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ON A SIX DIMENSIONAL PROJECTIVE REPRESENTATION OF $PSU_4(3)$

J. H. LINDSEY, II

In the course of an investigation of six-dimensional complex linear groups, it was discovered that a central extension of Z_6 by $PSU_4(3)$ has a representation of degree six. In fact, this representation has as its image the unimodular subgroup $X(G)$ of index 2 of the following 6-dimensional matrix group: $\langle$all 6 by 6 permutation matrices; all unimodular diagonal matrices of order 3; $I_6 - Q/3$ where Q has all its entries equal to one$\rangle$. This matrix group leaves the following lattice invariant: $\{(a_1, \cdots, a_6) \mid a_i \in Z(\omega)$ where throughout this paper ω is a primitive third root of unity; $a_i - a_j \in \sqrt{-3}\, Z(\omega)$ for all i, j; $\sum_{i=1}^{6} a_i \in 3Z(\omega)\}$. The generators of the matrix group are similar to the following generators for an 8-dimensional complex linear group with Jordan-Holder constituents Z_2, the nontrivial simple constituent of $0_8(2)$, Z_2: $\langle$all 8 by 8 permutation matrices, all unimodular diagonal matrices of order 2, $I_8 - P/4$ where P has all entries equal to 1$\rangle$.

The projective representation of $PSU_4(3)$ can be used to construct a 12-dimensional representation $Y(H)$, a central extension of Z_6 by the Suzuki group, which leads to the known 24-dimensional projective representation of the Conway group. In fact, H has a subgroup K isomorphic to a central extension of $(Z_6 \times Z_3)$ by $PSU_4(3)$. Also, $Y \mid H$ has two six-dimensional constituents coming from the above matrix group where the constituents are related by an outer automorphism of $PSU_4(3)$ which does not lift to the central extension of Z_6 by $PSU_4(3)$ with the six-dimensional representation. We obtain two commuting automorphisms, α and β respectively, of G from $I_6 - Q/3$ and complex conjugation. For $PSU_4(3)$, the outer automorphism group is dihedral of order eight with its center corresponding to complex conjugation of $X(G)$. The entire automorphism group lifts to K. We may take the center of K to be $\langle a, b, c\rangle$ with a and b of order 3 and c of order 2, with $G \cong K/b$, and with $\alpha(a) = a$, $\alpha(b) = b^{-1}$, $\beta(a) = a^{-1}$, $\beta(b) = b^{-1}$. We can also find an automorphism γ of K with $\gamma(a) = b$ and $\gamma(b) = a$. We give the character table of K giving only one representative of each family of algebraically conjugate characters and classes. Irrational characters and classes are underlined. Only one class in each coset of $Z(K)$ is represented by the character tables. The characters in the table $\widetilde{U_4(3)}$ give the characters with $Z(K)$ in the kernel. The succeeding five character tables in order give the following linear characters, respectively, on $Z(K)$: $\theta(a) = \theta(b) = 1, \theta(c) = -1$; $\theta(a) = \omega$, $\theta(b) = \theta(c) = 1$; $\theta(a) = \omega^{-1}$, $\theta(b) = 1$, $\theta(c) = -1$; $\theta(a) =$

$\theta(b) = \omega$, $\theta(c) = 1$; $\theta(a) = \theta(b) = \omega$, $\theta(c) = -1$. **The characters with other actions are obtained by applying elements of the outer automorphism group. The automorphism α transposes π_7 with π_7^{-1}; and N_1 with N_1^{-1} in the character tables. The automorphism β transposes N_1 with N_1^{-1}; and N_2 with N_2^{-1}. The automorphism γ transposes T_1 with T_2; JT_1 with JT_2; N_1 with N_2; N_1^{-1} with N_2^{-1}; and possibly π_7 with π_7^{-1}. As $SU_4(3)/\Omega(ZSU_4(3))$ has the centralizer of some central involution isomorphic to the centralizer of some central involution J in G, presumably $SU_4(3)/\Omega_1(ZSU_4(3)) \cong G/0_3(Z(G))$.**

The first four character tables give the characters of the central extension of $\langle d\rangle = Z_6$ by $LF(3, 4)$ with a six dimensional, complex representation. Respectively, they give the following linear characters on $\langle a\rangle$: $\theta(a) = 1$, $\theta(a) = \omega$, $\theta(a) = -1$, $\theta(a) = -\omega$. The characters with $\theta(a) = \omega^{-1}$ or $\theta(a) = -\omega^{-1}$ come from complex conjugation of the second and fourth table respectively.

We let $\widetilde{U_4(3)} = PSU_4(3)$ and let S_p be a p-Sylow subgroup of whatever group is in question. The term "Blichfeldt" refers to the theorem in [1] that no primitive complex linear group contains an element with some eigenvalue within 60 degrees of all the other eigenvalues of the element. Where clear, we use χ_n to refer to the previously discussed character of G of degree n. Finally, $a(X, Y, Z)$ is the coefficient of the conjugacy class containing Z in the product of the classes containing X and Y.

This paper fills a gap in [9] concerning groups G with a faithful unimodular representation X with character χ of degree six and $\bar{G}$ simple of order $2^7 3^6 35$ where $Z = Z(G)$ and $\bar{G} = G/Z$. We also know by [9, § 8], that $C(S_5) = S_5 Z$, $C(S_7) = S_7 Z$, $4/t_5 = [N(S_5): C(S_5)] = 4$, and $6/t_7 = [N(S_7): C(S_7)] = 3$. Also, the principal 7-block $B_0(7)$ has degree equation $1 + 729 = 640 + 90$. Finally, by [9, § 8], $\chi(G) \subseteq Q(\omega)$, $3 \mid |Z|$, and we may take $X(S_3)$ to be

$$\left\langle \operatorname{diag}(1, 1, \omega, 1, 1, \omega^{-1}), \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \oplus I_3, I_3 \oplus \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \right\rangle .$$

I learned from the referee that this representation was discovered earlier by Mitchell, [10]. Mitchell also showed that this linear group and the first orthogonal group on six indices with modulus three have isomorphic nonsolvable Jordan-Holder constituents. Hammill, [6] and Todd, [12] also worked on this linear group with the latter constructing the character table of $\widetilde{U_4(3)}$.

2. The character table. By the above, $|Z| = 6$ since χ_{20}, the character of the skew-symmetric tensors of $X \otimes X \otimes X$, does not have a constituent of degree 90 or 640. There is a character $\chi_{640'}$ completing the 2-block of χ_{640} in G/Z_3. Since $\chi_{640'}$ is the 7-exceptional character in the block $B_1(7)$ with characters whose kernel is Z_3, and G/Z_3 does not have a character of degree 6, χ_{20} is irreducible. Degrees divisible exactly by 2 or 4 and $\equiv \pm 1 \pmod 7$ and $\equiv 0$ or $\pm 1 \pmod 5$ are 6, 36, 90, 20, and 540. The possibilities are $660 - 36 = 624$, $660 + 90 = 750$, $660 + 20 = 680$, and $660 - 540 = \underline{120}$. The degree equation is $20 + 640 = 120 + 540$. By [5], 3-7 block separation in G/Z_3, these characters are in the same 3-block of G/Z_3. Let $T \in Z(S_3)$, $\chi(T) = -3$. Then (mod 3) $|G|\chi_{540}(T)/(540)|C(T)| \equiv |G|\chi_{20}(T)/(20)|C(T)| \equiv (-7)|G|/(20)|C(T)| =$ some 3-unit, so $\chi_{540}(T)$ is divisible exactly by 27 and $|C(T)|/|Z| > 3^6$.

