PACIFIC JOURNAL OF MATHEMATICS
Vol. 36, No. 2, 1971

ON A PARTITION PROBLEM OF H. L. ALDER

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We study $\Delta_d(n) = q_d(n) - Q_d(n)$, where $q_d(n)$ is the number of partitions of n into parts differing by at least d, and $Q_d(n)$ is the number of partitions of n into parts congruent to 1 or $d + 2 \pmod{d + 3}$. We prove that $\Delta_d(n) \to +\infty$ with n for $d \geqq 4$, and that $\Delta_d(n) \geqq 0$ for all n if $d = 2^s - 1, s \geqq 4$.

In 1956, H. L. Alder proposed the following problem [1].

"Let $q_d(n) =$ the number of partitions of n into parts differing by at least d; let $Q_d(n) =$ the number of partitions of n into parts congruent to 1 or $d + 2 \pmod{d + 3}$; let $\Delta_d(n) = q_d(n) - Q_d(n)$. It is known that $\Delta_1(n) = 0$ for all positive n (Euler's identity), $\Delta_2(n) = 0$ for all positive n (one of the Rogers-Ramanujan identities), $\Delta_3(n) \geqq 0$ for all positive n (from Schur's theorem which states $\Delta_3(n) =$ the number of those partitions of n into parts differing by at least 3 which contain at least one pair of consecutive multiples of 3). (a) Is $\Delta_d(n) \geqq 0$ for all positive d and n? (b) If (a) is true, can $\Delta_d(n)$ be characterized as the number of a certain type of restricted partitions of n as is the case for $d = 3$?"

This problem was again mentioned in [2; p. 743] as still being open. A recent general result on partitions with difference conditions [3] allows us to give some partial answers to Alder's problem.

First we derive a partition theorem which is somewhat analogous to the type of result asked for by Alder.

THEOREM 1. *Let ν be the largest integer such that $2^{\nu+1} - 1 \leqq d$. Let $\mathscr{L}_d(n)$ denote the number of partitions of n into distinct parts $\equiv 1, 2, 4, \cdots,$ or $2^\nu \pmod{d}$. Then*

$$q_d(n) \geqq \mathscr{L}_d(n) .$$

We may utilize some asymptotic formulae of Meinardus [4], [5] to prove

THEOREM 2. *For any $d \geqq 4$, $\lim_{n\to\infty} \Delta_d(n) = +\infty$*

Finally, Theorem 1 may be utilized to prove a result which settles Alder's problem in an infinite number of cases

THEOREM 4. *If $d = 2^s - 1$ and $s = 1, 2,$ or $\geqq 4$, then $\Delta_d(n) \geqq 0$ for all n.*

The proof of Theorem 4 relies on the following result which is of independent interest.

THEOREM 3. *Let* $S = \{a_i\}_{i=1}^{\infty}$ *and* $T = \{b_i\}_{i=1}^{\infty}$ *be two strictly increasing sequences of positive integers such that* $b_1 = 1$ *and* $a_i \geqq b_i$ *for all* i. *Let* $\rho(S; n)$ (resp. $\rho(T; n)$) *denote the number of partitions of* n *into parts taken from* S (resp. T). *Then*

$$\rho(T; n) \geqq \rho(S; n)$$

for all n.

2. Proof of Theorem 1. In Theorem 1 of [3] set $N = d$, $a(1) = 1$, $a(2) = 2, \cdots, a(\nu + 1) = 2^{\nu}$. Thus in the notation of [3], $D(A_N; n)$ becomes $\mathscr{L}_d(n)$. Now $D(A_N; n) = E(A'_N; n)$ where the latter partition function is the number of partitions of n:

$$n = b_1 + b_2 + \cdots + b_s,$$

$$b_i \equiv 1, 2, 3, 4, \cdots, 2^{\nu+1} - 1 \pmod{d}$$

with

$$b_i - b_{i+1} \geqq dw(\beta_d(b_{i+1})) + v(\beta_d(b_{i+1})) - \beta_d(b_{i+1}) .$$

