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BV-FUNCTIONS ON SEMILATTICES

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It has been shown that the cone C of completely monotonic functions on a commutative semigroup G with identity induces a vector lattice ordering on the vector space $E = C - C$ spanned by C. An intrinsic characterization of the absolute value of the functions in E is desirable. In the present work we offer such a characterization when each member of G is idempotent, i.e. G is a semilattice. A notion of variation and bounded variation (BV) of arbitrary functions on G is introduced. We show that E is precisely the family of BV-functions and that if $f \in E$, then our concept of variation of f agrees with the usual absolute value as given by $f \vee (-f)$.

In case the natural order on G is linear, then C is the cone of nonnegative, nondecreasing functions and our notions of variation and bounded variation reduce to the classical concepts. More generally, we show that our variation reduces (not trivially) to the variation defined by Birkhoff [2] for BV-valuations on a distributive lattice with largest element.

1. Completely monotonic functions on semilattices. In order to set the stage for our investigations, it will be necessary for us to recall [cf. 1] how the integral representation of a completely monotonic function simplifies when the underlying semigroup is a semilattice.

If f is a real-valued function defined on a commutative semigroup G with identity 1, then the difference operators Δ_n, for n a nonnegative integer, are defined inductively by $\Delta_0 f(a) = f(a)$, and $\Delta_n f(a; b_1, \cdots, b_n) = \Delta_{n-1} f(a; b_1, \cdots, b_{n-1}) - \Delta_{n-1} f(ab_n; b_1, \cdots, b_{n-1})$. The function f is said to be *completely monotonic* if $\Delta_n f(a; b_1, \cdots, b_n) \geqq 0$ for all choices of $a, b_1, \cdots, b_n \in G$. Let $C = C(G)$ denote the family of all completely monotonic functions on G and $C_1 = \{f \in C: f(1) = 1\}$. Then C is a convex cone with base [9] C_1, in the linear space R^G of all real valued functions on G. If we equip R^G with the topology of pointwise convergence, then the span, $E = C - C$, of C becomes a locally convex linear topological space and C_1 is compact. From [5] we see that C_1 is an r-simplex, i.e. every $f \in C_1$ admits a unique representing measure which is supported by the extremal points (ext C_1) of C_1, and ext C_1 is closed. A nontrivial homomorphism from G into the multiplicative semigroup of real numbers in the closed unit interval is called an *exponential*. The set of exponentials on G

is denoted by $\exp G$. It is known [cf. 1 or 5] that $\operatorname{ext} C_1 = \exp G$. It follows that every $f \in E$ admits a unique representing measure μ_f supported by $\exp G$. Thus the map $f \to \mu_f$ is an isomorphism between E and the vector lattice of all regular Borel measures on $\exp(G)$. In particular E is a vector lattice with positive cone C.

In the present work we will always assume G is an idempotent semigroup, i.e. a semilattice. If we define $a \leqq b$ whenever $a = ab$ then "$\leqq$" induces a natural ordering on G. A nonvoid subsemigroup F of G whose complement F' is an ideal is called a *filter*, or equivalently F, is a nonvoid subsemigroup of G such that $b \in F$ whenever $a \in F$ and $a \leqq b$. A routine check reveals that the exponentials are just the characteristic functions of the filters. Let $\mathscr{F}$ denote the set of all filters in G, and $\mathscr{F}(a) = \{F \in \mathscr{F} \mid a \in \mathscr{F}\}$. Then $\mathscr{F}(ab) = \mathscr{F}(a) \cap \mathscr{F}(b)$ (in fact the map $a \to \mathscr{F}(a)$ is a semilattice isomorphism). The map $\chi_F \to F$ is then a bijection of $\exp G$ onto $\mathscr{F}$. If we impose on $\mathscr{F}$ the topology induced by this latter map then sets of the form $\mathscr{F}(a)$ and $\mathscr{F}(a)'$ form a subbase for $\mathscr{F}$. Note $\mathscr{F}$ is a zero dimensional compact Hausdorff space. Let $f \in E$ and μ be its representing measure. If we transfer μ to $\mathscr{F}$ then we have

$$f(a) = \int_{\exp G} \phi(a) d\mu = \mu(\{\phi \in \exp G \mid \phi(a) = 1\}) = \mu(\mathscr{F}(a))$$

and more generally,

$$\begin{aligned} \Delta_n f(a; b_1, \cdots, b_n) &= \int_{\exp G} \phi(a)(1 - \phi(b_1)) \cdots (1 - \phi(b_n)) d\mu \\ &= \mu(\{\phi \in \exp G \mid \phi(a) = 1, \phi(b_1) = \cdots = \phi(b_n) = 0\}) \\ &= \mu(\mathscr{F}(a) \cap \mathscr{F}(b_1)' \cap \cdots \cap \mathscr{F}(b_n)') . \end{aligned}$$

This last formula will be used in § 2.

