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THE REGULARITY OF MINIMAL SURFACES DEFINED OVER SLIT DOMAINS

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Let Ω denote the disc $x_1^2 + x_2^2 < r^2$ in the $x = (x_1, x_2)$ plane from which the segment $\{0 \leqq x_1 < r, x_2 = 0\}$ has been deleted. Suppose that $u(x) \in C^0(\bar{\Omega})$ is a solution to the minimal surface equation in Ω((1) below) and attains boundary values $f(x_1) \in C^{1,\alpha}(0 < \alpha < 1)$ on the slit $\{0 \leqq x_1 < r, x_2 = 0\}$. We shall prove here that the gradient of u, $Du = (u_{x_1}, u_{x_2})$, is continuous at the origin $x = 0$.

There is a corresponding result for harmonic functions, due to H. Lewy [7], which we paraphrase here. If $u(x) \in C^0(\bar{\Omega})$ is harmonic and attains boundary values $f(x_1) \in C^{1,\alpha}(0 < \alpha < 1)$ on the slit $\{0 \leqq x_1 < r, x_2 = 0\}$, then

$$\liminf_{h \uparrow 0} \frac{1}{h}(u(h, 0) - u(0, 0)) = \begin{cases} \infty, \text{ or} \\ -\infty, \text{ or} \\ f'(0) . \end{cases}$$

When the last alternative holds, $Du(x)$ is continuous at $x = 0$. The harmonics $u_{\pm}(x) = \pm\rho^{1/2}\sin\theta/2$, $x = \rho e^{i\theta}$, illustrate the occurrence of the ∞ and $-\infty$ as possible limit values. The result to be proven here is, then, another example of the greater regularity possessed by solutions of the minimal surface equation (cf Bers [2], Nitsche [9], and [4]).

As an application, we consider the problem of minimizing the non-parametric area integrand among functions constrained to lie above a given function defined on a segment in a domain. More precisely, let P be a bounded, open, convex domain with smooth boundary, σ a closed straight segment in P, and $f(x)$ a continuous nonnegative convex function on σ which vanishes at the endpoints of σ. Denote by

$$\mathscr{K} = \{v(x) \in C^{0,1}(\bar{P}) \colon v(x) \geqq f(x) \text{ on } \sigma \text{ and } v = 0 \text{ on } \partial P\} .$$

The problem is then

(A) Prove that there exists a $u(x) \in \mathscr{K}$ such that

$$\iint_P \sqrt{1 + |Du(x)|^2}\, dx = \min_{v \in \mathscr{K}} \iint_P \sqrt{1 + |Dv(x)|^2}\, dx .$$

Evidently, a solution to A, if it exists, satisfies (1) in the set

$\{x \in P: u(x) > f(x)\}$. Johannes C. C. Nitsche [10], considering, in fact, a larger class of surfaces than $\mathscr{K}$ above has proven:

(B) If P is symmetric with respect to a line and σ lies on this line of symmetry, then there exists a solution to A.

Furthermore, he has shown:

(C) If a solution to A exists, it is unique. Moreover the set $\tau = \{x \in P: u(x) = f(x)\}$ is a (connected) sub-interval of σ.

Using the theorem to be proved here in addition to some similar elementary considerations, we may prove

THEOREM I. *If $u(x)$ is a solution to A where $f \in C^{1,\alpha}(\sigma)$, $0 < \alpha < 1$, then $\partial u/\partial x_1$ is continuous in $\bar{P}$ and $\partial u/\partial x_2$ is continuous in $\bar{P}$-τ and upon one-sided approach to τ. In addition $|\partial u/\partial x_1|$ is bounded by a constant depending only on P, σ, and f.*

For the solution of B, Nitsche has shown the second part of Theorem I([10], p. 105). We remark briefly on the proof of Theorem I at the conclusion of this paper. Primarily, we wish to prove

THEOREM II. *Let $u(x) \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfy*

$$(1) \quad (1 + u_{x_2}^2)u_{x_1x_1} - 2u_{x_1}u_{x_2}u_{x_1x_2} + (1 + u_{x_1}^2)u_{x_2x_2} = 0 \ \textit{in} \ \Omega$$
$$u(x_1, 0) = f(x_1),\ 0 \leqq x_1 < r\ ,$$

where $f(x_1) \in C^{1,\alpha}([0, r])$, $0 < \alpha < 1$.
Then $Du(x)$ is continuous at $x = 0$.

