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LIE ALGEBRAS OF GENUS ONE AND GENUS TWO

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The genus of a finite dimensional algebra is the difference between its dimension and the number of elements in a minimal generating set for the algebra. In this paper the classification of finite dimensional Lie algebras of genus one and genus two is accomplished in four steps. First, it is shown that every such algebra is either solvable or contains a simple subalgebra. Second, the algebras containing a simple subalgebra are determined. Third, nonminimal genus one and two Lie algebras are shown to have a one, two dimensional ideal, J, respectively, with genus zero quotient. Fourth, the different possible L module structures for J are analyzed and completely determined except for the genus two index two nilpotent algebras.

The purpose of this paper is to classify and solve the isomorphism problem for finite dimensional Lie algebras of genus 1 and genus 2 over arbitrary fields. The classification problem was attacked for Lie algebras with genus less than three in Bond [1]. The classification obtained was somewhat unsatisfactory for genus 1 and genus 2 algebras because the results were presented by specifying different possible multiplication tables in terms of minimal generating sets for the algebras and therefore provided no ready access to the isomorphism problems. Nevertheless, the classification of genus 1 algebras was fairly short and had interesting consequences and therefore was made the basis of a short paper, which because of the perceptive comments of the referee grew into the present paper.

The concept of genus seems to have originated with Knebelman [5]. He obtained among other results a valid classification of the structure constants for genus 0 and 1 Lie algebras. His paper contains errors, which have been documented in Patterson [8] and Marshall [7]. Wallace [9] has obtained a classification of Lie algebras of dimension four over the complex numbers. It should be remarked that not unlike Wallace, who was unaware of Knebelman's work, the author was unaware of both Wallace's and Knebelman's work until quite recently.

Patterson [8] establishes the most far reaching results. His main theorem for algebras implies an interesting theorem about Lie algebras, (Theorem 2, § 1) which seems to have gone unnoticed.

The author first became aware of the term "genus of a Lie algebra" through [7]. The results obtained in [7] are indirectly related to ours.

Throughout this paper use is made of the results in [2]. It should be pointed out that a minimal set of generators of a finite dimensional algebra is always a weak minimal generating set (a concept particular to [2]) for the algebra. The converse is false in general. The results needed from [2] are proved under the hypothesis of the existence of weak minimal generating sets with specific properties and hence hold under the stronger hypothesis of the existence of a minimal generating set with the same properties.

The following notation will be in force throughout the paper. L will denote a finite dimensional Lie algebra over an arbitrary field denoted by F. The elements of L will be denoted by small Roman letters and the elements of F by small Greek letters. $J(a, b, c) = 0$ will stand for the Jacobi identity $(ab)c+(bc)a+(ca)b=0$. $\langle a, b, c, \cdots\rangle$ will denote the subalgebra of an algebra A generated by the elements $a, b, c, \cdots$ of A. $\langle S, T, \cdots, a, b, \cdots\rangle$ will denote the subalgebra generated by the subsets of elements $S, T, \cdots$ of A and the elements $a, b, \cdots$ of A.

1. The genus of an algebra. A subset S of elements of a finite dimensional algebra A is said to minimally generate A if S generates A as an algebra and no subset of A having fewer elements than S generates A. A finite dimensional algebra A has genus k if A has a minimal generating set with m elements and dim $L=k+m$.

PROPOSITION 1. *If B is a subalgebra of an algebra A then genus $B \leqq$ genus A.*

Proof. Let M be a minimal generating set of B consisting of m elements. Let $A = B \dotplus C$, C a subspace of A. Then for any basis N of C, $M \cup N$ generates A. Therefore

$$\begin{aligned} \text{genus } A &\geqq \dim A - (\dim C + m) \\ &= (\dim A - \dim C) - m \\ &= \dim B - m = \text{genus } B. \end{aligned}$$

COROLLARY 1. *If A has genus k then the dimension of a subalgebra generated by t elements of A is at most $t + k$.*

PROPOSITION 2. *Let I be an ideal of an algebra A. Then either $A = B \dotplus I$, B a subalgebra of A or genus $A/I <$ genus A.*

Proof. Let $\varphi\colon A \to A/I$ be a homomorphism with $\ker \varphi = I$. Let M be a set of elements of A with $\varphi(M)$ minimally generating A/I. Let B be the subalgebra of A generated by M. Then genus $A \geqq$

genus $B \geqq$ genus A/I. If genus $B =$ genus A/I then $\dim B = \dim A/I$ and hence $\varphi \mid B$ is an isomorphism.

