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THE FUNCTIONS OF BOUNDED INDEX AS A SUBSPACE OF A SPACE OF ENTIRE FUNCTIONS

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Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ be entire functions. Define $d(f, g) = \text{Sup}\,\{|a_0 - b_0|, (|a_n - b_n|)^{1/n}\, n = 1, 2, \cdots\}$. It is the purpose of this note to show that, in the topology generated by d, the entire functions of bounded index, B, are of the first category.

Further, for Γ, the corresponding space of all entire functions, and $B_n = \{f \in B \,|\, \text{the index of } f \text{ is} \leqq n\}$ is shown that $B - B_n$ is dense in Γ for any nonnegative integer n. It is also shown that $\Gamma - B$ is dense in Γ. (For definition and main results see [2], [3].)

LEMMA 1. *For any $f \in \Gamma$, $N \geqq 0$, and $\varepsilon > 0$ there exists a $\delta > 0$ such that if $g \in \Gamma$ and $d(f, g) < \delta$ then $d(f^{(k)}, g^{(k)}) < \varepsilon$ for $k = 0, 1, \cdots, N$.*

Proof. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \Gamma$, $N \geqq 0$, and $\varepsilon > 0$ be given. Let

$$T > \text{Sup}\left\{\left(\frac{(n+k)!}{n!}\right)^{1/n} \,|\, n = 1, 2, \cdots \text{ and } k = 0, 1, \cdots, N.\right\}.$$

It is straightforward to verify that if $g(z) = \sum_{n=0}^{\infty} b_n z^n \in \Gamma$ and $d(f, g) < \dfrac{\varepsilon}{T + \varepsilon}$ then

$$\begin{aligned} d(f^{(k)}, g^{(k)}) &= \text{Sup}\left\{k!\,|a_k - b_k|, \left(\frac{(n+k)!}{n!}\,|a_{n+k} - b_{n+k}|\right)^{1/n} n = 1, 2, \cdots\right\} \\ &< T \cdot \frac{\varepsilon}{T + \varepsilon} < \varepsilon \text{ for } k = 0, 1, \cdots, N. \end{aligned}$$

REMARK. If $f \in \Gamma - B$ then f is said to be of unbounded index and the index of $f = \infty$.

LEMMA 2. *If n is a nonnegative integer and f is of index $> n$* (bounded or unbounded) *then there exists a $\delta > 0$ such that if $g \in \Gamma$ and $d(f, g) < \delta$ then $g \in \Gamma - B_n$.*

Proof. Let n be given such that $n \geqq 0$. Let $f \in \Gamma$ be given such that the index of f (bounded or unbounded) is $> n$. Let k be a positive integer $> n$ and z_1 a complex number such that f is of index k at the point z_1. Let $\delta_1 > 0$ be such that for

$$j < k, \frac{|f^{(k)}(z_1)|}{k!} - \delta_1 > \frac{|f^{(j)}(z_1)|}{j!} .$$

Let $R \geqq |z_1|$. It is known that for every $j \leqq k$ there exists an ε_j such that if $g_j \in \Gamma$ and $d(f^{(j)}, g_j) < \varepsilon_j$ then $|f^{(j)}(z) - g_j(z)| < \delta_1/2$ for $|z| \leqq R$, and in particular at z_1 [1; p. 220]. In Lemma 1 we let $N = k$ and $\varepsilon = \text{Min}\,\{\varepsilon_0, \varepsilon_1, \cdots, \varepsilon_k\}$. Hence there exists a δ such that for $g \in \Gamma$ and $d(f, g) < \delta$ we have

$$\frac{|g^{(k)}(z_1)|}{k!} > \frac{|g^{(j)}(z_1)|}{j!} \text{ for } j = 0, 1, \cdots, k-1 .$$

Thus g is of index $\geqq k > n$.

LEMMA 3. *If $p(z)$ is a polynomial of degree n then $h(z) = e^z + p(z)$ is of index $\leqq n + 1$.*

Proof. Let $k > n + 1$. Thus,

$$\frac{|h^{(k)}(z)|}{k!} = \frac{|e^z|}{k!} < \frac{|e^z|}{(n+1)!} = \frac{|h^{(n+1)}(z)|}{(n+1)!}$$

and hence h is of index $\leqq n + 1$.

THEOREM 1. *For any n, B_n* is nowhere dense in B and thus $B = \bigcup_{k=0}^{\infty} B_k$ *is of the first category.*

Proof. Let n be given. Lemma 2 shows that B_n is closed. Thus let $f \in B_n$ and $\varepsilon > 0$. Let

$$e^{z^2} = \sum_{k=0}^{\infty} b_k z^k,\ f(z) = \sum_{k=0}^{\infty} a_k z^k, \text{ and } f_j(z) = \sum_{k=0}^{j} a_k z^k + \sum_{k=j+1}^{\infty} b_k z^k .$$

Since the order of f_j is two, for every j, we have that $f_j \in \Gamma - B$ [3]. Let i be such that $d(f, f_i) < \varepsilon/2$ and let $f_i = \sum_{k=0}^{\infty} c_k z^k$. For every $j > 0$ let $g_j(z) = \sum_{k=0}^{j} c_k z^k + \sum_{k=j+1}^{\infty} z^k/k!$. By the previous lemma the index of g_j is $\leqq j + 1$. Thus, for every j, $g_j \in B$. In Lemma 2 we let $\delta < \varepsilon/2$ be such that if $g \in \Gamma$ and $d(f_i, g) < \delta$ then the index of g is $\geqq n + 1$. Let m be such that $d(f_i, g_m) < \delta$. Thus $d(f, g_m) < \varepsilon$ and $g_m \in B - B_n$. Hence, for every integer n, B_n is nowhere dense in B and $B = \bigcup_{k=0}^{\infty} B_k$ is of the first category.

THEOREM 2. *The following are dense in Γ:*
(a) *$\Gamma - B$; and*
(b) *$B - B_n$, for any integer n.*

Proof. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \Gamma$.

(a) Let

$$e^{z^2} = \sum_{k=0}^{\infty} b_k z^k, \text{ and } f_j = \sum_{k=0}^{j} a_k z^k + \sum_{k=j+1}^{\infty} b_k z^k .$$

As in the proof of Theorem 1, $f_j \in \Gamma - B$ for every j and $\lim_{j\to\infty} d(f, f_j) = 0$.

(b) Now let

$$f_j(z) = \sum_{k=0}^{j} a_k z^k + \sum_{k=j+1}^{\infty} \frac{z^k}{k!} .$$

By Lemma 3, f_j is of bounded index for every j. For each j, if the index of f_j is $> n$ let $g_j = f_j$. If the index of f_j is $\leqq n$ then by Theorem 1 there exists a function $g \in B - B_n$ such that $d(f_j, g) < 1/j$. Let $g_j = g$. Hence the $\lim_{j\to\infty} d(f, g_j) = 0$ and for every j, $g_j \in B - B_n$.

In conclusion it should be noted that the polynomials could be excluded from the class of entire functions, Γ, and the proofs of the preceeding Lemmas and Theorems would remain valid.

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