PACIFIC JOURNAL OF MATHEMATICS
Vol. 36, No. 2, 1971

VOLTERRA TRANSFORMATIONS OF THE WIENER MEASURE ON THE SPACE OF CONTINUOUS FUNCTIONS OF TWO VARIABLES

WILLIAM N. HUDSON

The transformation of Wiener integrals over the space C_2 of continuous functions of two variables by a Volterra operator T is investigated. The operator T is defined for functions $x \in C_2$ by

$$Tx(s, t) = x(s, t) + \int_0^s \int_0^t K(u, v)x(u, v)dudv ,$$

where the kernel $K(u, v)$ is continuous. A stochastic integral analogous to K. Ito's is defined and used to determine a Jacobian $J(x)$ for T such that if $F(x)$ is a Wiener measurable functional, Γ a Wiener measurable set, and m Wiener measure, $\int_\Gamma F(x)dm = \int_{T^{-1}(\Gamma)} F(Tx)J(x)dm.$

Let C_2 be the collection of real valued functions f defined on $D = [0, 1] \times [0, 1]$ such that $f(0, t) = f(s, 0) = 0$. The space C_2 is topologized by the sup-norm. In [3], Yeh defined a measure m on C_2 over the Borel σ-algebra and extended it to the Caratheodory σ-algebra relative to m. It is the purpose of this paper to investigate the transformation of the measure m when the elements of C_2 are transformed by a Volterra integral operator of the second kind. The effect of such transformations in the Wiener space of continuous functions of one variable was studied by Cameron and Martin in [1].

Let $0 = s_0 < s_1 < \cdots < s_m \leqq 1$ and $0 = t_0 < t_1 < \cdots < t_n \leqq 1$ and let E be a nm-dimensional Borel set. We denote by $\mathfrak{F}(s_1, \cdots, s_m, t_1, \cdots, t_n)$ the σ-algebra of sets of the form $\{x \in C_2: [x(s_1, t_1), \cdots, x(s_m, t_n)] \in E\}$ and let $\mathfrak{F}_0 = \cup \mathfrak{F}(s_1, \cdots, s_m, t_1, \cdots, t_n)$ where the union is over all such partitions of D. The measure m is given on $\mathfrak{F}(s_1, \cdots, s_m, t_1, \cdots, t_n)$ by

$$\begin{aligned} &m\{x \in C_2: [x(s_1, t_1), \cdots, x(s_m, t_n)] \in E\} \\ (1.1) \qquad &= K(s_1, \cdots, s_m, t_1, \cdots, t_n) \\ &\cdot \int_E (mn) \int W(s_1, \cdots, s_m, t_1, \cdots, t_n, u_{11}, \cdots, u_{mn}) du_{11}, \cdots, du_{mn} , \end{aligned}$$

where

$$\begin{aligned} &K(s_1, \cdots, s_m, t_1, \cdots, t_n) \\ &\quad = \{(2\pi)^{-mn}[s_1(s_2 - s_1) \cdots (s_m - s_{m-1})]^n[t_1(t_2 - t_1) \cdots (t_n - t_{n-1})]^m\}^{\frac{1}{2}} , \end{aligned}$$

$$W(s_1, \cdots, s_m, t_1, \cdots, t_n, u_{11}, \cdots, u_{mn})$$
$$= \exp\left\{-\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{(u_{ij} - u_{i-1,j} - u_{i,j-1} + u_{i-1,j-1})^2}{(s_i - s_{i-1})(t_j - t_{j-1})}\right\},$$

and $u_{0,i} = u_{i,0} = 0$. Yeh showed that m was a probability measure over $(C_2, \mathfrak{F}_0)$ and considered the Caratheodory extension $\mathfrak{F}$ of the algebra $\mathfrak{F}_0$ relative to m. It is well known that $\mathfrak{F}$ contains the Borel σ-algebra.

We consider the stochastic process $X(s, t, x) = x(s, t)$, $x \in C_2$. $X(s, t)$ is analogous to ordinary Brownian motion and proceeding accordingly, we define a stochastic integral analogous to Ito's and denote such integrals of a process $f(s, t, x)$ on C_2 by $\int_D f(s, t, x)dX$.

Next, the Volterra operator T defined by

$$(1.2) \qquad (Tx)(s, t) = x(s, t) + \int_0^s\int_0^t K(u, v)x(u, v)dudv$$

is considered. The kernel $K(s, t)$ of T is assumed to be continuous over the unit square D. It is well-known that T is a one-to-one map of C_2 onto C_2 with a bounded inverse. We can now state our main results.

2. Statement of main results.

THEOREM 1. *Let $F(x)$ be bounded and continuous on C_2 and vanish outside a bounded subset of C_2. Let $K(s, t)$ be continuous over the unit square D. Then*

$$(2.1) \qquad \int_{C_2} F(x)dm(x) = \int_{C_2} F(Tx)J(x)dm(x)$$

where $(Tx)(s, t) = x(s, t) + \int_0^s\int_0^t K(u, v)x(u, v)dudv$, $x \in C_2$ *and* $J(x)$ *is given by the formula*

$$(2.2) \quad J(x) = \exp\left\{-\int_D K(u, v)X(u, v)dX - \frac{1}{2}\int_D K(u, v)^2 x(u, v)^2 dudv\right\}.$$

The first integral in the expression for $J(x)$ is the stochastic integral of the process $K(s, t)X(s, t, x)$ with respect to the process $X(s, t, x) = x(s, t)$.

THEOREM 2. *Let T and $J(x)$ be as in Theorem 1. Then for every $\Gamma \in \mathfrak{F}$, $T^{-1}(\Gamma) \in \mathfrak{F}$ and $T(\Gamma) \in \mathfrak{F}$ and*

$$(2.3) \qquad m(\Gamma) = \int_{T^{-1}(\Gamma)} J(x)dm(x) \qquad \textit{and}$$

$$m(T(\Gamma)) = \int_{\Gamma} J(x) dm(x) . \tag{2.4}$$

Furthermore, if $F(x)$, $x \in C_2$ *is measurable with respect to* $\mathfrak{F}$, *then*

$$\int_{\Gamma} F(x) dm(x) = \int_{T^{-1}(\Gamma)} F(Tx) J(x) dm(x) \tag{2.5}$$

$$\int_{T(\Gamma)} F(x) dm(x) = \int_{\Gamma} F(Tx) J(x) dm(x) \tag{2.6}$$

in the sense that the existence of one side implies the existence of the other and the equality of the two.