2. Variation and bounded variation.

DEFINITION 2.1. (a) The tuples $(a; c_1, \cdots, c_m)$ and $(b; c_{m+1}, \cdots, c_{m+n})$ are said to be *related* if $c_i ab = ab$, i.e., if $c_i \geqq ab$ for some $i = 1, \cdots, m + n$.

(b) Let f be a real-valued function defined on G. The *variation* $|f|$, and *positive variation* f^+ of f at a point a in G are defined by

$$|f|(a) = \sup \sum_{i=1}^{k} |\Delta_{n_i} f(ab_{i,0}; b_{i,1}, \cdots, b_{i,n_i})|$$

and

$$f^+(a) = \sup \sum_{i=1}^{k} [\Delta_{n_i} f(ab_{i,0}; b_{i,1}, \cdots, b_{i,n_i})] \vee 0 ,$$

where the suprema are taken over all finite sets of mutually related tuples $\{(b_{i,0}; b_{i,1}, \cdots, b_{i,n_i})\}_{i=1}^{k}$.

(c) The function f defined on G is said to be of *bounded variation* (a BV-function) if $|f|(1)$ is finite.

Observe that for a given family of tuples as in (b), it is possible that $n_i = 0$ for some i with $1 \leqq i \leqq k$. But since the tuples are pairwise related, we can have $n_i = 0$ for at most one such index i.

The main result of § 2 is Theorem 2.2 where we characterize E as the BV-functions on G. The following three propositions are necessary for our proof of this theorem.

PROPOSITION 2.2. *The tuples $(a; c_1, \cdots, c_m)$ and $(b; c_{m+1}, \cdots, c_{m+n})$ are related if and only if the sets $\mathscr{F}(a) \cap \mathscr{F}(c_1)' \cap \cdots \cap \mathscr{F}(c_m)'$ and $\mathscr{F}(b) \cap \mathscr{F}(c_{m+1})' \cap \cdots \cap \mathscr{F}(c_{m+n})'$ are disjoint.*

Proof. Suppose $\mathscr{F}(ab) \cap \mathscr{F}(c_1)' \cap \cdots \cap \mathscr{F}(c_{m+n})' = \phi$. Let $F = \{x \in G\colon ab \leqq x\}$ then F is a filter on $G, F \in \mathscr{F}(ab)$, and so $F \in \mathscr{F}(c_i)$ for some index i. Therefore, $ab \leqq c_i$, so the tuples are related.

Conversely, if the tuples are related then $ab \leqq c_i$ for some index i. Hence if F is a filter on G such that $F \in \mathscr{F}(ab)$, then $F \in \mathscr{F}(c_i)$, i.e., $\mathscr{F}(ab) \subseteqq \mathscr{F}(c_i)$, and so

$$\mathscr{F}(ab) \cap \mathscr{F}(c_1)' \cap \cdots \cap \mathscr{F}(c_{m+n})' = \phi .$$

Recall that a *semialgebra* on a set X is a family $\mathscr{S}$ of subsets of X such that

(i) if S_1 and S_2 are members of $\mathscr{S}$, then $S_1 \cap S_2$ is a member of $\mathscr{S}$; (ii) if $S \in \mathscr{S}$, then there is a finite subfamily of pairwise disjoint members of $\mathscr{S}$ whose union is S'; and (iii) $X \in \mathscr{S}$. An *algebra* on X is a nonempty family of subsets of X which is closed under the operations of forming finite unions and taking complements. It is easy to verify that the algebra generated by a semialgebra $\mathscr{S}$ consists of those sets which can be expressed as the union of a finite subfamily of pairwise disjoint members of $\mathscr{S}$.

The proof of the next result is straightforward, so will be omitted.

PROPOSITION 2.3. *The family $\mathscr{S}$ of sets $\mathscr{F}(a_0) \cap \mathscr{F}(a_1)' \cap \cdots \cap \mathscr{F}(a_n)'$, where the tuple $(a_0, \cdots, a_n)$ ranges over all finite subsets of the semilattice G, is a semialgebra on the set $\mathscr{F}$ of all filters on G.*

The algebra on $\mathscr{F}$ generated by the semialgebra $\mathscr{S}$ will be denoted by $\mathscr{A}$.

PROPOSITION 2.4. *If f is a real-valued function defined on the semilattice G, then the function μ defined by*

$$\mu_f(\mathscr{F}(a_0) \cap \mathscr{F}(a_1)' \cap \cdots \cap \mathscr{F}(a_n)') = \Delta_n f(a_0; a_1, \cdots, a_n)$$

can be extended to a finitely additive regular set function on $\mathscr{A}$.