THEOREM 1. *Let L be a finite dimensional solvable genus k Lie algebra over any field. Then L is at most $(k+2)$-step solvable.*

Proof. If $k = 0$ then A is metabelian by Theorem 1 of [2].

Suppose the theorem has been proved for all finite dimensional Lie algebras of genus $< k$, k a positive integer.

Let L be of minimal dimension among the Lie algebras of genus k. Then genus $L^2 <$ genus L. Hence L^2 is at most $(k+1)$-step solvable, i.e., L is at most $(k+2)$-step solvable.

Suppose the theorem has been proved for all genus k algebras of dimension less than n. Let $\dim L = n$. Let I be a minimal nonzero ideal of L. Then I is abelian since I^2 is also an ideal of L.

If genus $L/I <$ genus L then L/I is at most $(k+1)$-step solvable. Hence L is at most $(k+2)$-step solvable.

If genus $L/I =$ genus L then the last proposition implies $L = S \dotplus I$, S a subalgebra. Any minimal generating set of S extended by any basis of I is a minimal generating set for L since genus $S =$ genus L. It follows that Proposition 1, [2] applies. Hence Theorem 2, [2] applies with $S = B$ and $I = C$. Consider K of Theorem 2, [2]. If $K = S$ then $KI \subset K \cap I = 0$. Hence $L^2 = S^2$ and L is at most $(k+2)$-step solvable by the induction hypothesis applied to S. If K is not S then $K \dotplus Fb = S$, F the ground field, with $KC \subset B$ and $bs - s$ in B. But, $KC \subset I$ and $bs - s$ in I, so that $KC = 0$ and $bs = s$. Therefore from $L = K \dotplus Fb \dotplus C$ it follows that $L^2 = (K \dotplus Fb \dotplus C)^2 \subset K^2 \oplus C$. Hence $(L^2)^2 = (K^2)^2$ and the theorem follows from the induction hypothesis applied to K.

Note that no inequality bounding the genus k by the length of the derived series is possible. Indeed, a metabelian Lie algebra can have any genus greater than one. Let A be a cyclic $F[x]$ module with respect to a linear transformation T of A. Let $L = A \dotplus Ft$ and define $t \cdot a = T(a)$, $t^2 = 0$, $ab = 0$, for all a and b in A. Then L is 2-generated and hence has genus $\dim A - 1$.

Recall that a Lie algebra L is said to be perfect if $L^2 = L$. The remaining results of this section relate assumptions on the genus of a Lie algebra to simplicity of the algebra.

PROPOSITION 3. *Let P be a proper perfect subalgebra of a Lie algebra L over an infinite field F and x an element of L not in P. Then genus $\langle P, x\rangle >$ genus P.*

Proof. Suppose $\langle P, x\rangle = P \dotplus Fx$. If M minimally generates P while $M \cup \{x\}$ does not minimally generate $\langle P, x\rangle$ then genus $\langle P, x\rangle >$ genus P. If $M \cup \{x\}$ minimally generates $\langle P, x\rangle$ apply Theorem 2, [2] to $\langle P, x\rangle$ with $B = P$ and $C = Fx$ to conclude $B = B^2 \subset K$. Hence P is an ideal in $\langle P, x\rangle$. Apply Theorem 4, [2] and derive a contradiction to P perfect.

COROLLARY 2. *Let S be a simple subalgebra of a Lie algebra L over an infinite field F. If genus S = genus L then $S = L$.*

PROPOSITION 4. *Let B be a subalgebra of a simple finite dimensional Lie algebra S over any field F. If genus B = genus S then $B = S$.*

Proof. Suppose $B \neq S$. Let $S = B \dotplus C$, C a subspace of S. If dimension C is at least two by Theorem 2, [2] S has an ideal. If $\dim C = 1$ then $B = K \dotplus Fb$, otherwise K is an ideal, from which it follows that $S^2 = (K \dotplus Fb \dotplus Fc)^2 \subset K + Fbc \neq S$, a contradiction.