3. Definition of the stochastic integral. In this section the basic definition of the stochastic integral is given and some fundamental properties are listed. The proofs are omitted since they are strictly analogous to those of K. Itô in [2].

Let $(\Omega, \mathfrak{B}, P)$ be a probability space and let $\{X(s, t): s, t \in [0, 1]\}$ be a stochastic process with two time parameters defined over $(\Omega, \mathfrak{B}, P)$. If for any pair (m, n) of positive integers, and any set $S = \{a_1, \cdots a_m, b_1, \cdots, b_m, c_1, \cdots, c_n, d_1, \cdots, d_n\}$ of real numbers in $[0, 1]$ such that $a_1 < b_1 \leqq a_2 < b_2 \leqq \cdots \leqq a_m < b_m$ and $c_1 < d_1 \leqq c_2 < d_2 \leqq \cdots \leqq c_n < d_n$, the "increments" $X(b_i, d_j) - X(a_i, d_j) - X(b_i, c_j) + X(a_i, c_j)$ $i = 1, \cdots, m$, $j = 1, \cdots, n$ are independent random variables, the process $X(s, t)$ will be called biadditive. If a biadditive process $X(s, t)$ is Gaussian and has the additional properties that for all $(s, t) \in D$ $E(X(s, t)) = 0$, $\mathrm{var}\,(X(s, t)) = st$, and $X(0, t) = X(s, 0) = 0$, then $X(s, t)$ will be said to be a generalized Brownian motion. The process $X(s, t, x) = x(s, t)$, $x \in C_2$ defined on $(C_2, \mathfrak{F}, m)$ is an example of a generalized Brownian motion.

Now let $X(s, t)$ be a fixed generalized Brownian motion and denote the increments $X(b, d) - X(a, d) - X(b, c) + X(a, c)$ by $\Delta(a, c, b, d)$. Let $\mathfrak{D}$ denote the Borel subsets of D. For each choice of $(s, t) \in D$, let $\mathfrak{A}(s, t)$ be a sub σ-algebra of $\mathfrak{B}$ which contains $\sigma\{X(u, v): u \leqq s$ and $v \leqq t\}$, the σ-algebra generated by $X(s, t)$ up to (s, t), and which is independent of $\sigma\{\Delta(s, t, u, v): u \geqq s$ or $t \geqq v\}$. Assume also that if $s \leqq s'$ and $t \leqq t'$, $\mathfrak{U}(s, t) \subset \mathfrak{U}(s', t')$. Let $\mathfrak{M}$ denote the class of stochastic processes $f(s, t, \omega)$ defined on $(\Omega, \mathfrak{B}, P)$ with domain of definition D which satisfy

(i) $f(s, t, \omega)$ is $\mathfrak{D} \times \mathfrak{B}$ measurable and

(ii) $f(s, t, \cdot)$ is $\mathfrak{U}(s, t)$ measurable.

$\mathfrak{M}_0$ will denote the subset of $\mathfrak{M}$ such that if $f(s, t, \omega) \in \mathfrak{M}_0$, there are real numbers $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_m = 1$ and $0 = \beta_0 < \beta_1 < \cdots < \beta_n = 1$ such that $f(s, t) = f(\alpha_{j-1}, \beta_{k-1})$ whenever $\alpha_{j-1} \leqq s < \alpha_j$ and $\beta_{k-1} \leqq t < \beta_k$. $\mathfrak{M}_1$ will denote the subset of $\mathfrak{M}$ such that $f(s, t, \omega) \in \mathfrak{M}_1$ whenever

$\int_{D\times\Omega}(f(u, v, \omega))^2 dP \times \ell < \infty$ where ℓ denotes two dimensional Lebesgue measure. $\mathfrak{M}_2$ will denote the set of all $f \in \mathfrak{M}$ such that for almost all $\omega \in \Omega$, $\int_D f(u, v, \omega)^2 d\ell < \infty$. We define the stochastic integral successively for $f \in \mathfrak{M}_0$, then for $f \in \mathfrak{M}_1$, and finally for $f \in \mathfrak{M}_2$.

DEFINITION. For $f(s, t) \in \mathfrak{M}_0$, the stochastic integral of $f(s, t)$ with respect to $X(s, t)$ is denoted by $\left(\int f dX\right)(s, t)$ and is defined by

$$\begin{aligned}\left(\int f dX\right)(s, t) = &\sum_{p=1}^{j-1}\sum_{q=1}^{k-1} f(\alpha_{p-1}, \beta_{q-1})\Delta(\alpha_{p-1}, \beta_{q-1}, \alpha_p, \beta_q)\\ &+ \sum_{p=1}^{j-1} f(\alpha_{p-1}, \beta_{k-1})\Delta(\alpha_{p-1}, \beta_{k-1}, \alpha_p, t)\\ &+ \sum_{q=1}^{k-1} f(\alpha_{j-1}, \beta_{q-1})\Delta(\alpha_{j-1}, \beta_{q-1}, s, \beta_q)\\ &+ f(\alpha_{j-1}, \beta_{k-1})\Delta(\alpha_{j-1}, \beta_{k-1}, s, t)\ .\end{aligned}$$

where the α's and β's are taken as in the definition of $\mathfrak{M}_0$.

The following properties follow from this definition for $f \in \mathfrak{M}_0$ in the same way as for the usual stochastic integral.

THEOREM 3.1. *If* $f \in \mathfrak{M}_0$, *then* $\left(\int f dX\right)(s, t)$ *has the following properties.*

(1) *For* $f, g \in \mathfrak{M}_0$, $\omega_0 \in \Omega$, *if* $f(s, t, \omega_0) = g(s, t, \omega_0)$ *on* D, *then* $\left(\int f dX\right)(s, t, \omega_0) = \left(\int g dX\right)(s, t, \omega_0)$ *on* D.