Proof. For each element a in G, let $\tilde{a}$ denote the characteristic function of the set $\mathscr{F}(a)$. We claim that the family $\{\tilde{a}: a \in G\}$ of real-valued functions on $\mathscr{F}$ is linearly independent. To see this, suppose $\alpha_1\tilde{a}_1 + \cdots + \alpha_n\tilde{a}_n = 0$, where the α_i are real numbers. When given the relative order of G, the set $\{a_1, \cdots, a_n\}$ must have a maximal element. Assume without loss of generality that a_n is such an element, and put $F = \{x \in G: a_n \leqq x\}$. Then F is a filter on G with $a_n \in F$, but $a_i \notin F$ for $i < n$, i.e., $F \in \mathscr{F}(a_n)$, but $F \notin \mathscr{F}(a_i)$ for $i < n$. Therefore, $(\alpha_1\tilde{a}_1 + \cdots + \alpha_n\tilde{a}_n)(F) = \alpha_n = 0$. Continuing in this way, we conclude that $\alpha_1 = \cdots = \alpha_n = 0$.

We define a linear functional $\bar{f}$ on the linear subspace of $R^{\mathscr{F}}$ generated by $\{\tilde{a}: a \in G\}$ by putting $\bar{f}(\sum \alpha_i\tilde{a}_i) = \sum \alpha_i f(a_i)$. Since the family $\{\tilde{a}: a \in G\}$ is linearly independent, and since the mapping $a \rightarrow \tilde{a}$ is one-to-one, the function $\bar{f}$ is well-defined.

Now let $\{\mathscr{F}(a_{i,0}) \cap \mathscr{F}(a_{i,1})' \cap \cdots \cap \mathscr{F}(a_{i,n_i})'\}_{i=1}^k$ be a family of pairwise disjoint members of $\mathscr{S}$ whose union is

$$\mathscr{F}(a_{0,0}) \cap \mathscr{F}(a_{0,1})' \cap \cdots \cap \mathscr{F}(a_{0,n_0})' .$$

The characteristic function of the ith such set, where $0 \leqq i \leqq k$, is

$$\tilde{a}_{i,0}(1 - \tilde{a}_{i,1}) \cdots (1 - \tilde{a}_{i,n_i}) ,$$

and therefore,

$$\sum_1^k \tilde{a}_{i,0}(1 - \tilde{a}_{i,1}) \cdots (1 - \tilde{a}_{i,n_i}) = \tilde{a}_{0,0}(1 - \tilde{a}_{0,1}) \cdots (1 - \tilde{a}_{0,n_0}) .$$

When the function $\bar{f}$ is applied to both sides of the last equation we obtain

$$\sum_1^k \Delta_{n_i} f(a_{i,0}; a_{i,1}, \cdots, a_{i,n_i}) = \Delta_{n_0} f(a_{0,0}; a_{0,1}, \cdots, a_{0,n_0}) ,$$

and hence

$$\sum_1^k \mu_f(\mathscr{F}(a_{i,0}) \cap \mathscr{F}(a_{i,1})' \cap \cdots \cap \mathscr{F}(a_{i,n_i})')$$
$$= \mu_f(\mathscr{F}(a_{0,0}) \cap \mathscr{F}(a_{0,1})' \cap \cdots \cap \mathscr{F}(a_{0,n_0})') .$$

It follows that the set function μ_f on $\mathscr{S}$ has a well-defined and unique finitely additive extension to $\mathscr{A}$. This extension is also denoted by μ_f.

The regularity of μ_f on $\mathscr{A}$ is a consequence of the fact that the elements of the semialgebra $\mathscr{S}$ and hence the elements of the algebra $\mathscr{A}$ are open-and-closed.

The following is the main result of the section. The symbol $|\nu|$ is used to denote the variation of the measure ν [4]. Recall that E is the vector lattice spanned by the completely monotonic functions.

THEOREM 2.5. *Let f be a real valued function defined on the semilattice G with identity. Then f is a member of E if and only if it is of bounded variation on G. Moreover, if f is a BV-function and if ν_f is the regular Borel measure which represents f, then*

$$|f|(a) = |\nu_f|(\mathscr{F}(a)) ,$$
$$f^+(a) = \nu_{f^+}(\mathscr{F}(a)) \text{ for each } a \in G .$$

Finally $|f| = f \vee (-f)$ and $f^+ = f \vee 0$.