COROLLARY 3. *A simple Lie algebra of genus* 1 *over any field has dimension* 3.

Proof. There exist a, b in S generating a 3 dimensional subalgebra.

We state without proof Theorem 1, [8] by E. M. Patterson:

Let A be an algebra of genus k and dimension n over an arbitrary field F. If $n \geqq 3k + 3$ there exist elements $a_1, a_2, \cdots, a_{k+s}$ of A, with $s \leqq 2k$ such that the multiplication in A is given by $xy = \phi(y)x + \psi(x)y + \sum_{\lambda=1}^{k+s} \theta^\lambda(x, y)a_\lambda$, where ϕ and ψ are linear functions, θ^λ are bilinear functions, and $a_1, \cdots, a_k$ are generated by $a_{k+1}, \cdots, a_{k+s}$. If F is not the field $GF(2)$ the theorem is also true for $n = 3k + 2$.

If A is assumed to be a Lie algebra this result implies:

THEOREM 2. *Let L be a Lie algebra of genus k and dimension n over an arbitrary field F. If $n \geqq 3k + 4$ there exists an ideal J with $\dim J \leqq 2k$ such that genus $L/J = 0$. If F is not the field $GF(2)$, the theorem is true also for $n = 3k + 3$.*

Proof. The multiplication in L is given by $xy = \phi(y)x + \psi(x)y + \sum_{\lambda=1}^{k+s} \theta^\lambda(x, y)a_\lambda$, $s \leqq 2k$, ϕ and ψ linear functions, the θ^λ bilinear functions, $a_1, \cdots, a_{k+s}$ fixed linearly independent elements of L. Suppose further that the a^λ have been chosen so that the number of $\theta^\lambda \neq 0$

is minimal among all such multiplication rules for L.

If x, y and the a_λ are linearly independent $xy = -yx$ implies $\phi(x) = -\psi(x)$. Then from $\phi(x+a) = -\psi(x+a)$ and $\phi(x) = -\psi(x)$ conclude by linearity $\phi(a) = -\psi(a)$. Hence $\phi = -\psi$. Take x, y, z and the a_λ linearly independent elements of L. Then the identity $J(x, y, z) = 0$ implies $\sum_{\lambda=1}^{k+s} \theta^\lambda(x, y)\ \phi(a_\lambda) = 0$ (the coefficient of z). Since this sum is zero for any x, y linearly independent of the a's by bilinearity it is zero for all x and y in L.

Suppose for some μ that $\phi(a_\mu) \neq 0$. Define

$$b_\lambda = \begin{cases} a_\lambda - \phi(a_\mu)^{-1}\phi(a_\lambda)a_\mu & \text{for } \lambda \neq \mu \\ a_\lambda & \text{for } \lambda = \mu\ . \end{cases}$$

Then $xy = \phi(y)x - \phi(x)y = \sum_{\substack{\lambda=1 \\ \lambda\neq\mu}}^{k+s} \theta^\lambda(x, y)b_\lambda$ for all x and y in L. This contradicts the choice of the a_λ. Hence for all λ, $\phi(a_\lambda) = 0$, i.e. the a_λ span an ideal J. Obviously, genus $L/J = 0$.

COROLLARY 4. *If L is a genus 1 Lie algebra over any field F and* $\dim L \geqq 7$ *then L has a 1-dimensional ideal with genus 0 quotient.*

Proof. By Patterson, Theorem 2, [8], the s occurring in the last theorem can be taken equal to zero when $k = 1$.

2. The structure of finite dimensional Lie algebras of genus 1 and 2.

PROPOSITION 5. *There are no perfect 4-dimensional Lie algebras over any field F.*

Proof. If genus $L = 0$ then L is not perfect by Theorem 1, [2]. If genus $L = 1$ let $L = B \dotplus Fc$, B a 2-generated subalgebra of L. Apply Theorem 2, [2]. Either $B = K$ and c is not in L^2 or $B = K \dotplus Fb$ and $L^2 \subset K + Fbc$.