(2) *For* $f, g \in \mathfrak{M}_0$, α, β *real numbers*, $(s, t) \in D$ $\left(\int(\alpha f+\beta g)dX\right)(s, t) = \alpha\left(\int f dX\right)(s, t) + \beta\left(\int g dX\right)(s, t)$.

(3) *For almost every* ω, $\left(\int f dX\right)(s, t, \omega)$ *is continuous over* D.

(4) $E\left[\left(\int f dX\right)(s, t)\right] = 0$ *for all* $(s, t) \in D$ *and*

(5) $\operatorname{var}\left[\left(\int f dX\right)(s, t)\right] = \|f\|_{s,t}^2$ *where* $\|f\|_{s,t}^2$

$$\equiv \int_{[0,s]\times[0,t]\times\Omega} f(s', t', \omega)^2 d\ell \times P\ .$$

As in the one variable case, $\mathfrak{M}_0 \cap \mathfrak{M}_1$ is dense in $\mathfrak{M}_1$ with respect to the Hilbert norm on $L_2(D \times \Omega)$. Denoting this norm by $\|\cdot\|$ and using property (5) above, we make the following definition for $f \in \mathfrak{M}_1$.

DEFINITION. For $f \in \mathfrak{M}_1$, the stochastic integral of f is defined to be

$$\left(\int f dX\right)(s, t) = \lim_{n\to\infty}\left(\int f_n dX\right)(s, t)$$

where the limit on the right is that of convergence in the norm of $L_2(\Omega)$ and $\{f_n\}$ is any sequence of functions in $\mathfrak{M}_0 \cap \mathfrak{M}_1$ such that $\lim_{n\to\infty} \|f_n - f\| = 0$.

The following properties hold for functions in $\mathfrak{M}_1$.

THEOREM 3.2. *Let f and g be in $\mathfrak{M}_1$. The following statements are true.*

(1) *If $f(s, t, \omega) = g(s, t, \omega)$ on D for $\omega \in A \in \mathfrak{B}$, then $\left(\int f dX\right)(s, t, \omega) = \left(\int g dX\right)(s, t, \omega)$ for $\omega \in A_0 \subset A$ where $A_0 \in \mathfrak{B}$ and $P(A_0) = P(A)$.*

(2) *If α and β are any two real numbers and $(s, t) \in D$ then almost surely*

$$\left(\int (\alpha t + \beta g) dX\right)(s, t) = \alpha\left(\int f dX\right)(s, t) + \beta\left(\int g dX\right)(s, t) .$$

(3) *For every point (s, t) in D, $\operatorname{var}\left[\left(\int f dX\right)(s, t)\right] = \|f\|_{s,t}$ and $E\left[\left(\int f dX\right)(s, t)\right] = 0$.*

Let $\chi_{[0,n]}(t)$ denote the indicator function of $[0, n]$, i.e., $\chi_{[0,n]}(t) = 1$ if $0 \leqq t \leqq n$ and $\chi_{[0,n]}(t) = 0$ otherwise. In order to define the stochastic integral for a function $f \in \mathfrak{M}_2$, we observe that $\int_D f^2 d\ell \leqq n$ implies $\int_0^1\int_0^t f^2(u, v) du dv \leqq n$ for all $t \in [0, 1]$ and so $f(s, t, \omega) = f_{n+m}(s, t, \omega)$ for $m = 0, 1, 2, \cdots$ where $f_n(s, t) = \chi_{[0,n]}\left(\int_0^1\int_0^t f^2(u, v) du dv\right) f(s, t)$. Let $F_n = \left\{\omega\colon \int_D f^2 d\ell \leqq n\right\}$. Then $F_n \in \mathfrak{B}$ and $F_1 \subset F_2 \subset F_3 \subset \cdots$ and from the definition of $\mathfrak{M}_2$, $P(\bigcup_1^\infty F_n) = 1$. Using property (1) of the last theorem, we see that there is a set $F_{n,0} \subset F_n$ such that $F_{n,0} \in \mathfrak{B}$, $P(F_{n,0}) = P(F_n)$ and for $\omega \in F_{n,0}$, $(s, t) \in D$,

$$\left(\int f_n dX\right)(s, t, \omega) = \left(\int f_{n+m} dX\right)(s, t, \omega) \qquad m = 1, 2, \cdots .$$

DEFINITION. Let f be a function in $\mathfrak{M}_2$. The stochastic integral of f is defined to be

$$\left(\int f dX\right)(s, t, \omega) = \left(\int f_n dX\right)(s, t, \omega) \qquad \text{if } \omega \in F_{n,0}$$

and is defined to be zero if $\omega \notin \bigcup_{n=1}^\infty F_{n,0}$.

The next theorem gives easy properties of such integrals.

THEOREM 3.3. *Let f and g be in $\mathfrak{M}_2$. The following statements hold for stochastic integrals of f and g.*

(1) *If $f(s,t,\omega)=g(s,t,\omega)$ on D for $\omega\in A\in\mathfrak{B}$, then $\left(\int f dX\right)(s,t,\omega)=\left(\int g dX\right)(s,t,\omega)$ on D for $\omega\in A_0\in\mathfrak{B}$ where $A_0\subset A$ and $P(A_0)=P(A)$.*

(2) *If α and β are real numbers, then*

$$\left(\int(\alpha f+\beta g)dX\right)(s,t)=\alpha\left(\int f dX\right)(s,t)+\beta\left(\int g dX\right)(s,t)$$

holds almost surely.