Proof. For each $f \in C$, let μ_f be the finitely additive set function on $\mathscr{A}$ defined in Proposition 2.4 by

$$\text{(i)} \qquad \mu_f(\mathscr{F}(a) \cap \mathscr{F}(b_1)' \cap \cdots \cap \mathscr{F}(b_n)') = \Delta_n f(a; b_1, \cdots, b_n)$$

then the map $f \rightarrow \mu_f$ is an isomorphism of the cone C onto the cone of all nonnegative finitely additive set functions on $\mathscr{A}$. This map therefore admits a unique extension to an isomorphism of E onto the vector lattice of all finitely additive BV-set functions on $\mathscr{A}$. Therefore, this extension is order preserving and necessarily assumes the form (i). The definition of variation of a set function [4], Proposition 2.2 and (i) above imply:

$$\text{(ii)} \qquad \begin{aligned} (f \vee -f)(a) &= |\mu_f|(\mathscr{F}(a)) \\ &= \sup \sum |\mu_f(\mathscr{F}(ab_{i,0}) \cap \mathscr{F}(b_{i,1})' \cap \cdots \cap \mathscr{F}(b_{i,n_i})')| \\ &= \sup \sum |\Delta_{n_i} f(ab_{i,0}; b_{i,1}, \cdots, b_{i,n_i})| \\ &= |f|(a) \end{aligned}$$

where the first supremum is taken over all finite collections of mutually disjoint members of $\mathscr{A}$ and the second is taken over all finite collections of mutually related tuples. Since a similar argument shows that

(iii) $$(f \vee 0)(a) = f^+(a)$$

for all $f \in E$, the first and last assertions of the theorem are established.

To establish the intermediate formulas, let μ be a nonnegative finitely additive set function on $\mathscr{A}$ and let f be defined by $f(a) = \mu(\mathscr{F}(a))$. Then $f \in C$, so there exists a unique nonnegative regular Borel measure ν, such that ν represents f. It follows that ν is an extension, and in fact the only extension, of μ. Thus the map $\mu \to \nu$ defines an isomorphism of the cone of finitely additive set functions on $\mathscr{A}$ onto the nonnegative regular Borel measures. It now follows that every finitely additive BV-measure μ admits a unique extension ν, namely the unique representing measure for the BV-function f defined by $f(a) = \mu(\mathscr{F}(a))$. Since this extension is order preserving we must have $|\mu|(A) = |\nu|(A)$ and $\mu^+(A) = \nu^+(A)$ for all $A \in \mathscr{A}$. The desired formulas now follow from (ii) and (iii).

3. Valuations.

Recall [2] that a *valuation* on a lattice $L = (L; \bigvee, \bigwedge)$ is a real-valued function f defined on L which satisfies the identity

$$f(a) + f(b) = f(a \vee b) + f(a \wedge b) .$$

Now assume that L is a distributive lattice with largest element 1. Denote by $V = V(L)$ the set of all nonnegative, nondecreasing valuations on L and let $V_1 = \{f \in V: f(1) = 1\}$.

Throughout this section, the symbol G will denote the semilattice $(L; \bigwedge)$, and $C = C(G)$ will denote the cone of all completely monotonic functions defined on G.

PROPOSITION 3.1. *If f is a valuation on a distributive lattice L, then*

$$\Delta_n f(a; b_1, \cdots, b_n) = \Delta_1 f(a; b_1 \vee \cdots \vee b_n)$$

for each integer $n \geqq 1$.

Proof. The assertion is obviously true if $n = 1$. If $n = 2$, then

$$\begin{aligned}\Delta_2 f(a; b_1, b_2) &= f(a) - f(a \wedge b_1) - f(a \wedge b_2) + f(a \wedge b_1 \wedge b_2)\\ &= f(a) - f((a \wedge b_1) \vee (a \wedge b_2))\\ &= f(a) - f(a \wedge (b_1 \vee b_2))\\ &= \Delta_1 f(a; b_1 \vee b_2) .\end{aligned}$$

Assume that the result is true for $n = k \geqq 2$; then

$$\begin{aligned}\Delta_{k+1}f(a; b_1, \cdots, b_{k+1}) &= \Delta_k f(a; b_1, \cdots, b_k) - \Delta_k f(a \wedge b_{k+1}; b_1, \cdots, b_k) \\ &= \Delta_1 f(a; b_1 \vee \cdots \vee b_k) - \Delta_1(a \wedge b_{k+1}; b_1 \vee \cdots \vee b_k) \\ &= \Delta_2 f(a; b_1 \vee \cdots \vee b_k, b_{k+1}) \\ &= \Delta_1 f(a; b_1 \vee \cdots \vee b_{k+1}) .\end{aligned}$$

PROPOSITION 3.2. *The set V is a closed extremal* [4] *subset of C.*

Proof. It is immediate that V_1 is closed in C_1. (Recall that C_1 carries the topology of pointwise convergence.)