It remains to show that a 2-generated 4-dimensional Lie algebra cannot be perfect. Let a, b, ab, and $a(ab)$ be a basis for L and adjust b so that $b(ab)$ is in $Fa \dotplus Fb \dotplus Fab$. Suppose $a(a(ab)) = \alpha a + \beta b + \gamma ab + \delta a(ab)$ and $b(ab) = \alpha' a + \beta' b + \gamma' ab$. Then $b(a(ab) = \beta' ab + \gamma' a(ab)$ and $(ab)(a(ab)) = a(b(a(ab))) - b(a(a(ab))) = (\alpha\gamma' - \gamma\alpha')a + (\beta\gamma' - \gamma\beta')b + (\alpha - \delta\beta')ab + \beta' a(ab)$ so that the whole multiplication table is determined. Observe L is a Lie algebra if and only if the parameters are so chosen that $J(a, ab, a(ab)) = 0$ and $J(b, ab, a(ab)) = 0$. The first Jacobi identity is equivalent to (1) $2\alpha - \delta\beta' = 0$, (2) $2\beta\gamma' - \alpha\delta = 0$, (3) $2\beta\beta' - \delta(\beta\gamma' - \gamma\beta') = 0$, and (4) $\beta'\alpha + \beta\alpha' - \delta(\alpha\gamma' - \gamma\alpha') = 0$.

Use 1) to rewrite the coefficient of ab in $ab(a(ab))$ *as* $-\alpha$. Then find that the second Jabobi identity is equivalent to (5) $2\beta'\gamma + \alpha'\delta = 0$, (6) $\alpha\beta' + \alpha'\beta + 2\gamma'(\gamma'\beta - \beta'\gamma) = 0$, and (7) $2\alpha\alpha' + 2\gamma'(\gamma'\alpha - \alpha'\gamma) = 0$.

First suppose $\alpha' \neq 0$. Replace a by $a' = \alpha' a + \beta' b$ and obtain with change of notation $\beta' = 0$. Then the equation (5) implies $\delta = 0$ and (4) that $\beta = 0$ and hence b is not in L^2.

Second suppose $\alpha' = 0$. First, we show if $\operatorname{char} F \neq 2$ then $\alpha = 0$. It would then follow that a is not in L^2. By (7) $\alpha\gamma' = 0$ so that (4) implies $\alpha\beta' = 0$, which implies by multiplication of (1) by α that $\alpha = 0$. Finally suppose $\alpha \neq 0$. Then $\operatorname{char} F = 2$ and (2) implies $\delta = 0$ and (4) $\beta' = 0$. Here $\alpha a + \beta b$ is the only linear combination of a and b to occur in L^2.

THEOREM 3. *Let L be a finite dimensional Lie algebra over any field F. Suppose genus $L \leqq 2$. Then L is either solvable or contains a simple subalgebra.*

Proof. This theorem is proved by induction on the dimension of L. Every Lie algebra of dimension less than 3 is solvable. Suppose the theorem proved for all Lie algebras of genus k of dimensions $\leqq n$, k fixed.

Let L be an $(n+1)$-dimensional genus k Lie algebra containing no simple subalgebras. A minimal ideal I of L is solvable and hence abelian. If L/I is solvable then L is solvable. Otherwise L/I contains a simple subalgebra T. Let S be the subalgebra generated by the preimage of a minimal generating set of T under the natural map of L onto L/I. S cannot be solvable for it has a nonsolvable homomorphic image. If $S \neq L$, S, and so L, would have a simple subalgebra. Hence $S = L$ and so $T = L/I$.

Genus $L/I <$ genus L since the preimage of a minimal generating set of L/I is a generating set of L. Since all genus 0 Lie algebras are solvable the theorem has been proved for $k = 0, 1$. If $k = 2$, $\dim L/I = 3$ by Proposition 2 and Corollary 3. Then $\dim L = 4$ since L is generated by two elements. Thus, by Proposition 5, $L^2 \neq L$ so that L^2 (and hence L) is solvable.

The very last argument fails for $k=3$. For there is a 2-generated 5-dimensional perfect Lie algebra with a 3-dimensional simple quotient algebra. For example: $L = S \dotplus A$, with S the 2 by 2 matrices of trace zero operating on the 2-dimensional underlying vector space of the abelian ideal A.

THEOREM 4. *If a finite dimensional genus 1 Lie algebra over any field F contains a simple subalgebra S then $S = L$.*

Proof. Suppose $0 \neq x$ in $L - S$. Then $\dim \langle S, x \rangle = 4$. Therefore $\langle S, x \rangle^2 \neq \langle S, x \rangle$ by Proposition 5 and hence equals S.