4. Lemmas for Theorem 2.1. Let $K_{ij}=K((i/n),(j/n))$ where $K(s,t)$ is a real-valued continuous function on D and define the transformation $T_n\colon C_2\to C_2$ by

$$\begin{aligned}(T_n x)(s,t)&=x(s,t)+\frac{1}{n^2}\sum_{i=1}^{[ns]}\sum_{j=1}^{[nt]}K_{i-1,j-1}x\left(\frac{i-1}{n},\frac{j-1}{n}\right)\\&+\frac{1}{n}\sum_{j=1}^{[nt]}K_{[ns],j-1}x\left(\frac{[ns]}{n},\frac{j-1}{n}\right)\left(s-\frac{[ns]}{n}\right)\\&+\frac{1}{n}\sum_{i=1}^{[ns]}K_{i-1,[nt]}x\left(\frac{i-1}{n},\frac{[nt]}{n}\right)\left(t-\frac{[nt]}{n}\right)\\&+K_{[ns],[nt]}x\left(\frac{[ns]}{n},\frac{[nt]}{n}\right)\left(s-\frac{[ns]}{n}\right)\left(t-\frac{[nt]}{n}\right).\end{aligned}\tag{4.1}$$

For $s=(\ell/n)$, $t=(k/n)$, we have for $\ell, k=1,2,\cdots,n$

$$(T_n x)\left(\frac{\ell}{n},\frac{k}{n}\right)=x\left(\frac{\ell}{n},\frac{k}{n}\right)+\frac{1}{n^2}\sum_{i=1}^{\ell-1}\sum_{j=1}^{k-1}K_{ij}x\left(\frac{i}{n},\frac{j}{n}\right).\tag{4.2}$$

LEMMA 1. *Let $H(\eta_{11},\cdots,\eta_{nn})$ be a real-valued bounded and continuous function on R^{n^2} and let $G(x)$, $x\in C_2$, be defined by*

$$G(x)=H\left(x\left(\frac{1}{n},\frac{1}{n}\right),\cdots,x\left(\frac{n}{n},\frac{n}{n}\right)\right),\tag{4.3}$$

then

$$\begin{aligned}\int_{C_2}G(x)dm=\int_{C_2}G(T_n x)\exp&\left\{-\sum_{i=1}^{n}\sum_{j=1}^{n}K_{i-1,j-1}x\left(\frac{i-1}{n},\frac{j-1}{n}\Delta_{ij}\right\}\right.\\&\cdot\exp\left\{-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}K_{i-1,j-1}^2x^2\left(\frac{i-1}{n},\frac{j-1}{n}\right)\frac{1}{n^2}\right\}dm\end{aligned}\tag{4.5}$$

where

$$\Delta_{ij} = x\left(\frac{i}{n}, \frac{j}{n}\right) - x\left(\frac{i-1}{n}, \frac{j}{n}\right) - x\left(\frac{i}{n}, \frac{j-1}{n}\right) + x\left(\frac{i-1}{n}, \frac{j-1}{n}\right).$$

Proof. From the definition of m, we have

$$\begin{aligned} &\int_{C_2} G(x)dm = (2\pi^{-2}n)^{-n^2/2}\int_{R^{n^2}} H(\eta_{11}, \cdots, \eta_m) \\ &\cdot \exp\left\{-\frac{n^2}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(\eta_{ij} - \eta_{i-1,j} - \eta_{i,j-1} + \eta_{i-1,j-1})^2\right\}d\eta_{11}, \cdots, d\eta_{nn}. \end{aligned} \tag{4.6}$$

Let S_n denote the linear transformation of R^{n^2} onto itself defined by

$$\eta_{ij} = \xi_{ij} + \frac{1}{n^2}\sum_{m=1}^{i-1}\sum_{l=1}^{j-1} K_{ml}\xi_{ml} \qquad i, j = 1, 2, \cdots, n.$$

The Jacobian of S_n is equal to 1. Applying S_n to the right side of (4.6) we obtain

$$\begin{aligned} &\int_{C2} G(x)dm = (2\pi^{-2}n)^{-n^2/2} \\ &\cdot \int_{R^{n^2}} H\left(\xi_{11}, \cdots, \xi_{nn} + \frac{1}{n^2}\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} K_{ij}\xi_{ij}\right)J_n(\xi_{11}, \cdots, \xi_{nn})d\xi_{11}, \cdots, d\xi_{nn} \end{aligned} \tag{4.7}$$

where

$$\begin{aligned} &J_n(\xi_{11}, \cdots, \xi_{nn}) \\ &= \exp\left\{-\sum_{i=1}^{n}\sum_{j=1}^{n} K_{i-1,j-1}\xi_{i-1,j-1}(\xi_{ij} - \xi_{i-1,j} - \xi_{i,j-1} + \xi_{i-1,j-1})\right\} \\ &\cdot \exp\left\{-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} K^2_{i-1,j-1}\xi^2_{i-1,j-1}\frac{1}{n^2} - \frac{n^2}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(\xi_{ij}-\xi_{i-1,j}-\xi_{i,j-1}+\xi_{i-1,j-1})^2\right\}. \end{aligned}$$

On the other hand,

$$\begin{aligned} G(T_nx) &= H\left(T_nx\left(\frac{1}{n}, \frac{1}{n}\right), \cdots, T_nx\left(\frac{n}{n}, \frac{n}{n}\right)\right) \\ &= H\left(x\left(\frac{1}{n}, \frac{1}{n}\right), \cdots, x\left(\frac{n}{n}, \frac{n}{n}\right) + \frac{1}{n^2}\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} K_{ij}x\left(\frac{i}{n}, \frac{j}{n}\right)\right). \end{aligned}$$

Again from the definition of m, we see that the right side of (4.7) is equal to

$$\begin{aligned} \int_{C_2} G(T_nx)\exp\Bigg\{&-\sum_{i=1}^{n}\sum_{j=1}^{n} K_{i-1,j-1}x\left(\frac{i-1}{n}, \frac{j-1}{n}\right)\Delta_{ij} \\ &- \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} K^2_{i-1,j-1}x^2\left(\frac{i-1}{n}, \frac{j-1}{n}\right)\frac{1}{n^2}\Bigg\}dm \end{aligned}$$

which is precisely the right side of (4.5).