A 7-block whose characters have kernel Z_2 contains χ_{15} from the skew-symmetric tensors (irreducible since G/Z_2 has no representation of degree 6) and $\chi_{729'}$ completing a 3-block of defect 1. There is another degree divisible exactly by 3 which must be 384, 24, 15, 60, 120, 480, or 960. The degree 24 is impossible since

$$\chi_{24}(\pi_7)\overline{\chi_{24}(\pi_7)} = 2 ,$$

but $\chi_{24}\bar{\chi}_{24}$ cannot fit $\chi_0 + \chi_{729}$ in $B_0(7)$ inside. The possibilities are $729 + 15 - 384 = \underline{360}$, $744 + 15 = 759$, $744 + 60 = 804$, $744 + 120 = \underline{864}$, $744 + 480 = 1224$, and $744 + 960 = 1704$. Since G/Z_2 has no representation of degree 6, χ_{21} corresponding to the symmetric tensors of $X \otimes X$, is irreducible. In the case of 864 there is a 5-block with degree equation $864 + 864 + 729 = 21 + \cdots$ and the fifth degree is too large. Therefore, the 7-block has degree equation $15 + 729 = 384 + 360$. Suppose that G has an element J with $X(J)$ having eigenvalues $i, i, i, -i, -i, -i$. Then $\chi_{15}(J) = (0^2 - (-6))/2 = 3$. Also χ_{384} has 2-defect 0 and $\chi_{384}(J) = 0$. Since $t_7 = 2$, $a_{J,J,\pi_7} = 0$ in G/Z and G/Z_2, so

$$3^2/15 + \chi_{729'}(J)^2/729 - \chi_{360}(J)^2/360 = 0$$

and $3 + \chi_{729'}(J) = \chi_{360}(J)$. Then $9 | \chi_{729'}(J)$, $3 | \chi_{360}(J)$, $27 | \chi_{729'}(J)$, and $4 | \chi_{360}(J)$; so $\chi_{729'}(J) \equiv -27 \pmod{108}$. Then $\chi_{729'}(J) = -27$, otherwise $|\chi_{729'}(J)| > 80$ and the sum is negative. Then in $B_0(7)$,

$$\chi_0(J) = 1, \chi_{729}(J) = -27, \chi_{640}(J) = 0, \chi_{90}(J) = 1 - 27 = -26 ,$$

and $1^2/1 + 27^2/729 - 26^2/90 \neq 0$, a contradiction. Therefore, J cannot exist. We have a character $\chi_{384'}$ faithful on Z completing a 2-block containing χ_{384}. Then a 5-block faithful on Z contains characters of degree 6 and 384. Now $1 = (\chi_{15}, \chi_6\chi_6) = (\bar{\chi}_6\chi_{15}, \chi_6)$ so $\bar{\chi}_6\chi_{15}$ contains χ_6 as a constituent. Also $\bar{\chi}_6\chi_{15} - \chi_6$ has an irreducible constituent of degree $\equiv -1 \pmod 5$ and divisible by 6: 84 or 24. By the previous

$\chi_{24}\bar{\chi}_{24}(\pi_7)$ argument, 24 is impossible and the 5-block contains the degree 6, 384, and 84. We have another degree divisible exactly by 2: 6, 486, 126, or 1134. The possibilities are

$$384 + 84 - 6 - 6 = 456, 486 - 462 = 24$$

already shown to be an impossible degree,

$$462 - 126 = \underline{336}, \text{ and } 462 + 1134 = 1596 .$$

The degree equation is $6 + 126 + 336 = 84 + 384$. As with 84, $\bar{\chi}_6\chi_{21} - \chi_6$ is a character. Since $(\bar{\chi}_6\chi_{21}, \chi_6) = 1$, $\bar{\chi}_6\chi_{21} - \chi_6$ has no constituent of degree 6. Therefore, from the 5-block, all its constituents have degrees divisible by 30, and must be 120, 90, or 60. The degree 90 would imply the impossible degree 30. If 60, then a 7-block has degree equation $6 + 384 = 60 + 330$, impossible. Therefore, it is irreducible, and the 7-block is $6 + 384 = 120 + 270$. If J gives an involution in G/Z, then possibly replacing J by $-J$, $X(J)$ has eigenvalues $1, 1, 1, 1, -1, -1$ as $\chi(G) \subseteqq Q(\omega)$ and eigenvalues $i, i, i, -i, -i, -i$ are impossible. In $\bar{G} = G/Z$, $\langle\pi_5\rangle$ is self-centralizing and $a_{J,J,\pi_5} = 0$ or 5. Now $|C_G(J)| = |C_{\bar{G}}(\bar{J})|\,|Z|$ and $a_{J,J,\pi_5} = 0$ or 5 in $G/Z, G/Z_2, G/Z_3$, and G. Then looking successively at $G, G/Z_2, G/Z_3$, and G we see that $\sum \chi_i(J)^2\chi_i(\pi_5)/\chi_i(1)$ over each 5-block is 0 or $5\,|C_{\bar{G}}(\bar{J})|^2/|\bar{G}|$. By 2-block orthogonality on (I, J), $\chi_{384'}(J) = 0$. Also $\chi_6(J) = 2$, $\chi_{84}(J) = 2(4 - 6)/2 - 2 = -4$. Then $\chi_{126}(J) + \chi_{336}(J) = -4 - 2 = -6$. Let $a = \chi_{336}(J)$. We may find some J in $Z(S_2)$ with $2^7 | \sum = 4/6 + a^2/336 + (6 + a)^2/126 - 16/84$. Then $4 | a$ and we may let $a = 4b$. Multiply the sum by 63:

$$2^7 | 42 + 3b^2 + 8b^2 + 24b + 18 - 12 = 11b^2 + 24b + 48 .$$

Then $4 | b$ and if $c = b/4$, then $8 | 11c^2 + 6c + 3$. Then c is odd. Since $|6 + 16c| < 126$, we have $c = \pm 1, \pm 3, \pm 5$, or ± 7. Also $11c^2 \equiv 11 \equiv 3 \pmod 8$, so $6c \equiv 2 \pmod 8$ and $c \equiv 3 \pmod 4$. The possibilities are $11 - 6 + 3 = 8$, $99 + 18 + 3 = 120$ impossible by the factor 5 since $5 \nmid |C_{\bar{G}}(\bar{J})|$, $275 - 30 + 3 = 248$ divisible by 31 and impossible, $539 + 42 + 3 = 584$ divisible by 73. Therefore,

$$c = -1, 5\,|C_{\bar{G}}(\bar{J})|^2/|\bar{G}| = (8)(4)(4)/63 ,$$

and $|C_{\bar{G}}(\bar{J})| = 2^7 9$. Then J inverts a 5-element and there is only one such class of such J mod Z. If another involution J_1 does not invert a 5-element, then $2^7 | 0 = \sum \chi_i(J_1)^2\chi_i(\pi_5)/\chi_i(1)$, and the above leads to a contradiction. Therefore, G/Z has a unique class of involutions. Suppose that there is an element F with $X(F)$ having eigenvalues $1, 1, 1, 1, i, -i$. Then

$$\chi_{15}(F) = (4^2 - 2)/2 = 7, \chi_{21}(F) = (4^2 + 2)/2 = 9, \chi_{84}(F) = 28 - 4 = 24 ,$$

and $\chi_{120}(F) = 36 - 4 = 32$. However, $32^2 + 24^2 > 2^7 9 = |C_{\bar{G}}(\bar{F}^2)| \geqq |C_{\bar{G}}(\bar{F})$, a contradiction.