Consider D_x on L^2. Let $m(t)$ and $c(t)$ denote its minimal and characteristic polynomials, respectively. Then $\deg m(t) \leqq 2$, otherwise L^2 would have a 2-generated subspace of $\dim > 2$, contradicting genus $L = 1$. The classical decomposition of L^2 into cyclic subspaces for D_a then contains a 1-dimension subspace. Therefore, a linear factor, $t - \lambda$, which divides $m(t)$. Hence $c(t) = (t - \lambda)^3$. It follows that L^2 has two eigenvectors for λ, their product cannot be zero since L^2 is perfect and 3-dimensional, hence corresponds to eigenvalue 2λ. Thus $\lambda = 0$, x is central. The argument applies to $x + a$ for any a in L^2, a contradiction.

LEMMA 1. *A genus* 2 *Lie algebra* L *over any field with* $\dim L > 5$ *contains a proper genus* 2 *subalgebra.*

Proof. If a genus 2 Lie algebra L has more than six generators then there exist a, b, c, d with $\dim \langle a, b, c, d \rangle = 6$.

Suppose a, b, c, d, ab, cd is a basis for A. Consider

$$ac = \alpha a + \beta b + \gamma c + \delta d + \mu ab + \kappa cd \ .$$

If $\mu \neq 0$, a, b, and c generate a genus 2 subalgebra. If $\kappa \neq 0$, a, d, c generate a genus 2 subalgebra. Hence $ac = \alpha a + \gamma c$, or done. Similarly, $bc = \beta_1 b + \gamma_1 c$, $ad = \alpha_2 a + \delta_2 d$, $bd = \beta_3 b + \delta_3 d$, or done. Then $a + c$, b, and d generate a genus 2 subalgebra.

PROPOSITION 6. *A* 5*-dimensional simple Lie algebra over any field is* 2*-generated.*

Proof. By Corollary 3 genus $L \neq 1$ and by earlier work genus $L \neq 0$. There remains the possibility that L has genus 2, i.e., L is minimally 3-generated. So suppose a, b, and c minimally generate L.

If B is a 4-dimensional subalgebra of L then B has genus less than 2 by Proposition 4.

Suppose a, b, ab are linearly independent. If ac and bc are both in $Fa \dotplus Fb \dotplus Fc \dotplus Fab$ then so also is $(ab)c = a(bc) - b(ac)$; for $a(ab)$ and $b(ab)$ are in $Fa \dotplus Fb \dotplus Fab$. Therefore with a change of notation, if necessary, L has basis a, b, c, ab, cb.

If $ac = \alpha a + \beta b + \gamma c + \delta ab + \mu cb$ replace a by $a + \mu b$ and c by $c - \delta b$ and find to avoid L being 2-generated that $ac = \alpha' a + \gamma' c$. By symmetry between a and c it can be supposed that $ac = \varepsilon c$, $\varepsilon = 0$ or 1.

Let

$$\begin{aligned} a(ab) &= \alpha a + \beta b + \gamma ab & c(cb) &= \mu c + \delta b + \kappa cb \\ b(ab) &= \alpha' a + \beta' b = \gamma' ab & b(cb) &= \mu' c + \delta' b + \kappa' cb \\ c(ab) &= \alpha'' a + \beta'' b + \gamma'' ab + \mu'' c + \kappa'' cb\,. \end{aligned}$$

Suppose $\varepsilon = 0$. To avoid c in $\langle a, b + c\rangle$ from $a(a(b + c))$ conclude $\beta = 0$ and from $(b + c)(a(b + c))$ conclude $\kappa'' = 0$. To avoid a in $\langle c, b + a\rangle$ from $c(c(b + a))$ conclude $\delta = 0$ and from $(b + a)(c(b + a))$ conclude $\gamma' = 0$. Then $J(c, a, ab) = 0$ implies $\beta'' = 0$. Either $\beta' \neq 0$ or $\delta' \neq 0$ because b is in L^2, say $\beta' \neq 0$ without loss of generality. Replace c by $c - \beta'^{-1}\delta' a$ and find that it can be supposed that $\delta' = 0$, hence $\mu = 0$. $J(a, b, ab) = 0$ implies $\alpha = \beta'$, $\gamma = 0$, and $\gamma' = 0$. From $b((a + c)b)$ conclude $\eta' = 0$. Note $c(ab) = a(cb)$. Calculate $(cb)(ab)$ using $J(c, b, ab) = 0$ and using $J(cb, a, b) = 0$ find that $\beta' = -2\mu''$, so that char $F \neq 2$ and $2\alpha'' = 0$; hence $\alpha'' = 0$. Then the subalgebra $Fc \dotplus Fbc$ is an ideal of L because each generator carries it into itself.