LEMMA 2. *Let X be a random variable on a probability space $(\Omega, \mathfrak{B}, P)$ which is distributed normally with mean 0 and variance v. Let Y be a random variable on $(\Omega, \mathfrak{B}, P)$ which is measurable with respect to a σ-algebra $\mathfrak{A} \subset \mathfrak{B}$. If the σ-algebra $\sigma\{X\} \subset \mathfrak{B}$ generated by X and the σ-algebra $\mathfrak{A}$ are independent, then*

$$E\Big(\exp\Big\{XY - \frac{1}{2}vY^2\Big\} \mid \mathfrak{A}\Big) = 1 . \tag{4.8}$$

Proof. To prove the lemma we show that for every $A \in \mathfrak{A}$

$$\int_A \exp\Big\{XY - \frac{1}{2}vY^2\Big\}dP = \int_A dP .$$

Let $P_{\mathfrak{A}}$ be the restriction of P to $\mathfrak{A}$. Let us write $\sigma\{\mathfrak{M}\}$ to mean the σ-algebra generated by a collection of sets $\mathfrak{M}$. Consider the transformation T of the measure space $(\Omega, \mathfrak{B})$ into the measure space $(R^1 \times \Omega, \sigma\{\mathfrak{B}^1 \times \mathfrak{A}\})$ defined by $T(\omega) = (X(\omega), \omega)$. This is a measurable transformation since $T^{-1}(\sigma\{\mathfrak{B}^1 \times \mathfrak{A}\}) = \sigma\{T^{-1}(\mathfrak{B}^1 \times \mathfrak{A})\}$ which is contained in $\mathfrak{B}$ since $T^{-1}(\mathfrak{B}^1 \times \mathfrak{A})$ is. Let U be the transformation of $(R^1 \times \Omega, \sigma\{\mathfrak{B}^1 \times \mathfrak{A}\})$ into $(R^1, \mathfrak{B}^1)$ defined by

$$U(\xi, \omega) = \exp\{\xi Y(\omega) - 1/2\, Y^2(\omega)\} .$$

This too is a measurable transformation since Y is $\mathfrak{A}$-measurable.

Let P_T be the probability measure on $\sigma\{\mathfrak{B}^1 \times \mathfrak{B}\}$ induced by T. For $B \in \mathfrak{B}^1$ and $A \in \mathfrak{B}$, we have from the independence of $\sigma\{X\}$ and $\mathfrak{A}$

$$P_T(B \times A) = P(T^{-1}(B \times A)) = P\{X \in B\}P(A) = P_X(B)P_{\mathfrak{A}}(A)$$

where P_X is the probability measure on $\mathfrak{B}^1$ induced by X, i.e., the normal distribution with mean 0 and variance v. Thus P_T is the product measure of P_X and $P_{\mathfrak{A}}$.

Now for $A \in \mathfrak{A}$ we have by Tonelli's Theorem

$$\begin{aligned}\int_A \exp\Big\{XY - \frac{1}{2}vY^2\Big\}dP &= \int_\Omega (UT)(\omega)dP = \int_{R^1 \times A} U(\xi, \omega)dP_T \\ &= \int_A \Big[\int_{R^1} \exp\Big\{\xi Y(\omega) - \frac{1}{2}vY^2(\omega)\Big\}dP_X\Big]dP_{\mathfrak{A}} \\ &= \int_A \Big[\frac{1}{\sqrt{2\pi v}}\int_{R^1} \exp\Big\{-\frac{(\xi - vY(\omega))^2}{2v}\Big\}d\xi\Big]dP_{\mathfrak{A}} \\ &= \int_A dP_{\mathfrak{A}} .\end{aligned}$$

This completes the proof of the lemma.

LEMMA 3. *Let $X(s, t, x)$ be the stochastic process on the probability* *space* $(C_2, \mathfrak{F}, m)$ *and the domain of definition* $D = [0, 1] \times [0, 1]$ *defined by* $X(s, t, x) = x(s, t)$ *for* $x \in C_2$ *and* $(s, t) \in D$. *Let* $g(s, t)$ *be a real valued function on* D *and let* $f_n(s, t, x)$ *be a stochastic process on* $(C_2, \mathfrak{F}, m)$ *and* D *defined by*

$$(4.9) \qquad f_n(s, t, x) = g\Big(\frac{[ns]}{n}, \frac{[nt]}{n}\Big)x\Big(\frac{[ns]}{n}, \frac{[nt]}{n}\Big) \quad \text{for} \quad x \in C_2 .$$

Then the stochastic integral $\left(\int f_n dX\right)(s, t, x)$ *of the process* $f_n(s, t, x)$ *with respects to the generalized Brownian motion* $X(s, t, x)$ *satisfies*

$$(4.10) \qquad E\Big[\exp\Big\{\Big(\int f_n dX\Big)(1, 1, x) - \frac{1}{2}\int_D f_n^2(s, t, x)d\swarrow\Big\}\Big] = 1 .$$

Proof. Since f_n is a stochastic step function,

$$\Big(\int f_n dX\Big)(1, 1) = \sum_{i=1}^{n}\sum_{j=1}^{n} f_n\Big(\frac{i-1}{n}, \frac{j-1}{n}\Big)\Delta_{ij}$$

where

$$\Delta_{ij} = X\Big(\frac{i}{n}, \frac{j}{n}\Big) - X\Big(\frac{i-1}{n}, \frac{j}{n}\Big) - X\Big(\frac{i}{n}, \frac{j-1}{n}\Big) + X\Big(\frac{i-1}{n}, \frac{j-1}{n}\Big) .$$

Let

$$T_{ij} = f_n\Big(\frac{i-1}{n}, \frac{j-1}{n}\Big)\Delta_{ij} - \frac{1}{2}f_n^2\Big(\frac{i-1}{n}, \frac{j-1}{n}\Big)\frac{1}{n^2} .$$

Since

$$\int_D f_n^2(s, t)d\swarrow = \sum_{i=1}^{n}\sum_{k=1}^{n} f_n^2\Big(\frac{i-1}{n}, \frac{j-1}{n}\Big)\frac{1}{n^2} ,$$

$$Z_{ij} \equiv \exp\Big\{\int f_n dX\Big(\frac{i}{n}, \frac{j}{n}\Big) - \frac{1}{2}\int_0^{i/n}\int_0^{j/n} f_n^2(s, t)dsdt\Big\}$$
$$= \exp\Big\{\sum_{p=1}^{i}\sum_{q=1}^{j} T_{pq}\Big\} .$$

Let $\mathfrak{A}_{ij}$ denote the σ-algebra $\sigma\{X(s, t): s \leqq i/n \text{ or } t \leqq j/n\}$. Then $f_n(i-1)/n, (j-1)/n$ is $\mathfrak{A}_{i-1,\, j-1}$-measurable for $i, j = 1, 2, \cdots, n$. The random variable Δ_{ij} is normally distributed with mean 0 and variance $1/n^2$. Furthermore $\sigma\{\Delta_{ij}\}$ and $\mathfrak{A}_{i-1,j-1}$ are independent. By Lemma 2

$$(4.11) \qquad E(\exp\{T_{ij}\} \mid \mathfrak{A}_{i-1,j-1}) = 1 .$$

To prove the lemma, we must show that $EZ_{11} = 1$. Now for $m = 1, \cdots n$ $X(m-1)/n, 0) = 0$ and hence $f_n(m-1)/n, 0) = 0$ and $T_{m1} = 0$. We have $Z_{n1} = \exp\{\sum_{p=1}^{n} T_{p1}\} = 1$. The proof will proceed by induction.