Here $\beta_d(m)$ is the least positive residue of $m \bmod d$, $w(m)$ is the number of powers of 2 in the binary representation of m and $v(m)$ is the least power of 2 in the binary representation of m. Consequently if $b_{i+1} \equiv 2^j \pmod{d}$, $0 \leqq j \leqq \nu$,

$$dw(\beta_d(b_{i+1})) + v(\beta_d(b_{i+1})) - \beta_d(b_{i+1}) = d \cdot 1 + 2^j - 2^j = d .$$

If $b_{i+1} \not\equiv 2^j \pmod{d}$ $0 \leqq j \leqq \nu$, then

$$\begin{aligned} &dw(\beta_d(b_{i+1})) + v(\beta_d(b_{i+1})) - \beta_d(b_{i+1}) \\ &\qquad \geqq 2 \cdot d + 1 - (2^{\nu+1} - 1) \geqq 2d + 1 - d = d + 1 . \end{aligned}$$

Thus the difference condition is always $b_i - b_{i+1} \geqq d$ or stronger. Therefore $E(A'_N; n) \leqq q_d(n)$ and Theorem 1 follows.

3. Proof of Theorem 2. Meinardus has proved a general theorem on asymptotic formulae for partitions with repetitions [4]. Following the notation of Meinardus [4; pp. 388–389], we see that to treat $Q_d(n)$, we must have his

$$a_n = \begin{cases} 1 \text{ if } n \equiv 1, d + 2 \pmod{d + 3} \\ 0 \text{ otherwise} . \end{cases}$$

Under these circumstances, Meinardus's $D(s)$ satisfies

$$D(s) = (d + 3)^{-s}\left(\zeta\left(s, \frac{1}{d + 3}\right) + \zeta\left(s, \frac{d + 2}{d + 3}\right)\right) ,$$

where $\zeta(s, a) = \sum_{n=1}^{\infty}(n + a)^{-s}$, the Hurwitz zeta function [6; Ch. XIII], α, the abscissa of convergence for $D(s)$ is 1, and A, the residue at $s = 1$ is $2/d + 3$.

$$g(\tau) = \frac{e^{-\tau} + e^{-(d+2)\tau}}{1 - e^{-(d+3)\tau}} .$$

One may now easily verify that Meinardus's analytic conditions on $D(s)$ and $g(\tau)$ are fulfilled, thus

$$\log Q_d(n) \sim 2\pi\sqrt{\frac{n}{3d + 9}} . \tag{3.1}$$

In [5], Meinardus has derived the asymptotic formula

$$\log q_d(n) \sim 2\sqrt{A_d n} , \tag{3.2}$$

where

$$A_d = \frac{d}{2}\log^2 \alpha_d + \sum_{r=1}^{\infty}\frac{(\alpha_d)^{rd}}{r^2} ,$$

and α_d is real >0, $\alpha_d^d + \alpha_d - 1 = 0$.

If we put $\alpha_d = e^{-\lambda_d}$, so that $e^{-d\lambda_d} + e^{-\lambda_d} = 1$, then

$$\begin{aligned} A_d &= \frac{d}{2}\lambda_d^2 + \sum_{r=1}^{\infty}\frac{\alpha_d^{d\cdot r}}{r^2} > \frac{d}{2}\lambda_d^2 + \alpha_d^d \\ &= \frac{d}{2}\lambda_d^2 + 1 - e^{-\lambda_d} > \frac{d}{2}\lambda_d^2 + \lambda_d - \frac{1}{2}\lambda_d^2 \\ &= \frac{d-1}{2}\lambda_d^2 + \lambda_d . \end{aligned}$$

Now the following table shows that

$$A_d > \pi^2/(3d + 9) \text{ for } 4 \leqq d \leqq 14$$

TABLE 1.

d	$\lambda_d >$	$\frac{d-1}{2}\lambda_d^2 >$	$A_d >$	$\frac{\pi^2}{3d+9} <$
4	0.32	0.153	0.473	0.471
5	0.28	0.15	0.43	0.42
6	0.25	0.15	0.40	0.37
7	0.22	0.14	0.36	0.33
8	0.20	0.14	0.34	0.30
9	0.19	0.14	0.33	0.28
10	0.18	0.14	0.32	0.26
11	0.16	0.12	0.28	0.24
12	0.15	0.12	0.27	0.22
13	0.15	0.13	0.28	0.21
14	0.14	0.12	0.26	0.20