Suppose $\varepsilon = 1$. From $(b + c)(a(b + c))$ conclude $\eta' = 1$. Note $a(cb) = (ac)b + c(ab)$. Calculate the coefficient of cb in $J(c, a, ab) = 0$ and find that $\beta + \gamma = 1$. Then c is in $\langle a, b + c\rangle$ follows from $a(a(b + c))$.

LEMMA 2. *Let L be a 5-dimensional Lie algebra over any field F. Let B be a proper ideal of L containing a proper simple subalgebra S. Then genus $L = 3$.*

Proof. S must be genus 1 and 3-dimensional and hence B must be 4-dimensional and have genus 2.

Let a, b, ab, and $a(ab)$ be a basis for B. Some linear combination of a and b is an element of B^2. Suppose $b + \lambda a$ is an element of B^2 for some λ in F. If ac $= \alpha'' a + \beta'' b + \gamma'' ab + \delta''a(ab)$ set $c' = c - \delta'' ab - \gamma'' b$ and find that $ac' = \alpha a + \beta b$. If $\beta \neq 0$ a and c' generate L. If $\beta = 0$ let $b' = b + c'$. Claim: a and b' generate L. Since the multiplication table of B is determined by the products $b(ab)$ and $a(a(ab))$ b occurs in one of these products. The algebra generated by a and b' contains ab and $a(ab)$ because $ab' = ab + \alpha a$ and $a(ab') = a(ab)$.

Suppose $b + \lambda a$ is not an element of B^2 for any λ in F. Suppose a, b, chosen so that $b(ab) = \alpha' a + \gamma' ab$ and $(*)a(a(ab)) = \alpha a + \gamma ab + \delta a(ab)$. Then $S = Fa \dotplus Fab \dotplus Fa(ab)$.

Observe if s, t, and st are a basis of a simple Lie algebra L with $s(st) = \alpha s + \beta t + \gamma st$ and $t(st) = \alpha' s + \beta' t + \gamma' st$ then from $J(s, t, st) = 0$ it follows that $\gamma(\alpha' s + \beta' t) + \gamma'(\alpha s + \beta t) = 0$. This is a linear relation between the contributions of $s(st)$ and $t(st)$ to $Fs \dotplus Ft$ and therefore $\gamma = 0$ and $\gamma' = 0$, otherwise S would not be perfect.

Therefore since a and ab generate S from $(*)$ it follows that $\delta = 0$ and from $(ab)(a(ab)) = \gamma'\alpha a + \gamma'\gamma ab + (\gamma'\delta - \gamma')a(ab)$ that $\gamma'\delta - \gamma' = 0$, which implies $\gamma' = 0$. Hence $(ab)(a(ab)) = 0$, contradicting the simplicity of S.

THEOREM 5. *Let L be a finite dimensional genus 2 Lie algebra over any field F. Suppose L contains a simple subalgebra. Then* $\dim L = 4$ *and L^2 is simple 3-dimensional, Furthermore, if* $\operatorname{char} F \neq 2$ *then $L = L^2 \dotplus Fz$, Fz the center of L.*

Proof. Lemma 1 and Proposition 4 imply a simple genus 2 Lie algebra is at most 5-dimensional. Proposition 5 implies there are no 4-dimensional simple Lie algebras and Proposition 6 that there are no genus 2 5-dimensional simple algebras. Therefore L must contain a simple genus 1 Lie algebra S.