3. The centralizer of an involution. Let J be an involution with $X(J) = I_4 \oplus -I_2$. Then $X|C(J) = U \oplus V$ and $\chi|C(J) = \theta + \phi$ where θ corresponds to U and $\theta(J) = 4$. If α is a field automorphism fixing ω, then $\theta^\alpha + \phi^\alpha = \theta + \phi$, $\theta^\alpha = \theta$, and $\phi^\alpha = \phi$ since θ^α and θ are the sums of irreducible characters of $\chi|C(J)$ with J in the kernel. Therefore, $\theta(C(J))$ and $\phi(C(J))$ are contained in $Q(\omega)$. Let K be the subgroup of $C(J)$ of elements k such that $(\det V(k))^{2^m} = 1$ for some m. Then $|K| = 2^7 9 |Z|/3 = 2^8 9$. Suppose $x \in \ker U$. Then x is a 2-element, otherwise, some power y of v has order 3 with $\theta(y) = 4$, $\phi(y) = -1$, and Jy contradicts Blichfeldt. If x has order 4, then $X(x)$ has eigenvalues $1, 1, 1, 1, i, -i$; already shown impossible. Therefore, $\ker U = \langle J \rangle$ and $|U(K)| = 2^7 9$.

Suppose U has 2-dimensional spaces S and T as spaces of imprimitivity or invariant spaces. Then H of index 1 or 2 in $U(K)$ has $\theta|H = \mu + \nu$ corresponding to the 2-dimensional spaces S and T. Let L be a 2-Sylow subgroup of $U(K)$. Unless $[U(K): H] = 2$ and $\mu|L \cap H$ and $\nu|L \cap H$ are irreducible, H has an abelian subgroup A of order 2^5, impossible (if A has an element of order 8, the linear characters of $\theta|A$ are algebraic conjugates and faithful, so $|A| = 8$. Therefore, irrational characters of $\theta|A$ occur in pairs and have image of order 4. Rational characters have image of order 2. Therefore, $|A| \leqq 16$). Therefore, μ and ν are irreducible and a 2-element $x \in C(J)$ transposes S and T. If $\mu \nsubseteq Q(\omega)$, then μ and ν are algebraic conjugates, μ is faithful on H, and $H \cap L$ has an abelian subgroup of index 2 and order at least 2^5, impossible. Therefore, $\mu, \nu \subseteqq Q(\omega)$ and $\mu|L \cap H$, $\nu|L \cap H$ are rational. Then

$$|\mu(L \cap H)|, |\nu(L \cap H| \leqq [2/(2-1)] + [2/2] + \cdots = 3 .$$

Since $|L \cap H| = 2^6$, $L \cap H = \ker \nu \times \ker \mu$. In 2 by 2 matrix blocks let $U(x) = \begin{pmatrix} 0 & W \\ Y & 0 \end{pmatrix}$. Then $U(x^2) = \begin{pmatrix} WY & 0 \\ 0 & YW \end{pmatrix}$ is contained in a conjugate in H of $\operatorname{Ker} \nu \times \operatorname{Ker} \mu = L \cap H$, a 2-Sylow subgroup of H. Therefore, $\begin{pmatrix} WY & 0 \\ 0 & I_2 \end{pmatrix} = U(y)$ is contained in H. Now $U(y^{-1}x) = \begin{pmatrix} 0 & Y^{-1} \\ Y & 0 \end{pmatrix}$. Changing coordinates by conjugation with $\begin{pmatrix} I_1 & 0 \\ 0 & Y \end{pmatrix}$ and replacing x by $y^{-1}x$, we may take $U(x) = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}$. Since $\mu|L \cap H$ is irreducible and $L \cap H = \operatorname{Ker} \nu \times \operatorname{Ker} \mu$, there is a 2-element y with $U(y) = -I_2 \oplus I_2$. Then $U((xy)^2) = -I_4$, so $V((xy)^2) \neq -I_2$. However, ϕ is rational and $1 = \det U(xy) = \det V(xy)$. Therefore, $\phi(xy) \pm 2$. If $\operatorname{Ker} \nu$ has an element T of order 3, then $\mu(T) = -1$, $\nu(T) = 2$, and $X(J(xy)^{-1}TxyT^{-1})$ has eigenvalues $\omega, \bar{\omega}, \omega, \bar{\omega}, -1, -1$; contrary to

Blichfeldt. Therefore, the representation corresponding to μ has image of order 72. Then $\mu \subseteqq Q(\omega)$ implies that there is a 3-element g with $\mu(g) = 2\omega$. Then $\nu(g) = 1 + \omega$, otherwise, $\nu(g) = 2\bar{\omega}$ and $X(J(xy)^{-1}gxyg^{-1})$ contradicts Blichfeldt. Now $\phi(g) = \omega + \bar{\omega}$, otherwise, $\phi(g) = 2$ and $X(J(xy)^{-1}gxyg)$ has eigenvalues $\omega, \bar{\omega}, \omega, \bar{\omega}, -1, -1$ and contradicts Blichfeldt. There exists a 2-element z with $\mu(z) = i + (-i)$, $\mu(z^2) = -2$, and $\nu(z) = 2$. Then $\phi(z) = i + (-i)$ and $\phi(z^2) = -2$, otherwise, $X(z)$ or $X(zJ)$ has eigenvalues $i, -i, 1, 1, 1, 1$. Then $\theta(z^{-1}g^{-1}zg) = 4$ implies that $z^{-1}g^{-1}zg \in \langle J \rangle$. As $Jz^{-1}g^{-1}z$ has order 6, it cannot equal g^{-1}, and $z^{-1}g^{-1}zg$ is the identity in G. Then $V(z)$ with eigenvalues $i, -i$ commutes with $V(g)$ with eigenvalues $\omega, \bar{\omega}$ contrary to $\phi \subseteqq Q(\omega)$.

Now suppose that U is monomial, but not imprimitive on 2-dimensional subspaces. Then there exists a 3-element g corresponding to a permutation of order 3. As before, $U(K)$ has no abelian subgroup of order 32, so the image of $U(K)$ under ρ, the natural permutation representation on four letters has order eight and must be S_4. Then $U(K)$ has an element T of order 3 in $\operatorname{Ker} \rho$ and conjugates of some commutator of T with a transposition show that $U(K)$ contains all diagonal matrices of order 3 and determinant 1. Then $27 \mid |U(K)|$, a contradiction.