For $d \geqq 15$, we have

$$e^{-d(2/d)} + e^{-2/d} > e^{-2} + 1 - 2/d > 1 ,$$

Hence, $\lambda_d > 2/d$ and

$$A_d > \frac{d-1}{2}\left(\frac{2}{d}\right)^2 + 2/d = \frac{1}{d}(4 - 2/d) > \frac{10}{3d} > \frac{\pi^2}{3d+9} .$$

Thus for all $d \geqq 4$,

$$A_d > \frac{\pi^2}{3d+9} .$$

Hence comparing (3.1) with (3.2) we find

$$\lim_{n\to\infty} (\log q_d(n) - \log Q_d(n)) = +\infty .$$

Thus $\lim_{n\to\infty} \Delta_d(n) = \lim_{n\to\infty} q_d(n)(1 - Q_d(n)/q_d(n)) = +\infty$. and we have Theorem 2.

I would like to thank the referee for aid in simplifying and extending Theorem 2.

4. **Proof of Theorem 3.** Let us define $S_i = \{a_1, a_2, \cdots a_i\}$ and $T_i = \{b_1, b_2, \cdots, b_i\}$. We shall proceed to prove by induction on i that $\rho(T_i; n) \geqq \rho(S_i; n)$; this will establish Theorem 3 for if we choose I such that $a_I > n$, $b_I > n$, then $\rho(T; n) = \rho(T_I; n) \geqq \rho(S_I; n) = \rho(S; n)$.

First we remark that $\rho(T_i; n)$ is a nondecreasing function of n; this is because $1 = b_1 \in T_i$ and thus every partition of $n-1$ into parts taken from T_i may be transformed into a partition of n merely by adjoining a 1.

Now $\rho(T_1; n) = 1$ for all n since $T_1 = \{1\}$. Since $S_1 = \{a_1\}$

$$\rho(S_1; n) = \begin{cases} 1 \text{ if } a_1 \mid n \\ 0 \text{ otherwise} . \end{cases}$$

Hence

$$\rho(T_1; n) \geqq \rho(S_1; n) .$$

Now assume that $\rho(T_{i-1}; n) \geqq \rho(S_{i-1}; n)$ for all n. Hence if we define $\rho(T_i; 0) = \rho(S_i; 0) = 1$,

$$\begin{aligned}
&\sum_{n=0}^{\infty} (\rho(T_i; n) - \rho(S_i; n))q^n \\
&= \prod_{j=1}^{i} \frac{1}{1-q^{b_j}} - \prod_{j=1}^{i} \frac{1}{1-q^{a_j}} \\
&= \left(\prod_{j=1}^{i-1} \frac{1}{1-q^{b_j}}\right)\left(\frac{1}{1-q^{a_i}} + \frac{q^{b_i} - q^{a_i}}{(1-q^{a_i})(1-q^{b_i})}\right) - \prod_{j=1}^{i} \frac{1}{1-q^{a_j}}
\end{aligned}$$

$$= \frac{1}{1-q^{a_i}}\left(\prod_{j=1}^{i-1}\frac{1}{1-q^{b_j}} - \prod_{j=1}^{i-1}\frac{1}{1-q^{a_j}}\right) + \frac{q^{b_i}-q^{a_i}}{(1-q^{a_i})}\prod_{j=1}^{i}\frac{1}{1-q^{b_j}}$$
$$= \frac{1}{1-q^{a_i}}\Big(\sum_{n=0}^{\infty}(\rho(T_{i-1}; n) - \rho(S_{i-1}; n))q^n$$
$$+ \sum_{n=0}^{\infty}(\rho(T_i; n-b_i) - \rho(T_i; n-a_i))q^n\Big) .$$

Now the coefficients of these two infinite series are nonnegative: the first by the induction hypothesis, and the second by the fact that $\rho(T_i; n)$ is a nondecreasing sequence. Since $(1-q^{a_i})^{-1} = \sum_{j=0}^{\infty} q^{ja_i}$, we see that all coefficients in the power series expansion of our last expression must be nonnegative. Hence

$$\rho(T_i; n) \geqq \rho(S_i; n) ,$$

and Theorem 3 is proved.