Consider $E(Z_{n,j-1} \exp\{\sum_{m=1}^{k} T_{mj}\})$. If $k = 1$

$$E(Z_{n,j-1} \exp\{T_{1j}\}) = E[E(Z_{n,j-1} \exp\{T_{1j}\} \mid \mathfrak{A}_{0,j-1})] .$$

Since $Z_{n,j-1}$ is $\mathfrak{A}_{0,j-1}$-measurable

$$\begin{aligned} E(Z_{n,j-1} \exp\{T_{1j}\}) &= E[Z_{n,j-1} E(\exp\{T_{1j}\} \mid \mathfrak{A}_{0,j-1})] \\ &= E[Z_{n,j-1}] \end{aligned}$$

by (4.11). Now suppose that

$$E\Big(Z_{n,j-1} \exp\Big\{\sum_{m=1}^{k} T_{mj}\Big\}\Big) = E(Z_{n,j-1}) .$$

Then

$$E\Big(Z_{n,j-1} \exp\Big\{\sum_{m=1}^{k+1} T_{mj}\Big\}\Big) = E\Big[E\Big(Z_{n,j-1} \exp\Big\{\sum_{m=1}^{k+1} T_{mj}\Big\} \mid \mathfrak{A}_{k,j-1}\Big)\Big] .$$

Since $Z_{n,j-1} \exp\{\sum_{m=1}^{k} T_{mj}\}$ is $\mathfrak{A}_{k,j-1}$-measurable

$$\begin{aligned} E\Big[Z_{n,j-1} \exp\Big\{\sum_{m=1}^{k+1} T_{mj}\Big\} \mid \mathfrak{A}_{k,j-1}\Big] &= Z_{n,j-1} \exp\Big\{\sum_{m=1}^{k} T_{mj}\Big\} E(\exp\{T_{k+1,j}\} \mid \mathfrak{A}_{k,j-1}) \\ &= Z_{n,j-1} \exp\Big\{\sum_{m=1}^{k} T_{mj}\Big\} . \end{aligned}$$

By the induction hypothesis, we have

$$E\Big(Z_{n,j-1} \exp\Big\{\sum_{m=1}^{k+1} T_{mj}\Big\}\Big) = E(Z_{n,j-1}) .$$

In particular for $k = n$, $Z_{nj} = Z_{n,j-1} \exp\{\sum_{m=1}^{n} T_{mj}\}$ and $EZ_{nj} = EZ_{n,j-1}$. It follows that $EZ_{nn} = E_{n1} = 1$.

Let L_n be the transformation of C_2 into C_2 defined for $(s, t) \in [(i-1)/n, i/n] \times [(j-1)/n, j/n]$ by

$$\begin{aligned} L_n x(s, t) = \frac{1}{n^2}\Big\{ &x\Big(\frac{i}{n}, \frac{j}{n}\Big)\Big(s - \frac{i-1}{n}\Big)\Big(t - \frac{j-1}{n}\Big) \\ &+ x\Big(\frac{i-1}{n}, \frac{j}{n}\Big)\Big(\frac{i}{n} - s\Big)\Big(t - \frac{j-1}{n}\Big) \\ &+ x\Big(\frac{i}{n}, \frac{j-1}{n}\Big)\Big(s - \frac{i-1}{n}\Big)\Big(\frac{j}{n} - t\Big) \\ &+ x\Big(\frac{i-1}{n}, \frac{j-1}{n}\Big)\Big(\frac{i}{n} - s\Big)\Big(\frac{j}{n} - t\Big)\Big\} . \end{aligned} \tag{4.12}$$

Clearly

$$|||L_n x||| = \max_{i,j=1,\cdots,n} \Big|x\Big(\frac{i}{n}, \frac{j}{n}\Big)\Big| \leq |||x||| \tag{4.13}$$

$$\lim_{n\to\infty} |||L_n x - x||| = 0 . \tag{4.14}$$

Then for T and T_n defined by (1.2) and (4.1) respectively, we have

$$\lim_{n\to\infty} |||L_n T_n x - Tx||| = 0 . \tag{4.15}$$

This follows from

$$|||L_n T_n x - Tx||| \leqq |||L_n T_n x - L_n Tx||| + |||L_n Tx - Tx|||$$

where

$$|||L_n T_n x - L_n Tx||| \leqq |||T_n x - Tx||| \text{ by (4.13), } \lim_{n\to\infty} |||T_n x - Tx||| = 0$$

from the uniform continuity of $K(s, t)$ on D, and $\lim_{n\to\infty} |||L_n Tx - Tx||| = 0$ by (4.14).

LEMMA 4. *Let $X(s, t, x)$, $g(s, t)$ and $f_n(s, t, x)$ be as defined in Lemma 3. Then the random variables $Z_n(x)$, $n = 1, 2, \cdots$, on $(C_2, \mathfrak{F}, m)$ defined by*

$$Z_n = \exp\left\{\left(\int f_n dX\right)(1, 1) - \int_D f_n^2(s, t) d\mathscr{A}\right\} \tag{4.16}$$

are uniformly integrable on C_2. If $g(s, t)$ is bounded on D, then for every $B \geqq 0$, the random variables $Y_n(x)$, $n = 1, 2, \cdots$, defined by

$$Y_n(x) = \chi_{[0,B]}(|||L_n x|||) \exp\left\{\left(\int f_n dX\right)(1, 1)\right\} \tag{4.17}$$

are uniformly integrable on C_2.