Now by Blichfeldt's classification of groups of degree 4, $U(K)$ modulo $Z(U(K))$ has a subgroup N of the tensor product of 2-dimensional representations W of $M = GL(2, 3)$. Also, N has index 2 or 1 in $U(K)$. Now $Z(U(K)) \subseteqq \langle -I_4 \rangle$ since det $U(k)$ for $k \in K$ is a 2^m-th root of 1 and $\theta \subseteqq Q(\omega)$. Let $U | N = A \otimes B$. Now $W(M) \otimes I_2$ does not appear as a subgroup modulo scalars of $U(K)$ since eigenvalues $\gamma, \gamma, \gamma^{-1}, \gamma^{-1}$ with $\gamma^2 = i$ or $i, i, 1, 1$ contradict 2-rationality of θ. Therefore, the image under A of $\operatorname{Ker} B$ in $M/Z(M)$ has order at most 12. The image of N under B in $M/Z(M)$ has order at most 24. This gives $|N| \leqq |Z(N)|(12)(24) \leqq 2^6 9$. We must have equality. Then an element x takes $A \otimes B$ to $B \otimes A$. Therefore, $N \supset W(SL(2, 3)) \otimes I_2$, $I_2 \otimes W(SL(2, 3))$ after elements of $W(SL(2, 3))$ are changed by scalar multiplication. Also, the quaternions $Q = SL(2, 3)'$ can have $W(Q)$ taken as the matrices in [1, § 57]. Since det U is a 2^m-th root of 1 we may also use the matrix in § 57 for a 3-element S in $W(SL(2, 3))$. Let g be a 3-element with $U(g) = S \otimes I_2$. Then $V(g)$ has eigenvalues $\omega, \bar{\omega}$; otherwise $\phi(g) = 2$ and gJ has eigenvalues $\omega, \bar{\omega}, \omega, \bar{\omega}, -1, -1$; contrary to Blichfeldt. If h is a 3-element with $U(h) = I_2 \otimes S$, then, similarly, $\phi(h) = -1$. Also $U(g)$ and $U(h)$ commute, $V(g)$ and $V(h)$ commute modulo $\langle J \rangle$, and $V(g)$ and $V(h)$ commute. Both may be taken as diagonal. There exists $E \in W(M)$ with $E^{-1}SE = S^{-1}$. Let $V(g) = \omega \oplus \bar{\omega}$. If necessary, we may replace h with h^{-1} and change coordinates of U by conjugation with $I_2 \otimes E$ to take $V(h) = \omega \oplus \bar{\omega}$. If $x \in C(J)$ with $U(x) \in W(Q) \otimes I_2$ and $U(x)$ of order 4, then $U(x)$ has eigenvalues $i, i, -i, -i$ and $V(x)$ cannot have eigenvalues $i, -i$. Possibly replacing x by Jx, we may take $\phi(x) = 2$. Because of equality in $|N| \leqq 2^6 9$, $U(K)$ contains a tensor product of elements in

$$W(GL(2, 3)) - W(SL(2, 3)) .$$

By [1, § 57], we may take this element $U(y)$ as $\alpha((\gamma \oplus \gamma^{-1}) \otimes (\gamma \oplus \gamma^{-1}))$ where $\gamma^2 = i$. Then $U(y)$ has eigenvalues $\alpha i, \alpha, \alpha, -\alpha i$. By 2-rationality, $\alpha = \pm 1$ and $U(y)$ is determined. The action of $U(y)$ on the group of order 3: $W(SL(2, 3)) \otimes W(SL(2, 3))/\langle W(Q) \otimes W(Q), S \otimes S^{-1}\rangle$ is non-trivial. Therefore,

$$V(y)^{-1} V(g) V(y) = V(y)^{-1} V(h) V(y) = V(g)^{-1}$$

(since $-V(g)^{-1}$ is not a 3-element). Since $1 = \det U(y) = \det V(y)$, we may choose coordinates so that $V(y) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. The element x flipping $W(SL(2, 3)) \otimes I_2$ to $I_2 \otimes W(SL(2, 3))$ is determined modulo $W(M) \otimes W(M)/\langle U(y), W(SL(2, 3)) \otimes W(SL(2, 3))\rangle$ and modulo scalars to be $1 \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus 1$. We may take x as $\alpha\left(1 \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus 1\right)$ or $\alpha\left(1 \oplus \begin{pmatrix} 0 & 1 \\ i & 0 \end{pmatrix} \oplus i\right)$. As θ is rational on 2-elements, 2α or $\alpha(1 + i)$ is rational. Therefore, $\alpha = \pm 1$, and we are in the first case, so $U(x)$ is determined. Then $-1 = \det U(x) = \det V(x)$ and $V(x)$ has eigenvalues $1, -1$. Since the action of $U(x)$ on $W(SL(2, 3)) \otimes W(SL(2, 3))/\langle W(Q) \otimes W(Q), S \otimes S^{-1}\rangle$ is trivial, $V(x)$ and $V(g)$ commute. Possibly replacing x by xJ we may take $V(x) = 1 \oplus -1$. Therefore, $C(J)$ and $\widetilde{X(C(J))}$ are completely determined. In fact $C(J)/Z$ is isomorphic to $C(I_2 \oplus -I_2)$ in $\widetilde{U_4(3)}$: $(W(SL(2, 3)) \otimes I_2) \oplus \cdots \to SL(2, 3) \oplus I_2$; $(I_2 \otimes W(SL(2, 3))) \oplus \cdots \to I_2 \oplus SL(2, 3)$; $\left(\begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}\right) \oplus \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \to \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}$ here both elements have the same action on the central product of $SL(2, 3)$ with itself, the square of the left element is $\left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) \oplus -I_2 \approx \left(\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \otimes \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\right) \oplus I_2$. The square of the right element is $-i\left(\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \oplus \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\right)$; $1 \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus 1 \oplus 1 \oplus -1 \to \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}$. Here both elements have order 2. Both elements have identical action on the central product of $SL(2, 3)$ with itself. The commutator of $X(x)$ with $X(y)$ is $I_4 \oplus -I_2$. The corresponding commutator in $\widetilde{U_4(3)}$ is $i \oplus i \oplus -i \oplus -i$. This shows that $C(J)/Z$ is isomorphic to the centralizer of an involution in $PSU_4(3)$. By Phan's characterization of $PSU_4(3)$, $PSU_4(3) \cong G/Z$.

4. The normalizer of $Z(S_3)$. Earlier, for

$$T = \operatorname{diag}(\omega, \omega, \omega, \bar{\omega}, \bar{\omega}, \bar{\omega}) ,$$

we showed that $|C(T)/Z| > 3^6$ and T is centralized by an involution in $\bar{G} = G/Z$. We may take T in $C(J)$ and $\bar{J}$ in the center of a Sylow-2-subgroup of $C(T)/Z$. As $\chi(T) = -3$, $U(T) = S^{\pm 1} \otimes I_2$ or $I_2 \otimes S^{\pm 1}$, say the former. Then

$$U(C(TJ)) = \langle U(T), U(Z), I_2 \otimes SL(2, 3)\rangle, |C(T)| = 3^6 8 |Z| ,$$