Note that if f is an element of C, then $0 \leqq \Delta_2 f(a \vee b; a, b) = f(a \vee b) - f(a) - f(b) + f(a \wedge b)$ and so $f(a) + f(b) \leqq f(a \vee b) + f(a \wedge b)$ for each pair of elements a, b in L.

Now let f be a valuation on L, and suppose that $f = \alpha f_1 + (1-\alpha)f_2$ with f_1 and f_2 elements of C, and $0 < \alpha < 1$. If for some pair a, b in L we have $f_j(a) + f_j(b) < f_j(a \vee b) + f_j(a \wedge b)$ for $j = 1$ or 2, then

$$\begin{aligned}f(a) + f(b) &= \alpha f_1(a) + (1-\alpha)f_2(a) + \alpha f_1(b) + (1-\alpha)f_2(b) \\ &< \alpha[f_1(a \vee b) + f_1(a \wedge b)] + (1-\alpha)[f_2(a \vee b) + f_2(a \wedge b)] \\ &= f(a \vee b) + f(a \wedge b) ,\end{aligned}$$

which is a contradiction. Thus, the cone V is an extremal subset of the cone C.

In particular this implies that the vector space $V - V$ is a sublattice of $E = C - C$. Thus by Theorem 2.5, if $f \in V$ then, the absolute value of f with respect to the vector lattice $V - V$ agrees with the variation $|f|$ of f defined in 2.1. In summary we may state:

THEOREM 3.3. *The vector space $V - V$ is a vector lattice which consists of all valuations of bounded variation. Its positive cone is the cone of all nonnegative nondecreasing valuations on L and $|f| = f \vee (-f)$ for all $f \in V - V$.*

From Proposition 3.1, if f is a valuation and $a \in L$ then

$$|f|(a) = \sup \sum_{i=1}^{k} |\Delta_1 f(a \wedge b_i; c_i)| + |f(a \wedge b)|$$

where the arguments of the summands range over all finite sets of mutually related pairs $\{a \wedge (b_i; c_i)\}_i$ such that $c_i \wedge (a \wedge b \wedge b_i) = a \wedge b \wedge b_i$.

It is not hard to see that this notion of variation agrees with the classical concept of variation of real valued functions on linearly ordered lattices. Birkhoff [2] has also generalized the classical concept of variation to valuations on a lattice. We will distinguish his definition from ours by referring to his as *linear variation.*

DEFINITION 3.4. (Birkhoff [2]). Let f be a valuation on a distributive lattice L. Then the *linear variation* $|f|_0$ is defined by

$$|f|_0(a) = \sup \sum_{i=1}^{n-1} |f(x_{i+1}) - f(x_i)| + |f(x_1)|$$

and the *linear positive* variation $(f)_0^+$ is defined by

$$(f)_0^+ = \sup \sum_{i=1}^{n-1} [f(x_{i+1}) - f(x_i)] \vee 0 + [f(x_1)] \vee 0 ;$$

where the suprema are taken over all ascending chains $x_1 \leqq x_2 \leqq \cdots \leqq x_n = a$.

In [2], using methods different from ours, Birkhoff essentially shows that Theorem 3.3 is valid when "variation" is replaced by linear variation." Hence it follows that $|f| = |f|_0$ whenever either $|f|(1) < \infty$ or $|f|_0(1) < \infty$. We will now prove a series of algebraic lemmas which will lead to an alternate proof of this fact; in this approach we use Theorem 3.3 rather than Birkhoff's results.

LEMMA 3.5. *If f is a valuation on L then $|f|_0 \leqq |f|$.*

Proof. Let $x_1 \leqq x_2 \leqq \cdots \leqq x_n = a$ be a chain in L. Then

$$\sum_{i=1}^{n-1} |f(x_{i+1}) - f(x_i)| + |f(x_1)| \leqq \sum_{i=1}^{n-1} |\Delta_1 f(x_{i+1}; x_i)| + |f(x_1)| .$$

If $n+1 > i > j$ then $x_i \geqq x_j = x_{i+1} \wedge x_j$ so that $\{(x_{i+1}; x_i) \mid i = 1, 2, \cdots, n\}$ is a mutually related set of ordered pairs.

For the remainder of this section we find it convenient to let $N(m, n)$ $(n \geqq m)$ be the class of all strictly increasing functions $\alpha \mid \{1, 2, \cdots, m\} \to \{1, 2, \cdots, n\}$.