Proof. For $M > 0$ let $A_{M,n} = \{x \in C_2 : Z_n(x) > M\}$. To show the uniform integrability of Z_n, $n = 1, 2, \cdots$, we show that for every $\varepsilon > 0$ there exists $M > 0$ independent of n such that

$$\int_{A_{M,n}} Z_n(x) dm < \varepsilon \qquad n = 1, 2, \cdots .$$

According to Lemma 3 applied to $2f_n$, $E(Z_n^2) = 1$ and so choosing $M > 1/\varepsilon$, we have

$$\int_{A_{M,n}} Z_n(x) dm \leqq \int_{A_{M,n}} \frac{1}{M} Z_n^2(x) dm \leqq \frac{1}{M} < \varepsilon$$

proving the uniform integrability of Z_n, $n = 1, 2, \cdots$.

Suppose $g(s, t)$ is bounded on D. Now

$$|||L_n x||| = \max_{(s,t)\in D} \left| X\left(\frac{[ns]}{n}, \frac{[nt]}{n}, x\right) \right|$$

and so if $x \in C_2$, $|||L_n x||| \leqq B$, and $B > 0$, then $|||f_n||| \leqq B|||g|||$ and $\int_D f_n^2(s,t)d_{\swarrow} \leqq B^2|||g|||^2$. Letting $\gamma = \exp\{|||g|||^2 B^2\}$ we have

$$Y_n(x) = \chi_{[0,B]}(|||L_n x||)Z_n(x)\exp\left\{\int_D f_n^2(s,t)d_{\swarrow}\right\} \leqq \gamma Z_n(x) .$$

For $K > \gamma\varepsilon^{-1}$

$$\int_{[Y_n > K]} Y_n(x)dm \leqq \gamma\int_{[Z_n > K/\gamma]} Z_n(x)dm < \varepsilon\gamma \qquad n = 1, 2, \cdots$$

proving the uniform integrability of Y_n, $n = 1, 2, \cdots$.

LEMMA 5. *If $x \in C_2$ and for some $M > 0$ $|||L_n x||| > M\exp\{|||K|||\}$, then $|||L_n T_n x||| > M$.*

Proof. As in the Volterra integral theory one can show that T_n defined by (4.1) transforms C_2 onto C_2 in a one-to-one manner, T_n and T_n^{-1} are bounded linear operators, and $||T_n^{-1}|| \leqq \exp\{|||K|||\}$.

Now for any $x \in C_2$ which satisfies $|||x||| > M\exp\{|||K|||\}$ for some $M > 0$, we have $|||x||| > M||T_n^{-1}||$. If $|||T_n x||| \leqq M$,

$$M||T_n^{-1}|| < |||x||| = |||T_n^{-1}T_n x||| \leqq ||T_n^{-1}|| \cdot |||T_n x||| \leqq ||T_n^{-1}||M$$

a contradiction. Thus for any $x \in C_2$, if $|||x||| > M||T_n^{-1}||$ then $|||T_n x||| > M$. In particular $|||L_n x||| > M\exp\{|||K|||\}$ implies $|||T_n L_n x||| > M$. But (4.1) and (4.12) imply that $T_n L_n x = L_n T_n x$ and hence $|||L_n T_n x||| > M$.

5. Proof of Theorem 1.

Since $L_n x$, $x \in C_2$, is determined by the values of x on the lattice points $(i/n, j/n)$ $i, j = 1, 2, \cdots, n$, we may define a function H on R^{n^2} by

$$H(\eta_{11}, \cdots, \eta_{nn}) = F(L_n x)$$

where $\eta_{ij} = x(i/n, j/n)$. The continuity of F and L_n implies the $\mathfrak{F}$-measurability of $F \circ L_n$. If R^{n^2} is topologized according to the sup-norm, it is easy to see that H is continuous. Since F is bounded, so is H. Let

$$G(y) = F(L_n y) = H\left(y\left(\frac{1}{n}, \frac{1}{n}\right), \cdots, y\left(\frac{n}{n}, \frac{n}{n}\right)\right) .$$

Then $G(T_n y) = F(L_n T_n y)$ and according to Lemma 1

$$\begin{aligned} \int_{C_2} F(L_n x)dm = \int_{C_2} & F(L_n T_n x) \\ (5.1) \qquad & \exp\left\{-\sum_{i=1}^n\sum_{j=1}^n K\left(\frac{i-1}{n}, \frac{j-1}{n}\right)x\left(\frac{i-1}{n}, \frac{j-1}{n}\right)\Delta_{ij}\right\} \\ & \exp\left\{-\frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n K^2\left(\frac{i-1}{n}, \frac{j-1}{n}\right)x^2\left(\frac{i-1}{n}, \frac{j-1}{n}\right)\frac{1}{n^2}\right\}dm . \end{aligned}$$

Since $\lim_{n\to\infty} |||L_n x - x||| = 0$ and F is continuous, $\lim_{n\to\infty} F(L_n x) = F(x)$. Since F is bounded,

$$\lim_{n\to\infty}\int_{C_2} F(L_n x)dm = \int_{C_2} F(x)dm \,. \tag{5.2}$$

We now show that the integral on the right side of 5.1 converges to

$$\int_{C_2} F(Tx)\exp\left\{-\left(\int K(s,t)X(s,t)dX\right)(1,1) - \frac{1}{2}\int_D K^2(s,t)x^2(s,t)d\measuredangle\right\}dm$$

which will complete the proof. Since F vanishes off a bounded set, there exists $M > 0$ such that if $|||x||| > M$ then $F(x) = 0$. Let $N = M\exp\{|||K|||\}$. Then

$$\int_{C_2} F(L_n T_n x)J_n(x)dm = \int_{C_2}\chi_{[0,N]}(|||L_n x|||)F(L_n T_n x)J_n(x)dm \tag{5.3}$$

where

$$J_n(x) = \exp\left\{-\sum_{i=1}^{n}\sum_{j=1}^{n} K\left(\frac{i-1}{n}, \frac{j-1}{n}\right)x\left(\frac{i-1}{n}, \frac{j-1}{n}\right)\Delta_{ij}\right\}$$
$$\cdot \exp\left\{-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} K^2\left(\frac{i-1}{n}, \frac{j-1}{n}\right)x^2\left(\frac{i-1}{n}, \frac{j-1}{n}\right)\frac{1}{n^2}\right\}.$$