and T is conjugate to T^{-1}. As the constituents of $X|C(T)$ are not algebraically conjugate, $X(C(T)) = \langle -I_6\rangle \times H$ where $H =$ the subgroup of $X(C(T))$ whose action on the homogeneous ω-space of $X(T)$ has determinant $=$ to a third root of 1. A Sylow-2-subgroup of H is Q, the quaternions. Let -1 have order 2 in $Z(G)$. Now $\langle \pm J\rangle = Z(Q)$ is represented faithfully in the ω or the $\bar{\omega}$ space of H, say the ω space with $\zeta =$ the corresponding constituent of $X|H$. If ζ is monomial, then $\pm J$, being a square in H, is diagonal and conjugating with $\begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{pmatrix} \oplus I_3$ (the first component is taken to correspond to ζ), we have $C(T)/Z$ contains an elementary abelian subgroup of order 4, a contradiction. Therefore, the representation corresponding to ζ is the Hessian group in [1, § 79], except that $\omega \oplus 1 \oplus 1$ has been changed by a scalar. As an element inverting T flips the constituents of $X|C(T)$, taking $H \supset S_3$ with $X(S_3)$ in the normal form given at the start of this chapter, $X(C(T)) \subset \{M_1 \oplus M_2 | M_i$ appears in the Hessian group in [1], except that diag $(1, 1, \omega)$ replaces $\omega^{-1/3}$ diag $(1, 1, \omega)\}$. As the normal subgroup K of order 27 of the Hessian group appears independently in each component, we may examine the components of $X(H)$ modulo K. Let i be the image of $\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \bar{\omega} \\ 1 & \bar{\omega} & \omega \end{pmatrix} / (\omega - \bar{\omega})$ in this homomorphism. Since Q is represented faithfully in the top component and some element in $X(C(T))$ flips the components, Q is represented faithfully in the bottom component. By changing coordinates by conjugating with a power of diag $(1, 1, 1, \omega, 1, 1,)$, we may assume that $X(C(T))$ contains $i \oplus \pm \bar{i}$ (i stands for a coset of 3 by 3 matrices and $\bar{i}$ is obtained by complex conjugation of the entries) where

$$j = (\text{diag } 1, 1, \bar{\omega}))i(\text{diag } (1, 1, \omega)), \ -1 = i^2 ,$$

and $k = ij$. If $X(C(T))$ contains $i \oplus -\bar{i}$, then, conjugating with $T_1 =$ diag $(1, 1, \omega, 1, 1, \bar{\omega}) \in S_3$, we have $j \oplus -\bar{j}$ and $k \oplus -\bar{k} \in X(C(T))$ and

$$(i \oplus -\bar{i})(j \oplus -\bar{j})(k \oplus -k) = -1 \oplus 1 \in X(C(T)) ,$$

contrary to $8 \| |H|$. Since diag $(1, 1, \omega)$, i, and K generate the Hessian group, $H = \langle K \oplus I_3, I_3 \oplus K, M \oplus \bar{M}$ where M is any matrix in the Hessian group changed as shown by scalars$\rangle$.

$X(N(\langle T\rangle))$ is obtained from $X(C(T))$ by addition of a 2-element

$X(x) = \begin{pmatrix} 0 & E \\ F & 0 \end{pmatrix}$ where E and F are 3 by 3 matrices normalizing the Hessian group, and, hence, in the Hessian group modulo scalar multiplication. By multiplication with an element in $C(T)$ we may take E as scalar and, changing coordinates by conjugation with a direct sum of 3 by 3 scalar matrices, we may take $E = I_3$. Again, we are only interested in F modulo K. If F is scalar, then by determinant, $F = -I_3$ and $X(x^2) = -I_6$, impossible. The other possibilities are $F =$ some scalar times -1, $\pm i$, $\pm j$, or $\pm k$ in the notation of the previous paragraph. If not -1, then replace x by $T_1^a x T_1^a$ to take $F =$ some scalar times $\pm i$. The scalar is $-I_3$ by determinant $= 1$. Then

$$(-I_6)X(x^2) = \begin{pmatrix} \pm i & 0 \\ 0 & \pm i \end{pmatrix}.$$

Possibly replacing this by its third power, we have $\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & \bar{i} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, contrary to $8 \| |H|$. Therefore, $F =$ some scalar times -1 and the scalar is -1 by determinant $= 1$. This completely determines $X(N(\langle T \rangle))$.

5. **The correlation between $X(C(J))$ and $X(N(\langle T \rangle))$ for $T \in C(J)$.** Take $X(T) = (S \otimes I_2) \oplus \omega \oplus \omega^{-1}$ in our normal form for $X(C(J))$. Let $GL(2, 3)$ and $SL(2, 3)$ be the 2-dimensional matrix groups in [1, § 57] and ϕ be an isomorphism from $SL(2, 3)$ to $SL(2, 3)/0_2(SL(2, 3)) \cong Z_3$ with $\phi(S) = 1$ and $0_2(SL(2, 3))$ isomorphic to the quaternions. Then $X(N(\langle JT \rangle)) = \langle X(JT) = (S \otimes I_2) \oplus -\omega \oplus -\omega^{-1}; (I_2 \otimes u) \oplus (\omega \oplus \omega^{-1})^{\phi(u)}$ for $u \in SL(2, 3)$; $Y = \left(y \otimes \begin{pmatrix} \gamma & 0 \\ 0 & \gamma^{-1} \end{pmatrix}\right) \oplus \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ for some

$$y \in \begin{pmatrix} \gamma & 0 \\ 0 & \gamma^{-1} \end{pmatrix} 0_2(SL(2, 3))$$

with $y^{-1}Sy = S^{-1}$; $-\omega I_6\rangle$. We get a subgroup of order at least $2^7 3^6$ of $X(G)$ generated by our normal form for $N(\langle T \rangle)$ and the image under conjugation by a matrix R of our normal form for $X(C(J))$ where R conjugates $X(JT)$ and $X(N(\langle JT \rangle))$, in our normal form for $X(C(J))$, to $X(JT)$ and $X(N(\langle JT \rangle))$, respectively, in our normal form for $X(N(\langle T \rangle))$. Therefore, R is determined modulo multiplication on the left by a matrix P fixing $X(JT)$ and $X(N(\langle JT \rangle))$ in the normal form for $X(C(J))$. As we are only interested in the image of $X(C(J))$ under conjugation by R. We are only interested in P modulo multiplication on the left by a matrix fixing $X(JT)$, $X(N(\langle JT \rangle))$, and $X(C(J))$. As $0_2(0^2(X(N(\langle JT \rangle)))) = \langle (I_2 \otimes u) \oplus I_2$ such that $u \in 0_2(SL(2, 3))\rangle$, by [7, Satz 3] and [1], $P = (A \otimes B) \oplus C$ where $B \in GL(2, 3)$, $A \in C_{GL(2, \boldsymbol{C})}(S)$, and $C \in GL(2, \boldsymbol{C})$ where $\boldsymbol{C}$ is the complex number field. If $B \notin SL(2, 3)$, then P conjugates $(S \otimes S^{-1}) \oplus I_2$ to $(S \otimes Sv) \oplus I_2$ for some

$$v \in 0_2(SL(2, 3)) ,$$

a contradiction, since the former, but not the latter is in $X(N(\langle JT \rangle))$. Therefore, multiplying P by an element in $X(N(\langle JT \rangle))$, we may take $B = I_2$. Also,

$$\begin{aligned}(A^{-1}yA)^{-1}S(A^{-1}yA) &= (A^{-1}yA)^{-1}(A^{-1}SA)(A^{-1}yA) \\ &= A^{-1}y^{-1}SyA = A^{-1}S^{-1}A = S^{-1} .\end{aligned}$$