To prove Theorem II, we shall utilize known properties of the conformal representation of the surface

$$S = \{(x, x_3): x_3 = u(x),\ x \in \Omega\}$$

together with Lemma 1 below. In brief, S may be viewed as a minimal surface whose boundary contains a spike. The boundary behavior of such surfaces is known. We quote here Theorems D and E. To compute u_{x_1}, u_{x_2} in terms of parameters (ξ, η) different from (x_1, x_2) involves the determination of three functional determinants, one of which, the Jacobian $J = \partial(x_1, x_2)/\partial(\xi, \eta)$, occurs as a denominator. The fact that S has a one-to-one projection onto a slit domain is used to show that J has "lowest order" among the three determinants.

We close with remarks about extensions to weaker boundary regularity.

2. The conformal representation and its properties. In this paragraph we introduce conformal parameters so that the minimal surface $S = \{(x, x_3): x_3 = u(x), x \in \Omega\}$ in (x_1, x_2, x_3) space may be considered to be a minimal surface with a spike (cf [4]). We then determine regularity properties of this representation.

Denote by G the open upper half $\zeta = \xi + i\eta$ plane. By a conformal representation of S we shall understand a triple of harmonic functions.

$$X(\zeta) = (x_1(\zeta), x_2(\zeta), x_3(\zeta)), \zeta \in G$$

continuous in $\bar{G}$ and admitting finite limits at $\pm\infty$, which is a one-to-one map of G onto S and satisfies the isothermal relations

$$X_\xi(\zeta)^2 = X_\eta(\zeta)^2 \text{ and } X_\xi(\zeta) \cdot X_\eta(\zeta) = 0, \zeta \in G .$$

According to a result of Beckenbach and Rado [1], such a representation for S exists because $u \in C^0(\bar{\Omega})$. We may assume that $X(0) = (0, 0, f(0))$ and that the curve $x_3 = f(x_1)$, $x_2 = 0$, $0 \leqq x_1 < r$, is the one-to-one continuous image of $-\xi_1 < \xi \leqq 0$ and the one-to-one continuous image of $0 \leqq \xi < \xi_2$, for some $\xi_1, \xi_2 > 0$.

For the discussion which follows, it is more convenient to consider the conformal representation

$$Y(\zeta) = (y_1(\zeta), y_2(\zeta), y_3(\zeta)), \zeta \in G$$

obtained from $X(\zeta)$ above through the Euclidean motion

$$\begin{aligned} y_1 &= x_1 \cos\beta + (x_3 - f(0)) \sin\beta \\ y_2 &= x_2 \\ y_3 &= -x_1 \sin\beta + (x_3 - f(0)) \cos\beta , \end{aligned} \tag{2}$$

where

$$\beta = \text{arc tan } f'(0) .$$

Note that $|\beta| < \pi/2$. Evidently, $dy_1/dx_1|_{x_1=0} > 0$ and $dy_3/dx_1|_{x_1=0} = 0$ on the curve $x_3 = f(x_1)$, $x_2 = 0$, $0 \leqq x_1 < r$.

After a conformal mapping of G onto itself, if necessary, the conformal representation $Y(\zeta)$ satisfies these conditions:

$y_1(\xi)$ is strictly decreasing from $\bar{y}$ to 0 for $-1 < \xi \leqq 0$

$y_1(\xi)$ is strictly increasing from 0 to $\bar{y}$ for $0 \leqq \xi < 1$,

for some $\bar{y} > 0$, and

$$y_2(\xi) = 0, y_3(\xi) = g(y_1(\xi)) \text{ for } |\xi| < 1$$

where $g(y_1)$ is the $C^{1,\alpha}$ function of y_1 obtained by setting $x_3 = f(x_1)$.