LEMMA 3.6. *If f is a valuation on L and $a_1, a_2, \cdots, a_{k+1} \in L$ then for a fixed n: $0 \leqq n < k$*

$$\begin{aligned} &f[\bigvee\{(a_{\alpha(1)} \wedge a_{\alpha(2)} \cdots \wedge a_{\alpha(n+1)}) \mid \alpha \in N(n+1, k)\}] \\ &\quad + f[\bigvee\{(a_{\alpha(1)} \wedge a_{\alpha(2)} \cdots \wedge a_{\alpha(n)} \wedge a_{k+1}) \mid \alpha \in N(n, k)\}] \\ &= f[\bigvee\{(a_{\alpha(1)} \wedge a_{\alpha(2)} \cdots \wedge a_{\alpha(n+1)}) \mid \alpha \in N(n+1, k+1)\}] \\ &\quad + f[\bigvee\{(a_{\alpha(1)} \wedge a_{\alpha(2)} \cdots \wedge a_{\alpha(n+1)} \wedge a_{k+1} \mid \alpha \in N(n+1, k)\}] . \end{aligned}$$

Proof. If we let a and b the respective arguments of f in the first and second terms to the left of the equality, then,

$$a \vee b = \bigvee\{a_{\alpha(1)} \wedge a_{\alpha(2)} \cdots \wedge a_{\alpha(n+1)} \mid \alpha \in N(n+1, k+1)\}\ .$$

Moreover, the generalized distributive law implies,

$$a \wedge b = \bigvee\{(a_{\alpha(1)} \wedge a_{\alpha(2)} \cdots \wedge a_{\alpha(n+1)} \wedge a_{\beta(1)} \wedge a_{\beta(2)} \cdots \wedge a_{\beta(n)} \wedge a_{k+1}) \\ \mid \alpha \in N(n+1, k), \beta \in N(n, k)\}$$

so that

$$a \wedge b = \bigvee\{a_{\alpha(1)} \wedge a_{\alpha(2)} \cdots \wedge a_{\alpha(n+1)} \wedge a_{k+1}) \mid \alpha \in N(n+1, k)\}\ .$$

The assertion now follows since $f(a) + f(b) = f(a \vee b) + f(a \wedge b)$.

LEMMA 3.7. *If f is valuation on L and if $a_1, a_2, \cdots, a_k \in L$ then*

$$\sum_{i=1}^{k} f(a_i) = \sum_{i=1}^{k} f[\bigvee\{a_{\alpha(1)} \wedge a_{\alpha(2)} \cdots \wedge a_{\alpha(i)}) \mid \alpha \in N(i, k)\}]\ .$$

Proof. The assertion is clearly valid if $k = 1$. If we assume its validity for k then

$$\begin{aligned}\sum_{i=1}^{k+1} f(a_i) &= \sum_{i=1}^{k} f(a_i) + f(a_{k+1}) \\ &= \sum_{i=1}^{k} f[\bigvee\{a_{\alpha(1)} \wedge a_{\alpha(2)} \cdots \wedge a_{\alpha(i)} \mid \alpha \in N(i, k)\}] + f(a_{k+1}) \\ &= f\Big[\bigvee_{i=1}^{k} a_i\Big] + f(a_{k+1}) \\ &\quad + \sum_{i=2}^{k} f[\bigvee\{a_{\alpha(1)} \wedge a_{\alpha(2)} \cdots \wedge a_{\alpha(i)} \mid \alpha \in N(i, k)\}]\ .\end{aligned}$$

Applying Lemma 3.6 to the first term of the above expression with $n = 0$ and $k = 1$ yields

$$\begin{aligned}\sum_{i=1}^{k+1} f(a_i) &= f\Big[\bigvee_{i=1}^{k+1} a_i\Big] + f\Big[\bigvee_{i=1}^{k} (a_i \wedge a_{k+1})\Big] \\ &\quad + \sum_{i=2}^{k} f[\bigvee\{a_{\alpha(1)} \wedge a_{\alpha(2)} \cdots \wedge a_{\alpha(i)} \mid \alpha \in N(i, k)\}]\ .\end{aligned}$$

Reapplying Lemma 3.6 to the second term to the right of equality and the first term of the summation with $n = 1$ and $k = k$ yields

$$\begin{aligned}\sum_{i=1}^{k+1} f(a_i) &= f\Big[\bigvee_{i=1}^{k+1} a_i\Big] + f[\bigvee\{(a_{\alpha(1)} \wedge a_{\alpha(2)} \mid \alpha \in N(2, k+1)\}] \\ &\quad + f[\bigvee\{(a_{\alpha(1)} \wedge a_{\alpha(2)} \wedge a_{k+1} \mid \alpha \in N(2, k)\}] \\ &\quad + \sum_{i=3}^{k} f[\bigvee\{(a_{\alpha(1)} \wedge a_{\alpha(2)} \cdots a_{\alpha(i)} \mid \alpha \in N(i, k)\}]\ .\end{aligned}$$

Successively repeating this process proves the assertion for $k + 1$.