Therefore, $A^{-1}yA \in N_{GL(2,3)}(\langle S \rangle) - C_{GL(2,3)}(\langle S \rangle)$ where

$$N_{GL(2,3)}(\langle S \rangle) = \langle y, S, ZGL(2, 3) \rangle .$$

Multiplying P on the left by a power of $X(T)$, we may take $A^{-1}yA$ in $\langle y, ZGL(2, 3) \rangle$ of order 4 and $A^{-1}yA \in yZGL(2, 3) = y\langle -I_2 \rangle$. Let $Q \in GL(6, \boldsymbol{C})$ be the matrix which acts as I_3 on the space where $X(T)$ acts as ωI_3, and acts as $-I_3$ on the space where $X(T)$ acts as $\omega^{-1}I_3$. Then for $W \in N_{GL(6,\boldsymbol{C})}(X(\langle T \rangle))$, $W^{-1}(X(T))W = X(T)^a$ and $Q^{-1}W^{-1}QW = (-1)^{[(a-1)/2]}I_6$ with a equal to either 1 or -1. Therefore, Q normalizes $X(N(\langle T \rangle))$ and $X(N(\langle JT \rangle))$. Also, $Q \in C(J)$, $C(T)$, and

$$C((I_2 \otimes 0_2(SL(2, 3))) \oplus I_2) ,$$

and $Q^{-1}Y^{-1}QY = -I_6$. If we are allowed the possibility of replacing P by QP, then we may take $A^{-1}yA = y$. Then, as $\langle y, S \rangle$ is an irreducible two dimensional group on which A acts trivially, A and $A \otimes I_2$ are scalar. As the homomorphism $C(J) \rightarrow U(C(J))$ has kernel J, and $A \otimes I_2$ centralizes $U(N(\langle JT \rangle))$, C centralizes $V(N(\langle JT \rangle))/\langle -I_2 \rangle$. Then C centralizes $V(T) = w \oplus w^{-1}$, and C is diagonal. Let

$$F = 1 \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus 1 \oplus 1 \oplus -1 .$$

Then $V(F)$ is centralized by C. As $V(C(J)) = \langle V(N(\langle JT \rangle)), V(F) \rangle$, C centralizes $V(C(J))/\langle -I_2 \rangle$, and P normalizes $X(C(J))$.

Therefore, $X(JT)$ and $X(N(\langle JT \rangle))$ determine $X(C(J))$ except possibly for conjugation of $X(C(J))$ by a matrix U which is $\pm I_3$ on the homogeneous spaces of $X(T)$. Now $\langle C(J), N(\langle T \rangle) \rangle$ has index in G dividing 35. As $B_0(7)$ has only $\bar{\chi}_0$ with degree < 35,

$$G = \langle C(J), N(\langle T \rangle) \rangle .$$

We put $X(N(\langle T \rangle))$ in our normal form. Then $X(JT)$ and $X(\langle NJT \rangle))$ determine $X(C(J))$ within conjugation by U. However,

$$\begin{aligned}U^{-1}\langle X(C(J)), X(N(\langle T \rangle)) \rangle U &= \langle U^{-1}X(C(J))U, U^{-1}X(N(\langle T \rangle))U \rangle \\ &= \langle U^{-1}X(C(J))U, X(N(\langle T \rangle)) \rangle\end{aligned}$$

so the similarity class of the representation is not affected by replacing $X(C(J))$ by $U^{-1}X(C(J))U$. Therefore, there, is at most one unimodular, 6-dimensional, complex, linear group projectively representing a simple group of order $2^7 3^6 35$.

6. Existence of $X(G)$. We shall show that $G_1 = \langle x, D, P\rangle$, where $x = V \oplus \bar{V}$ and $V = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \bar{\omega} \\ 1 & \bar{\omega} & \omega \end{pmatrix} / (\omega - \bar{\omega})$, $D = \langle$all diagonal matrices of order 3 and determinant 1$\rangle$, and $P = \langle$all permutation matrices$\rangle$ has a central extension of Z_6 by $U_4(3)$ as a subgroup of index 2. First we show it is finite. In fact, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus I_4$ has a total of 126 conjugates, $C_1 \cup C_2$, where C_1 consists of 45 monomial matrices and C_2 of 81 conjugates of $z = I_6 - Q/3$ where $Q = (q_{i,j})$ and $q_{i,j} = 1$. $\langle C_1\rangle$ has no invariant subspaces, so only scalars commute with all conjugates. If S_i are sets of matrices, define $S_1^{-1}S_2S_1 = \{y \mid y = s_1^{-1}s_2s_1 \text{ for } s_i \in S_i\}$. Then $C_2 = D^{-1}zD$. Let $M = DP = PD$. Now $M^{-1}C_1M = C_1$ and $M^{-1}C_2M = M^{-1}(D^{-1}zD)M = M^{-1}zM = D^{-1}(P^{-1}zP)D = D^{-1}zD = C_2$. It only remains show that $x^{-1}(C_1 \cup C_2)x = C_1 \cup C_2$. Let $\{U_i\}$ be the top 3 by 3 blocks of the 9 elements of C_1 whose bottom 3 by 3 block is I_3. Then $\{U_i\} = -I_3$ {2-elements in the normal subgroup of order 54 of the Hessian group, [1, § 79]}. As the top left 3 by 3 block of x is contained in the Hessian group, conjugation by x permutes these 9 elements. We may reverse the roles of the top left and the bottom right to show that x permutes 9 more elements of C_1. As $x^{-1}zx$ is a permutation matrix transposing 1 and 4, $x^{-1}zx$ has eigenvalues $-1, 1, 1, 1, 1, 1$. Suppose that $d = \text{diag}(d_1, \cdots, d_6) \in D$ with $d_1d_2d_3 = 1$. Then in each row and column of $d^{-1}xd$, and nonzero entries are distinct and have sum 0, or are identical. Then $u_d = (d^{-1}xd)^{-1}z(d^{-1}xd) = I_6 - C_d$ where nonzero entries of C_d are sixth roots of 1. As z and u_d are unitary, u_d has entries 1 or 0 on the diagonal and third roots of 1 off the diagonal and is monomial. Then $u_d \in C_1$ since u_d has eigenvalues $-1, 1, 1, 1, 1, 1$. Therefore, $x^{-1}dzd^{-1}x \in dC_1d^{-1} = C_1$ where d runs through 27 cosets of $\langle wI_6\rangle$. This gives the other $27 = 45 - 9 - 9$ elements in C_1 and $x^{-1}C_1 \cup C_2x \supset C_1$; $C_1 \cup C_2 \supset xC_1x^{-1} = x^{-1}(x^2C_1x^2)x = x^{-1}C_1x$ as $-I_6x^2 \in P$. It only remains to show that $x^{-1}dzd^{-1}x \in C_2$ where $d_1d_2d_3 = \omega$ or $\bar{\omega}$, say $\bar{\omega}$ without loss of generality. We may find e in $\langle D, \text{diag}(\omega, 1, 1, 1, 1, 1)\rangle$ with $(\omega - \bar{\omega})d^{-1}xde = (a_{i,j})$; $\{a_{1,j}, a_{2,j}, a_{3,j}\} = \{1, \bar{\omega}, \bar{\omega}\}$ counting multiplicity for $j = 1, 2, 3$; and $\{a_{4,j}, a_{5,j}, a_{6,j}\} = \{-1, -\omega, -\omega\}$ for $j = 4, 5, 6$. As $d^{-1}xde$ is unitary, the ± 1's appear in different rows. Then the product of the nonzero entries in the first and the fourth rows is still -1, and $e \in D$. Now $(\omega - \bar{\omega})d^{-1}xdeQ$ and $(\omega - \bar{\omega})Qd^{-1}xde$ have all their entries equal to $\bar{\omega} + \bar{\omega} + 1 = -\omega - \omega - 1$. Then $zd^{-1}xde = d^{-1}xdez$, $d^{-1}x^{-1}dzd^{-1}xd = eze^{-1}$, and $x^{-1}dzd^{-1}x = deze^{-1}d^{-1} \in DzD^{-1} = C_2$.