The conformal representation $Y(\zeta)$ is a representation of S as a minimal surface with the spike

$$\Gamma: y_3 = g(y_1), y_2 = 0, 0 \leqq y_1 < \bar{y}; g(0) = g'(0) = 0 .$$

Let $F_j(\zeta) = y_j(\zeta) + iy_j^*(\zeta)$, where $y_j^*(\zeta)$ denotes the harmonic conjugate to $y_j(\zeta)$, $F_j(0) = 0$, $j = 1, 2, 3$. It is well known, [12], that $F_j(\zeta)$ have absolutely continuous boundary values for $\operatorname{Im}\zeta = 0$. About the $F_j(\zeta)$ we state Theorems D and E which are Theorem 1 [4] together with its corollary and Theorem 4′ [5] respectively.

THEOREM D. *There is a neighborhood* $U = \{|\zeta| < R, \operatorname{Im}\zeta > 0\}$ *and a branch of* $z = F_1(\zeta)^{1/m}$, $m > 0$ *even integer, such that* $z = F_1(\zeta)^{1/m}$ *is a univalent map of* U *onto a domain in the* (ordinary) $z = x + iy$ *plane.*

The curve γ *which is the image of* $[-1, 1] \cap \bar{U}$ *under this mapping meets at a straight angle at* $z = 0$. *Its tangent has a modulus of continuity proportional to* $g'(y_1)$ *at* $z = 0$.

THEOREM E. *There is a neighborhood* $U = \{|\zeta| < R, \operatorname{Im}\zeta > 0\}$ *such that*

$$F_1(\zeta)^{1/m},\ F_j(\zeta) \in C^{1,\alpha}(\bar{U}),\ j = 2, 3\ ,$$

where $m > 0$, *even, is the integer determined in Theorem D.*

For the proof of E, we refer to Theorem 4 in [5]. In addition to the facts just quoted, we require

LEMMA 1. *The functions* F_j' *admit the expansions*

$$F_j'(\zeta) = a_j\zeta + b_j(\zeta),\ \zeta \in \bar{U},\ j = 1, 2, 3$$

where a_1 *is real,* a_2, a_3 *are imaginary,* $|a_1| \geqq |a_2| > 0$ *and* $|b_j(\zeta)| \leqq C|\zeta|^{1+\alpha}$ *for* $\zeta \in \bar{U}$, $C > 0$, *a constant.*

The asymptotic expansion of the $F_j'(\zeta)$ provided by Theorem E, and stated explicitly in Lemma 1, is similar to those in [11], which is for minimal surfaces, and [3] which is for surfaces satisfying certain assumptions about their mean curvature. Both of these require the boundary to be of class C^2 and "regular," although the constants corresponding to a_j and C above depend only *a priori* on the given data. However, the existence of the tangent plane to a minimal surface when the boundary is suitably smooth has been known for some time [8].

3. Proof of Theorem II assuming Lemma 1. In terms of the given (x_1, x_2, x_3) coordinates, the mapping $\zeta \to x(\zeta) = (x_1(\zeta), x_2(\zeta))$ is a one-to-one harmonic mapping. In view of (2), its Jacobian is

$$\begin{aligned} J &= \operatorname{Im}(F_1'(\zeta)\cos\beta - F_3'(\zeta)\sin\beta)\,\overline{F_2'(\zeta)} \\ &= i\,a_1a_2\cos\beta|\zeta|^2 + \operatorname{Im}\{a_1\zeta\overline{b_2(\zeta)} + \bar{a}_2\bar{\zeta}b_1(\zeta) + b_1(\zeta)\overline{b_2(\zeta)}\}\cos\beta \\ &\quad - \operatorname{Im}\{a_3\zeta\overline{b_2(\zeta)} + \bar{a}_2\bar{\zeta}b_3(\zeta) + b_3(\zeta)\overline{b_2(\zeta)}\}\sin\beta \\ &= i\,a_1a_2\cos\beta\,|\zeta|^2 + B_1(\zeta),\ \zeta\in\bar{U}\,. \end{aligned}$$

Here we have used that a_1 is real and a_2, a_3 are imaginary. After two similar computations, we find that

$$\frac{\partial(x_2, x_3)}{\partial(\xi, \eta)} = -\,ia_1a_2\sin\beta\,|\zeta|^2 + B_2(\zeta),\ \zeta\in\bar{U}$$

and

$$\frac{\partial(x_1, x_3)}{\partial(\xi, \eta)} = ia_1a_3\,|\zeta|^2 + B_3(\zeta),\ \zeta\in\bar{U}\,.$$

The $B_j(\zeta)$ satisfy $|B_j(\zeta)| \leqq C|\zeta|^{2+\alpha}$ for a constant $C > 0$.