LEMMA 3.8. *If* $\{(a_i, b_i) \mid i = 1, 2, \cdots, k\}$ *and* (a_{k+1}) *are* $k+1$ *mutually related tuples such that* $b_i \leqq a_i$ *for each* $i = 1, 2, \cdots, k$ *then*

$$\bigvee\{(b_{\alpha(1)} \wedge b_{\alpha(2)} \cdots \wedge b_{\alpha(n)} \mid \alpha \in N(n, k)\}$$
$$\geqq \bigvee\{(a_{\alpha(1)} \wedge a_{\alpha(2)} \cdots \wedge a_{\alpha(n+1)} \mid \alpha \in N(n+1, k+1)\}$$

Proof. Let $a = (a_{\alpha(1)} \wedge a_{\alpha(2)} \cdots \wedge a_{\alpha(n+1)})$ for some $\alpha \in N(n+1, k+1)$.

Case 1. $\alpha(n+1) \neq k+1$. We may assume the existence of $j\,(j = 1, 2, \cdots, n)$ such that $b_{\alpha(j)} \ngeqq a$. Then $b_{\alpha(j)} \ngeqq a_{\alpha(j)} \wedge a_{\alpha(l)} \geqq a$ for any $l = 1, 2, \cdots, n+1$. Therefore if $j \neq l$ the mutual relatedness implies

$$b_{\alpha(l)} \geqq a_{\alpha(j)} \wedge a_{\alpha(l)} \geqq a, \text{ so that } \bigwedge_{l \neq j} b_{\alpha(l)} \geqq a .$$

Case 2. $\alpha(n+1) = k+1$. Then by mutual relatedness,

$$b_{\alpha(j)} \geqq a_{\alpha(n+1)} \wedge a_{\alpha(j)} \geqq a, \text{ so that } \bigwedge_{j=1}^{n} b_{\alpha(j)} \geqq a .$$

The assertion easily follows from these two cases.

LEMMA 3.9. *If* f *is a valuation on* L *then*

$$(f_0^+) \geqq f^+ .$$

Proof. Consider the sum $\sum_{i=1}^{k} \Delta_1 f(a_i; b_i) + f(a_{k+1})$.
Without loss of generality we make the following assumptions:
(i) Each term in the summand is strictly positive.
(ii) $b_i < a_i \leqq a$ for each $i = 1, 2, \cdots, k$ and $a_{k+1} \leqq a$.
(iii) The set $\{(a_i; b_i) \mid i = 1, 2, \cdots, k\} \cup \{a_{k+1}\}$ is mutually related.
From Lemma 3.7 we have

$$\begin{aligned}
&\sum_{i=1}^{k} \Delta_1 f(a_i; b_i) + f(a_{k+1}) \\
&= \sum_{i=1}^{k} (f[\bigvee\{(a_{\alpha(1)} \wedge a_{\alpha(2)} \cdots \wedge a_{\alpha(i)}) \mid \alpha \in N(i, k+1)\}] \\
&\quad - f[\bigvee\{(b_{\alpha(1)} \wedge b_{\alpha(2)} \cdots \wedge b_{\alpha(i)} \mid \alpha \in N(i, k)\}]) \\
&\quad + f(a_1 \wedge a_2 \cdots \wedge a_{k+1}) .
\end{aligned}$$

Lemma 3.8 and assumption (ii) above imply that the arguments of the terms in the above expression form a decreasing sequence. The assertion easily follows.

THEOREM 3.10. *If* f *is a valuation on a distributive lattice* L *then* $f^+ = (f)_0^+$. *Moreover* f *is of bounded variation* ($|f| < \infty$) *if and only if* f *is of bounded linear variation* ($|f|_0 < \infty$), *in this case,* $|f| = |f|_0$.

Proof. The first assertion follows from Lemmas 3.5 and 3.9. Lemma 3.5 also implies that $|f|_0 < \infty$ whenever $|f| < \infty$. Conversely if f is of bounded linear variation then $(f)_0^+(1) < \infty$ and $(-f)_0^+(1) < \infty$ so that

$$|f|(1) \leqq f^+(1) + (-f)^+(1) \leqq (f)_0^+(1) + (-f)_0^+(1) < \infty .$$

In view of Lemma 3.5, to complete the proof we need only verify that $|f| \leqq |f|_0$ if $|f|_0 < \infty$. Given $\varepsilon > 0$, there exists a chain $x_1 < x_2 < \cdots < x_n = a$ in L such that

$$(f)_0^+(a) \leqq \sum_{i=1}^{n-1} [f(x_{i+1}) - f(x_i)] \vee 0 + f(x_1) \vee 0 + \varepsilon/2 .$$