Since $K(s, t)$ and $x(s, t)$ are continuous on D

$$\lim_{n\to\infty}\exp\left\{-\frac{1}{2}-\sum_{i=1}^{n}\sum_{j=1}^{n} K^2\left(\frac{i-1}{n}, \frac{j-1}{n}\right)x^2\left(\frac{i-1}{n}, \frac{j-1}{n}\right)\frac{1}{n^2}\right\}$$
$$= \exp\left\{-\frac{1}{2}\int_D K^2(s,t)x^2(s,t)d\measuredangle\right\}.$$

Let

$$f_n(s,t) = K\left(\frac{[ns]}{n}, \frac{[nt]}{n}\right)X\left(\frac{[ns]}{n}, \frac{[nt]}{n}\right)$$

and $f(s, t) = K(s, t)X(s, t)$. Then

$$\sum_{i=1}^{n}\sum_{j=1}^{n} K\left(\frac{i-1}{n}, \frac{j-1}{n}\right)x\left(\frac{i-1}{n}, \frac{j-1}{n}\right)\Delta_{ij} = \left(\int f_n dX\right)(1, 1, x)\,.$$

By Theorem 3.2, since $f_n \in \mathfrak{M}_1$ and $f \in \mathfrak{M}_1$

$$\operatorname{var}\left[\left(\int f_n dX\right)(1,1,x) - \left(\int f dX\right)(1,1,x)\right] = \operatorname{var}\left[\left(\int (f_n - f)dX\right)(1,1,x)\right]$$
$$\leqq \int_D E(f_n - f)^2 d\measuredangle = ||f_n - f_n||^2\,.$$

Now

$$||f - f_n|| \leqq ||X(s, t)|| \cdot \left|\left|\left| K(s, t) - K\left(\frac{[ns]}{n}, \frac{[nt]}{n}\right) \right|\right|\right|$$
$$+ |||K||| \cdot \left|\left|\left| X(s, t) - X\left(\frac{[ns]}{n}, \frac{[nt]}{n}\right) \right|\right|\right| .$$

Using the continuity of $X(s, t)$ and $K(s, t)$ on D and the fact that $X(s, t) - X([ns]/n, [nt]/n)$ is normally distributed with mean 0 and variance $st - ([ns]/n) \cdot ([nt]/n)$ it is easy to see that $\lim_{n\to\infty} ||f_n - f|| = 0$. It follows that $\left(\int f_n dX\right)(1, 1, x)$ converges in m-measure to $\left(\int f dX\right)(1, 1, x)$ and hence

$$\exp\left\{-\sum_{i=1}^{n}\sum_{j=1}^{n} K\left(\frac{i-1}{n}, \frac{j-1}{n}\right)x\left(\frac{i-1}{n}, \frac{j-1}{n}\right)\Delta_{ij}\right\}$$

converges in m-measure to $\exp\left\{-\left(\int K(s, t)x(s, t)dX\right)(1, 1)\right\}$.

The integrand on the right side of (5.3) converges in m-measure to

$$\chi_{[0,N]}(|||x|||)F(Tx) \exp\left\{-\left(\int K(s, t)X(s, t)dX\right)(1, 1) - \frac{1}{2}\int_D K^2(s, t)x^2(s, t)d\mathscr{A}\right\} .$$

This follows from the above and the fact that $|||L_n x||| \leqq |||x|||$ and $\lim_{n\to\infty} |||L_n x - x||| = 0$, which implies

$$\lim_{n\to\infty} \chi_{[0,N]}(|||L_n x|||) = \chi_{[0,N]}(|||x|||) .$$

Since for each $n = 1, 2, \cdots$, the integrands on the right side of (5.3) are bounded in absolute value by

$$\chi_{[0,N]}(|||L_n x|||)|||F||| \exp\left\{-\left(\int f_n dX\right)(1, 1)\right\} .$$

Lemma 4 implies that the integrands are uniformly integrable justifying the taking of limits inside the integral. Thus

$$\lim_{n\to\infty}\int_{C_2} F(L_n T_n x)J_n(x)dm = \lim_{n\to\infty}\int_{C_2} \chi_{[0,N]}(|||L_n x|||)F(L_n T_n x)J_n(x)dm$$
$$= \int_{C_2} \chi_{[0,N]}(|||x|||)F(Tx)J(x)dm$$

where

$$J(x) = \exp\left\{-\left(\int K(s, t)X(s, t)dX\right)(1, 1) - \frac{1}{2}\int_D K^2(s, t)x^2(s, t)d\mathscr{A}\right\} .$$

Now $|||x||| > N$ implies $|||Tx||| > M$ and hence $F(Tx) = 0$. We get finally

$$\lim_{n \to \infty} \int_{C_2} F(L_n T_n x) J_n(x) dm = \int_{C_2} F(Tx) J(x) dm$$

which upon substitution into (5.1) proves Theorem 1.

The proof of Theorem 2 is proved using Theorem 1 in exactly the same way as Theorem III is proved from Theorem II in [4] by J. Yeh and is therefore omitted.

The author wishes to thank Professor J. Yeh for suggesting this problem.

REFERENCES

1. R. H. Cameron and W. T. Martin, *Transformations of Wiener integrals under a general class of linear transformations*, Trans. Amer. Math. Soc. **58** (1945), 184-219.
2. K. Ito, *On the stochastic differential equations*, Memoirs Amer. Math. Soc. **4** (1951).
3. J. Yeh, *Wiener measure in a space of functions of two variables*, Trans. Amer. Math. Soc. **95** (1960), 433-450.
4. ———, *Cameron-Martin translation theorems in the Wiener space of functions of two variables*, Trans. Amer. Math. Soc. **107** (1963), 409-420.

Received April 21, 1970. The paper is part of the author's thesis. The research was supported in part by the National Science Foundation (Grant No. GP-13288) and in part by the Air Force (Grant No. 1321-67).

UNIVERSITY OF CALIFORNIA, SANTA BARBARA