Therefore, for x in the image of $\bar{U}$ under $x(\zeta)$,

$$\frac{\partial u}{\partial x_1}(x) = f'(0) + R(\zeta),\ \text{where}\ |R(\zeta)| \leqq \text{const.}\ \ |\zeta|^\alpha\,.$$

But an elementary computation reveals that $x_1^2 + x_2^2 \geqq \text{const.}\ \ |\zeta|^4$, for $|\zeta|$ sufficiently small. Hence

$$\left|\frac{\partial u}{\partial x_1} - f'(0)\right| \leqq \text{const}\,|x|^{\alpha/2}\ \text{for}\ x\in\bar{\Omega},\ |x|$$

sufficiently small. In the same way

$$\left|\frac{\partial u}{\partial x_2} - \frac{1}{\cos\beta}\frac{a_3}{a_2}\right| \leqq \text{const}\,|x|^{\alpha/2}\ \text{for}\ x\in\bar{\Omega}\ ,$$

$|x|$ sufficiently small. Here we have used the abbreviation "const." to denote a positive constant, not necessarily the same at each occurrence.

The question of determining an a priori limitation of $(\partial u/\partial x_2)(0)$ is different in nature, and will be considered elsewhere.

4. Proof of Lemma 1. The proof of Lemma 1 is divided into the two lemmas below. Note that the strict monotonicity of $y_1(\xi)$ in $-1 < \xi \leqq 0$ and $0 \leqq \xi < 1$ implies the existence of continuous functions $H_j(y_1)$, $j = 1, 2$, such that $y_1^*(\xi) = H_1(y_1(\xi))$ for $-1 < \xi \leqq 0$ and $y_1^*(\xi) = H_2(y_1(\xi))$ for $0 \leqq \xi < 1$.

LEMMA 2. (a) *$H_j(y_1)$ are absolutely continuous functions of y_1 and $|H_j'(y_1)| \leqq C_1|g'(y_1)|$, a.e., $0 \leqq y_1 \leqq \bar{y}$, $C_1 > 0$ constant.*

(b) $$\lim_{\xi\to 0}\left|\frac{\partial y_2^*}{\partial\xi}(\xi)\left(\frac{\partial y_1}{\partial\xi}(\xi)\right)^{-1}\right| \leqq 1$$

(c) $|F_j'(\xi)| \leqq C_2|F_1'(\xi)| \leqq C_3|\xi|^{m-1}$ *for* $|\xi| < 1, \xi \in \bar{U}, j = 2, 3,$ *where* $m \geqq 2$ *is the integer determined in Theorem D and* $C_2, C_3 > 0$ *are constants.* U *is the set of Theorem E.*

Proof. Let s denote the arc length of the minimal surface on $\Gamma: y_3 = g(y_1), y_2 = 0, 0 \leqq y_1 \leqq \bar{y}$. According to Tsuji's result [12],

$$0 \neq \left(\frac{ds}{d\xi}\right)^2 = (1 + g'(y_1)^2)(\partial y_1/\partial\xi)^2, \text{ a.e. for } |\xi| < 1 .$$

Therefore, $\partial y_1/\partial\xi \neq 0$ a.e. for $-1 < \xi < 1$. It follows that the inverse function $\xi = h(y_1)$ to $y_1(\xi)$ on $-1 < \xi \leqq 0$ is absolutely continuous for $0 \leqq y \leqq \bar{y}$. Since h is also monotone, $H_1(y_1) = y_1^*(h(y_1))$ is absolutely continuous for $0 \leqq y \leqq \bar{y}$.