G_1 is primitive since D contains any proper normal reducible subgroup of M and x does not preserve the monomial form of M. Furthermore, G_1 may be made unimodular by replacing odd permutation matrices by their products with iI_6. As $3^7 \mid |G_1|$, by [9]'s classification of groups of degree 6, G_1 contains a central extension of Z_6 by $U_4(3)$ as normal subgroup, G. However, G_1 contains an element with eigenvalues $-1, 1, 1, 1, 1, 1$ and G contains no element with eigenvalues $-i, i, i, i, i, i$. By [8], $7^2 \nmid |G_1/Z|$. By [4], $3F$, S_7 is self-centralizing in G_1/Z, otherwise G_1 has a normal p-subgroup not contained in Z for some prime p, a contradiction. Since $[N_G(S_7); C_G(S_7)] = 3$ and $[N_{G_1}(S_7); C_{G_1}(S_7)] \leqq 6$, $[G_1: G] \leqq 2$ and $[G_1: G] = 2$. For any unimodular finite linear group normalizing $X(G)$, applying this argument to G_2 in place of G_1 shows that $[G_2: X(G)] = 2$, so G_1 is maximal among finite unimodular 6-dimensional complex linear groups normalizing $X(G)$.

7. $LF(3, 4)$. From [9] we may have a six-dimensional group $X(G)$ with $G/Z(G)$ simple of order $2^6 3^2 35$, $\chi(G) \subseteqq Q(w)$, and $B_0(5)$ with degree equation: $1 + 63 = 64$. As S_5 is self-centralizing in $\bar{G} = G/Z$ and $B_0(5)$ does not contain the degree 6, $|Z| \neq 1$. If $|Z| = 2$, then $B_1(5)$ contains the degrees 6 and 64 from a 2-block of defect 1, impossible as $64 - 6 = 58$ cannot be a degree. If $|Z| = 3$, then $B_1(5)$ contains the degrees 6 and 63 from a 3-block of defect 1, impossible as $63 + 6 = 69$. Therefore, $|Z| = 6$. Let J be any involution in $\bar{G}$. Then 0 or $5 = a_{J,J,\tau_5} = |\bar{G}|(1 + \chi_{63}(J)^2/63 - \chi_{64}(J)^2/64)/|C_{\bar{G}}J)|^2$. Now, χ_{64} has 2-defect 0, so $\chi_{64}(J) = 0$ and $\chi_{63}(J) = 1 - \chi_{64}(J) = 1$. Then $5\,|C_{\bar{G}}(J)|^2 = 2^6 3^2 35(1 + 1/63) = 2^{12}5$ and $|C_{\bar{G}}(J)| = 2^6$. Therefore, $\underset{\sim}{C(J)}$ has a normal 2-Sylow-subgroup, and by [11], $\bar{G} \approx LF(3, 4)$. As $U_4(3)$ has a subgroup isomorphic to $LF(3, 4)$ and $LF(3, 4)$ has no projective representation of degree $\leqq 5$, by § 6, G exists with a representation of degree 6. By private communication with N. Burgoyne, G is unique, and the subgroup of the outer automorphism group with trivial action on Z has order 2. A group $G_1 \triangleright G$ with $[G_1: G] = 2$ comes from the product of a field and a graph automorphism.

APPENDIX.

G = Some Central Extension of Z_6 by $LF(3, 4)$.

	$\theta=(1+\sqrt{5})/2$			G/Z		$\phi=(1+\sqrt{-7})/2$		
Element	I	π_5	π_7	T	J	F_1	F_2	F_3
Order	1	5	7	3	2	4	4	4
$C(g)$	g	5	7	9	64	16	16	16
	1	1	1	1	1	1	1	1
	63	θ	0	0	-1	-1	-1	-1
	64	-1	1	1	0	0	0	0
	20	0	-1	2	4	0	0	0
	45	0	$-\phi$	0	-3	1	1	1
	35	0	0	-1	3	3	-1	-1
	35	0	0	-1	3	-1	3	-1
	35	0	0	-1	3	-1	-1	3

G/Z_2

21	1	0	0	5	1	1	1
63	θ	0	0	-1	-1	-1	-1
84	-1	0	0	4	0	0	0
15	0	1	0	-1	3	-1	-1
15	0	1	0	-1	-1	3	-1
15	0	1	0	-1	-1	-1	3
45	0	$-\phi$	0	-3	1	1	1

G/Z_3

I	π_5	π_7	T	J	F_1	F_2	F_3
36	1	1	0	-4	0	0	0
64	-1	1	1	0	0	0	0
28	θ	0	1	4	0	0	0
90	0	-1	0	-2	-2	0	0
10	0	$-\phi$	1	-2	2	0	0
70	0	0	-2	2	2	0	0

G

36	1	1	0	-4	0	0	0
42	$-\theta$	0	0	-2	2	0	0
90	0	-1	0	-2	-2	0	0
60	0	ϕ	0	4	0	0	0
6	1	-1	0	2	2	0	0

$\widetilde{U_4(3)}$, G/Z, $\omega^3 = 1$, $v = \omega - \bar{\omega}$.

Element	I	π_5	$\underline{\pi}_7$	J	T	F	T_1	JT	FT	JT_1	JT_2	$\underline{N}_1$	$\underline{N}_2$	T_2	E	F_1	T_3
Order	1	5	7	2	3	4	3	6	12	6	6	9	9	3	8	4	3
$C(g)$	g	5	7	$2^7 9$	$2^3 3^6$	96	$2^2 3^5$	72	12	36	36	27	27	$2^2 3^5$	8	16	81
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	90	0	−1	10	9	−2	9	1	1	1	1	0	0	9	0	2	0
	$\underline{640}$	0	$(-1+\sqrt{-7})/2$	0	−8	0	−8	0	0	0	0	1	1	−8	0	0	1
	729	−1	1	9	0	−3	0	0	0	0	0	0	0	0	−1	1	0
	35	0	0	3	8	3	8	0	0	0	3	−1	2	−1	−1	−1	−1
	189	−1	0	−3	27	5	0	3	−1	0	0	0	0	0	1	1	0
	896	1	0	0	32	0	−4	0	0	0	0	−1	−1	−4	0	0	−4
	21	1	0	5	−6	1	3	2	−2	−1	−1	0	0	3	−1	1	3
	$\underline{280}$	0	0	−8	10	0	10	−2	0	−2	1	1	$2\bar{\omega} - \omega$	1	0	0	1
	35	0	0	3	8	3	−1	0	0	3	0	2	−1	8	−1	−1	−1
	140	0	0	12	5	4	−4	−3	1	0	0	−1	−1	−4	0	0	5
	$\underline{280}$	0	0	−8	10	0	1	−2	0	1	−2	$2\bar{\omega} - \omega$	1	10	0	0	1
	560	0	0	−16	−34	0	2	2	0	2	2	−1	−1	2	0	0	2
	315	0	0	11	−9	−1	−9	−1	−1	−1	2	0	0	18	1	−1	0
	315	0	0	11	−9	−1	18	−1	−1	2	−1	0	0	−9	1	−1	0
	420	0	0	4	−39	4	6	1	1	−2	−2	0	0	6	0	0	−3
	210	0	0	2	21	−2	3	5	1	−1	−1	0	0	3	0	−2	3

G/Z_3, $\omega^3 = 1$, $i^2 = -1$.