5. Proof of Theorem 4. Since $d = 2^s - 1$, we see that the ν of Theorem 1 is just $s-1$. Now

$$\sum_{n=0}^{\infty}\mathscr{L}_d(n)q^n = \prod_{j=0}^{\infty}(1+q^{dj+1})(1+q^{dj+2})\cdots(1+q^{dj+2^\nu})$$
$$= \prod_{j=0}^{\infty}\frac{1}{(1-q^{2dj+1})(1-q^{2dj+d+2})(1-q^{2dj+d+4})\cdots(1-q^{2dj+d+2^\nu})} .$$

Thus $\mathscr{L}_d(n) = \rho(T; n)$ where $T = \{m \mid m \equiv 1, d+2, d+4, \cdots, \text{ or } d+2^{s-1} \pmod{2d}\}$. Clearly, $1 \in T$. We now show that for $s \geqq 4$ the i^{th} element of T (arranged in increasing magnitude) is no larger than the i^{th} element of S where $S = \{m \mid m \equiv 1, d+2 \pmod{d+3}\}$. Since $s \geqq 4$, the first four elements of T are

$$1, d+2, d+4, d+8\ (2d+5 > d+8 \text{ since } d \geqq 15) .$$

Thus the first four elements of T are less than or equal the first four elements of S respectively. In general the $(4m+1)-st$ element of T is $\leqq 2dm+1$ while the $(4m+1)-st$ element of S is $2m(d+3)+1$; for $2 \leqq j \leqq 4$ the $(4m+j)-$th element of T is $\leqq 2dm+d+2^{j-1}$ while the $(4m+j)-$ element of S is $\geqq 2m(d+3)+d+2$ and for $2 \leqq j \leqq 4$, $m \geqq 1$, $2dm+d+2^{j-1} \leqq 2dm+d+8 \leqq 2dm+d+6+2 \leqq 2m(d+3)+d+2$. Hence, the conditions of Theorem 3 are met, and therefore

$$q_d(n) \geqq \mathscr{L}_d(n) = \rho(T; n) \geqq \rho(S; n) = Q_d(n) .$$

Thus Theorem 4 is established.

6. Conclusion. By modification of the results in [3], it appears possible to apply the techniques of §4 to prove that $\Delta_d(n) \geqq 0$ for any $d \geqq 15$ which is a difference of powers of 2; however, since this approach does not yield a complete answer to Alder's problem it seems hardly worth undertaking.

Lengthier versions of the following table indicate that Alder's problem may be extended as follows.

Conjecture. $\Delta_d(n) > 0$ for $n \geqq d + 6$ if $d \geqq 8$.

n	$\Delta_3(n)$	$\Delta_4(n)$	$\Delta_5(n)$	$\Delta_6(n)$	$\Delta_7(n)$	$\Delta_8(n)$
1	0	0	0	0	0	0
2	0	0	0	0	0	0
3	0	0	0	0	0	0
4	0	0	0	0	0	0
5	0	0	0	0	0	0
6	0	0	0	0	0	0
7	0	0	0	0	0	0
8	0	0	0	0	0	0
9	1	0	0	0	0	0
10	0	1	0	0	0	0
11	0	1	1	0	0	0
12	0	1	1	1	0	0
13	0	0	2	1	1	0
14	0	0	1	2	1	1
15	1	0	1	2	2	1
16	1	0	0	2	2	2
17	1	1	0	1	3	2
18	1	2	0	1	2	3
19	1	2	1	0	2	3
20	1	2	2	0	1	3
21	2	2	3	1	1	2
22	2	2	3	2	0	2
23	2	3	3	3	1	1
24	2	4	3	4	2	1

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Received April 1, 1970. Partially supported by NSF Grant GP-9660.

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