Furthermore,

$$\left(\frac{ds}{d\xi}\right)^2 = \sum_1^3 \left(\frac{\partial y_j^*}{\partial\xi}\right)^2 \text{ for } |\xi| < 1 . \tag{3}$$

Hence for a constant $C_1 > 0$,

$$\sum_1^3 \left(\frac{y_{j\xi}^*}{y_{1\xi}}\right)^2 \leqq \sup\ (1 + g'(y_1)^2) = C_1^2 \text{ for } |\xi| < 1 .$$

Using the isothermal relation

$$\sum_1^3 y_{j\xi}(\xi)\, y_{j\xi}^*(\xi) = 0, |\xi| < 1 ,$$

we obtain that

$$H_1'(y_1) = -g'(y_1)\frac{\partial y_3^*}{\partial\xi}\left(\frac{\partial y_1}{\partial\xi}\right)^{-1}, \text{ a.e. for } -1 \leqq \xi \leqq 0 .$$

Hence

$$|H_1'(y_1)| \leqq C_1|g'(y_1)| \text{ a.e., } -1 \leqq \xi \leqq 0 .$$

Now from (3),

$$(y_{2\xi}^*(\xi))^2 \leqq (1 + g'(y_1)^2)y_{1\xi}(\xi)^2, |\xi| < 1 .$$

Hence (b) follows.

Finally

$$|F_j'(\xi)|^2 \leqq \sum_1^3 |F_j'(\xi)|^2 = 2\left(\frac{ds}{d\xi}\right)^2 \leqq 2(1 + g'(y_1)^2)|F_1'(\xi)|^2$$

which implies that

$$|F_j'(\xi)| \leqq \sqrt{2}\, C_1|F_1'(\xi)| \text{ for } |\xi| < 1 .$$

Now $F_1(\zeta)^{1/m} \in C^{1,\alpha}(\bar{U})$, for a suitable U, by Theorem E; hence,

$$F_1(\xi) = \frac{1}{m} a_1 \xi^m + A_1(\xi)$$

and $F_1'(\xi) = a_1 \xi^{m-1} + b_1(\xi)$, $|\xi| < 1$ and $\xi \in \bar{U}$, $|b_1(\xi)| \leqq$ const. $|\xi|^{m-1+\alpha}$ and $a_1 \neq 0$. That $a_1 \neq 0$ is insured by the existence of a tangent with a suitable modulus of continuity to the curve γ: $z = F_1(\xi)^{1/m}$, $\xi \in \bar{U}$, (cf Theorem D). Also, $|F_1'(\xi)| \leqq$ const. $|\xi|^{m-1}$, $\xi \in \bar{U}$, from which (c) follows.

LEMMA 3. *$F_2(\zeta)$ admits the representation*

$$F_2(\zeta) = \tfrac{1}{2} a_2 \zeta^2 + \sum_{k>2} c_k \zeta^k, \ |\zeta| < 1$$

where $a_2 \neq 0$, c_k are imaginary.

Also the integer $m = 2$.

Proof. Since Re $F_2(\xi) = y_2(\xi) = x_2(\xi) = 0$ for $|\xi| < 1$, F_2 admits a development as that above, perhaps with a linear term, with a_2, c_k imaginary. We must demonstrate that $a_2 \neq 0$ and $c_1 = 0$. This follows from a well known argument about harmonic mappings [2]. The mapping $\zeta \to (x_1(\zeta), y_2(\zeta))$ is a one-to-one harmonic map. Hence by a lemma of Lewy [6], $\partial(x_1, y_2)/\partial(\xi, \eta) \neq 0$ in $|\zeta| < 1$, Im $\zeta > 0$, and therefore $F_2'(\zeta) \neq 0$ in $|\zeta| < 1$, Im $\zeta > 0$. For λ real, we consider the inverse image

$$C = \{|\zeta| < 1, \ \mathrm{Im}\zeta > 0 \colon y_2(\zeta) = \lambda\}$$

of $y_2 = \lambda$ in Ω. If not empty, C is an analytic curve in Im$\zeta > 0$, $|\zeta| < 1$ since $\zeta \to (x_1, y_2)$ is an analytic homeomorphism whose Jacobian does not vanish. For $\zeta \in C$,

$$F_2'(\zeta) = \left(\frac{\partial y_2}{\partial t} + i \frac{\partial y_2^*}{\partial t}\right)\left(\frac{d\zeta}{dt}\right)^{-1} \neq 0 ,$$

where t denotes the tangent direction on C. Hence $dy_2^*/dt \neq 0$ on C, so that $F_2(\zeta)$ is monotone on C. Hence F_2 is univalent in $|\zeta| < 1$, Im $\zeta > 0$, from which it follows that

$$F_2(\zeta) = \frac{1}{n} a_n \zeta^n + \sum_{k>n} c_k \zeta^k, \text{ with } a_n \neq 0, \ n \leqq 2 .$$

