{n+1}, x_{n+2}, \cdots) .$$

In the notation of the proof of Theorem 5 $\theta_n \equiv \theta_{x_n,n}$, where $x_n = (1, 1, \cdots, \dot{1}, 0, 0, \cdots)$; (the last 1 is the n-th position.) We therefore know that θ_n is an isomorphism of G onto itself, and $P_n = P\theta_n \supseteqq P$. It can be shown further that $P_n \subseteqq P_{n+1}$, for each $n = 1, 2, \cdots$, and finally $Q = \bigcup_{n=1}^{\infty} P_n$. Thus (G, Q, P) is a strong sequential limit A-space, for Q is very essential over P.

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