But it follows from Theorem 3.3 that $|f|(a) = 2f^+(a) - f(a)$. Therefore

$$\begin{aligned} |f|(a) &= 2(f)_0^+(a) - f(a) \leqq 2\sum_{i=1}^{n-1} [f(x_{i+1}) - f(x_i)] \vee 0 \\ &\quad + 2(f(x_1) \vee 0) - f(a) + \varepsilon \\ &= 2\sum_{i=1}^{n-1} [f(x_{i+1}) - f(x_i)] \vee 0 + 2(f(x_1) \vee 0) \\ &\quad - \sum_{i=1}^{n-1} [f(x_{i+1}) - f(x_i)] f(x_1) + \varepsilon \\ &= \sum_{i=1}^{n} |f(x_{i+1}) - f(x_i)| + |f(x_1)| + \varepsilon \\ &\leqq |f|_0 + \varepsilon. \quad \text{Hence } |f| \leqq |f|_0 . \end{aligned}$$

Let $f \in V - V$. We turn our attention to constructing the representing measure for f. To this end we will define a filter on G to be *prime* if $a \vee b \in G$ implies either $a \in G$ or $b \in G$. The following proposition is essentially contained in [3].

PROPOSITION 3.11. *A function $f \in V_1$ is an extreme point of V_1 if and only if it is the characteristic function of a prime filter* (i.e. a zero-one lattice homomorphism).

Proof. A routine check reveals that the zero-one lattice homomorphisms agree with the characteristic functions of the prime filters. Clearly they are all extremal elements of V. To prove the converse, observe that Lemma 3.2 implies that V_1 is an extremal subset of C_1. The assertion follows since each extremal point must be an exponential (or the characteristic function of a filter) as well as a valuation.

If we let P denote the set of all prime filters and $P(a) = P \cap F(a)$ then $\mathscr{P}(1)$ is a compact zero dimensional subspace of $\mathscr{F}(1)$. But if $f \in V - V$ then its representing measure is supported by the extreme points of V_1, or in our setting, by $\mathscr{P}(1)$. Therefore Theorem 2.5

implies the first part of the following.

THEOREM 3.12. *If f is a valuation of bounded variation, then there exists a unique regular Borel measure ν_f on $\mathscr{P}(1)$ such that*

$$|f|(a) = |\nu_f|(\mathscr{P}(a))$$
$$f^+(a) = \nu_f^+(\mathscr{P}(a)) .$$

Moreover the map $a \rightarrow \mathscr{P}(a)$ is a lattice isomorphism while the map $f \rightarrow \nu_f$ in a vector lattice isomorphism.

Proof. The only assertion which we have not proved is that the map $a \rightarrow \mathscr{P}(a)$ is biunique. But this is a known result from lattice theory. In fact it can be shown that every filter is the intersection of prime filters.

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It has come to our attention that S. E. Newman has recently announced results [7], [8] which are related to the work in § 2. In particular, using combinatorial technique, he has introduced a formally different notion of BV-functions on a semilattice. His emphasis is not, as is ours, on the lattice theoretic properties of the BV-functions (eg. he has not announced a concept of variation as such) but rather on the multiplicative structure of these functions. Some of his work carries over to our setting quite naturally. For example his theorem that the product of two BV-functions is again a BV function follows for BV-functions in our sense from [5, Cor. 1.6].

REFERENCES

1. H. Bauer, *Konvexitat in Topologischen Vektorräumen*, Lecture notes University of Hamburg 1963/64.

2. G. Birkhoff, *Lattice Theory*, Amer. Math. Soc., Colloquium Publication 25, New York (1948).

3. G. Choquet, *Theory of capacities*, Ann. Inst. Fourier (Grenoble) **5** (1954), 131-295.

4. N. Dunford and J. T. Schwartz, *Linear Operators*, Part I, Interscience, New York-London, 1958.

5. N. Fine and P. H. Maserick, *On the simplex of completely monotonic functions on a commutative semigroup*, Canad. J. Math., XXII (1970), 317-326.

6. J. Kist and P. H. Maserick, *Functions of bounded variation on a commutative idempotent semigroup*, Abstract 653-29, Notices of Amer. Math. Soc., **15** (1968).

7. S. E. Newman, *Functions of bounded variation on idempotent semigroups*, Abstract 663-458, Notices of Amer. Mach. Soc., **16** (1969).
8. ———, *Measure algebras and functions of bounded variation on idempotent semigroups*, Bull. Amer. Math. Soc., **75** (1969), 1396-1400.
9. R. R. Phelps, *Lectures on Choquet Theory*, van Nostrand, Toronto-New York-London 1966.

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