By the previous lemma

$$|F_2'(\xi)| \leqq c_1 |F_1'(\xi)| \leqq c_2 |\xi|^{m-1}, \ m \geqq 2 \text{ even} .$$

Therefore $2 \geqq n \geqq m \geqq 2$ or $m = n = 2$.

Proof of Lemma 1. Since $m = 2$, we know that

$$F'_j(\zeta) = a_j\zeta + b_j(\zeta),\ \zeta \in \bar{U},\ \text{with } |b_j(\zeta)| \leqq C|\zeta|^{1+\alpha}$$

for $j = 1, 2, 3$. By Lemma 2(b) and Lemma 3,

$$|a_1| \geqq |\mathrm{Re}\, a_1| \geqq |a_2| > 0\ .$$

It remains to show that a_1 is real and a_3 is imaginary. Using Lemma 2(a),

$$H_1'(y_1) = \frac{\mathrm{Im}\, F'_1(\xi)}{\mathrm{Re}\, F'_1(\xi)} = \frac{\mathrm{Im}\, a_1 + \mathrm{Im}\, b_1(\xi)\xi^{-1}}{\mathrm{Re}\, a_1 + \mathrm{Re}\, b_1(\xi)\xi^{-1}},\ \xi < 0\ ,$$

and $|H_1'(y_1)| \leqq C_1|g'(y_1)| \to 0$ as $y_1 \to 0$. Hence $\mathrm{Im}\, a_1 = 0$. Now according to the isothermal relations

$$\sum F'_j(\zeta)^2 = 0\ ,$$

hence $a_1^2 + a_2^2 + a_3^2 = 0$. Since a_1 is real, a_2 is imaginary, and $|a_1| \geqq |a_2|$, the relation implies that $(a_3)^2 \leqq 0$. Hence a_3 is imaginary.

We wish to remark here that by assuming only that $f'(x_1)$ satisfies $\int_0^a t^{-1}|f'(t)|dt < \infty$, some $a > 0$, it is possible to prove that $\partial u/\partial x_1$ is continuous as $x \to 0$ in any sector $0 < \tau \leqq \arg x \leqq 2\pi - \tau$. The proof is by the same argument, except that Theorem E must be replaced by a fact analogous to the existence of the angular derivative as proved by S. Warschawski [13]. This fact, whose proof requires a generalization of a classical theorem of Lindelöf, is not difficult to prove.

We now remark briefly on the proof of Theorem I. The technique by which continuity of $Du(x)$ was shown at the end points of the segment τ in Theorem II may be utilized in a simpler fashion to show that $u_{x_1}(x)$ and $u_{x_2}(x)$ are continuous at each interior point of τ. Continuity of $u_{x_2}(x)$ is understood to mean continuity upon one-sided approach to τ. In fact, the functions analogous to $F'_j(\zeta)$ in Lemma 1 admit an expansion of the form "$a_j + b_j(\zeta)$" with $|b_j(\zeta)| \leqq c|\zeta|^\alpha$, suitable $c > 0$, where the a_j satisfy the conclusions of Lemma 1.

Given $x^0 \in \partial P$, $Du(x^0)$ may be estimated by the slopes of the plane tangent to the space curve ∂P at $(x_1^0, x_2^0, 0)$ and some point of the curve $x_3 = f(x_1)$, $x_2 = 0$. This estimate depends only on the given data. Finally, we observe that $u_{x_1}(x)$ satisfies a maximum principle in $P - \tau$. Hence $\sup_{\bar{P}}|u_{x_1}(x)| \leqq \max(\sup_{\partial P}|Du(x)|, \sup|f'(x_1)|)$.

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