I	π_5	$\underline{\pi}_7$	J	T	F	T_1	JT	FT	JT_1	JT_2	$\underline{N}_1$	$\underline{N}_2$	T_2	E	T_3
20	0	-1	-4	-7	4	2	-1	1	2	2	-1	-1	2	0	2
$\underline{640}$	0	$(-1+\sqrt{-7})/2$	0	-8	0	-8	0	0	0	0	1	1	-8	0	1
120	0	1	8	12	0	-6	-4	0	2	2	0	0	-6	0	3
540	0	1	-12	-27	4	0	3	1	0	0	0	0	0	0	0
896	1	0	0	32	0	-4	0	0	0	0	-1	-1	-4	0	-4
56	1	0	8	2	0	11	2	0	-1	2	-1	2	2	0	2
$\underline{70}$	0	0	2	-11	2	7	-1	-1	-1	2	1	$1+3\omega$	-2	0	-2
56	1	0	8	2	0	2	2	0	2	-1	2	-1	11	0	2
504	-1	0	8	18	0	-9	2	0	-1	2	0	0	18	0	0
504	-1	0	8	18	0	18	2	0	2	-1	0	0	-9	0	0
$\underline{70}$	0	0	2	-11	2	-2	-1	-1	2	-1	$1+3\omega$	1	7	0	-2
70	0	0	2	16	2	7	-4	2	-1	-1	1	1	7	0	-2
$\underline{210}$	0	0	-10	21	2	3	-1	-1	-1	-1	0	0	3	$2i$	3
630	0	0	-14	-18	-6	9	-2	0	1	1	0	0	9	0	0
560	0	0	16	-34	0	2	-2	0	-2	-2	-1	-1	2	0	2

$G/Z_2,\ \omega^3 = 1,\ v = \omega - \bar{\omega} = \sqrt{-3}.$

I	π_5	π_7	J	T	F	T_1	JT	FT	JT_1	JT_2	N_2	E	F_1
15	0	1	−1	6	3	3	2	0	−1	2	$-v$	1	−1
21	1	0	5	3	1	6	−1	1	2	2	$\bar{\omega}v$	−1	1
729	−1	1	9	0	−3	0	0	0	0	0	0	−1	1
105	0	0	−7	15	5	3	−1	−1	−1	2	$-\bar{\omega}v$	−1	1
105	0	0	9	15	1	3	3	1	3	0	$-\bar{\omega}v$	1	1
384	−1	−1	0	24	0	12	0	0	0	0	$-v$	0	0
360	0	$(-1+\sqrt{-7})/2$	8	−18	0	−9	2	0	−1	2	0	0	0
756	1	0	−12	27	−4	0	3	−1	0	0	0	0	0
336	1	0	16	−6	0	6	−2	0	−2	−2	ωv	0	0
210	0	0	2	3	−2	15	−1	1	−1	2	v	0	−2
105	0	0	9	−12	1	12	0	−2	0	0	$-\omega v$	1	1
420	0	0	4	33	4	−6	1	1	−2	−2	$-\omega v$	0	0
945	0	0	−15	−27	1	0	−3	1	0	0	0	1	1
315	0	0	−5	−36	3	9	4	0	1	−2	0	−1	−1
630	0	0	6	9	2	−9	−3	−1	3	0	0	0	−2

$G, v = \omega - \bar{\omega} = \sqrt{-3}$.

I	π_5	$\underline{\pi}_7$	J	T	F	T_1	JT	FT	JT_1	JT_2	$\underline{N}_2$	E
6	1	−1	2	−3	2	3	−1	−1	−1	2	$-\omega v$	0
84	−1	0	−4	−15	4	6	−1	1	2	2	v	0
126	1	0	10	18	2	9	−2	2	1	−2	0	0
384	−1	−1	0	24	0	12	0	0	0	0	$-v$	0
336	1	0	−16	−6	0	6	2	0	2	2	ωv	0
120	0	1	8	−6	0	15	2	0	−1	2	$\bar{\omega}v$	0
$\underline{270}$	0	$(1+\sqrt{-7})/2$	−6	27	2	0	−3	−1	0	0	0	0
420	0	0	12	−21	−4	12	−3	−1	0	0	$\overline{-\omega v}$	0
210	0	0	6	−24	6	−3	0	0	−3	0	ωv	0
840	0	0	−8	−42	0	−3	−2	0	1	−2	$\bar{\omega}v$	0
630	0	0	18	9	2	−9	3	−1	3	0	0	0
840	0	0	−8	12	0	6	4	0	−2	−2	v	0
$\underline{630}$	0	0	2	9	−2	−9	−1	1	−1	2	0	$2i$

An Extension of Z_3 by $\widetilde{U_4(3)}$ (faithful characters).

I	π_5	$\underline{\pi_7}$	J	T	F	JT	FT	JT_1	JT_2	E	F_1
36	1	1	4	9	4	1	1	−2	−2	0	0
720	0	−1	16	18	0	−2	0	−2	−2	0	0
729	−1	1	9	0	−3	0	0	0	0	−1	1
$\underline{45}$	0	$(-1+\sqrt{-7})/2$	−3	−9	1	3	1	0	0	−1	1
189	−1	0	−3	27	5	3	−1	0	0	1	1
126	1	0	14	−9	2	−1	−1	2	2	0	2
756	1	0	−12	27	−4	3	−1	0	0	0	0
315	0	0	−5	18	3	−2	0	4	−2	−1	−1
315	0	0	−5	18	3	−2	0	−2	4	−1	−1
630	0	0	6	−45	2	3	−1	0	0	0	−2
315	0	0	11	18	−1	2	2	2	2	1	−1
945	0	0	−15	−27	1	−3	1	0	0	1	1

An Extension of Z_6 by $\widetilde{U_4(3)}$ (faithful characters).

I	π_5	$\underline{\pi_7}$	J	T	F	JT	FT	JT_1	JT_2	E
90	0	−1	−2	−18	6	−2	0	−2	−2	0
126	1	0	10	−9	2	1	−1	−2	4	0
126	1	0	10	−9	2	1	−1	4	−2	0
540	0	1	−12	−27	4	3	1	0	0	0
630	0	0	18	36	2	0	2	0	0	0
1260	0	0	4	−9	−4	1	−1	−2	−2	0
504	−1	0	8	−36	0	−4	0	2	2	0
720	0	−1	−16	18	0	2	0	2	2	0
270	0	$(1+\sqrt{-7})/2$	−6	27	2	−3	−1	0	0	0
$\underline{126}$	1	0	−6	−9	−2	−3	1	0	0	$2i$

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HARVARD UNIVERSITY