Suppose B is a 5-dimensional subalgebra of L containing S. Then genus $B \leqq$ genus L and hence B is not simple. Let I be a proper ideal of B of maximal dimension. By Lemma 2 and Proposition 1, I cannot contain S so it follows that $B = S \dotplus I$. S has dimension 3 so that if $B = S \oplus I$ then S would be an ideal of larger dimension than I. This means that I cannot be trivial as an S module and hence I must be abelian since the other 2-dimensional Lie algebra has a solvable derivation algebra. Since $S^2 = S$ the elements of S must operate on I as transformations of trace 0. Considering that S has dimension 3 while I has dimension 2 we see that B is isomorphic to the semi-direct sum of a 2-dimensional abelian Lie algebra V and the Lie algebra of transformation of trace 0 on V.

This being the case the characteristic of F cannot be 2 since then the 2×2 transformations of trace 0 are solvable.

Let $V = Fe \dotplus Ff$ and $a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $b = \begin{pmatrix} 00 \\ 10 \end{pmatrix}$, and $c = \begin{pmatrix} 01 \\ 00 \end{pmatrix}$ be a basis for S. Then $a + b$ and $c + 2e + f$ generate B. To see this observe $(a + b)(c + 2e + f) - (c + 2e + f) = c - a$. Therefore genus $B \geqq 3$, contradicting Proposition 1.

If $\operatorname{char} F \neq 2$ then S has a basis so that $a(ab) = \alpha b$ and $b(ab) = \beta a$ (Jacobson, page 14, [4]). Let $L = S \dotplus Fx$. Then $xa = \alpha' a + \beta' b + \gamma' ab$ and $xb = \alpha'' a + \beta'' b + \gamma'' ab$. Set $z = x + \gamma' b - \gamma'' a + \alpha^{-1}\beta' ab$, then $za = \alpha' a$ and $zb = \alpha'' a + \beta'' b$. Thus $z(ab) = (\alpha' + \beta'')ab$. From $J(z, a, ab) = 0$ conclude $\alpha' = 0$ and $\alpha'' = 0$ and from $J(z, b, ab) = 0$ that $\beta'' = 0$.

3. A characterization of nonminimal genus 1 and 2 Lie algebras.

A Lie algebra L will be called a minimal genus k Lie algebra, if genus $B =$ genus L, B a subalgebra of L, implies $B = L$.

Suppose L is a finite dimensional Lie algebra with a k-dimensional ideal J with genus $L/J = 0$. Then genus $L \leqq k$ because if $\dim L - k$ elements were to generate L their cosets would generate L/J. It is surprising that conversely nonminimal finite dimensional genus 1 and 2 Lie algebras always have ideals J of dimension 1 and 2, respectively, with genus 0 quotients, except possibly for $F = GF(2)$.

There do exist solvable minimal genus 1 and 2 Lie algebras over some fields without such ideals. Indeed, the construction of metabelian Lie algebras in section one of arbitrarily high genus, provide such an example whenever A can be also taken as an irreducible $F[x]$-module. This is equivalent to requiring that the characteristic polynomial of T is irreducible.

It has been shown that a genus 1 Lie algebra L, $\dim L \geqq 7$, had a 1-dimensional ideal J with genus $L/J = 0$ (Corollary 4). The next theorem shows this to be the case for $\dim L \geqq 4$, which is clearly the best possible result, since there are 3-dimensional genus 1 simple Lie algebras over every field.

THEOREM 6. *Let L be a nonminimal finite dimensional genus* 1 *Lie algebra over any field F. Then L has a* 1-*dimensional ideal J with genus* $L/J = 0$.

Proof. Let $L = B \dotplus C$, C a subspace and B a 3-dimensional subalgebra. Apply Theorem 2, [2] to conclude $B = \langle a, b\rangle$ with (1) $ac - c$, bc in B for all c in C, $B^2 \subset Fb \dotplus Fab$, or (2) $BC \subset B$. (1) If $ac = c + \alpha' a + \beta' b + \gamma' ab$, c in C, then $ac' = c' + (\beta' + \gamma')b$, where $c' = c - \gamma' b + \alpha' a$, so that $ac' = c'$, otherwise genus $\langle a, c'\rangle > 1$. Therefore C can be taken so that $ac = c$ for all c in C. Suppose $bc = \alpha a + \beta b + \gamma ab$. Then $b(c + \gamma a) = \alpha a + \beta b$ and therefore $\alpha = 0$, otherwise genus $\langle c + \gamma a, b\rangle > 1$. Next $(a + b)(c + \gamma a) = (c + \gamma a) - \gamma a + \beta b$ so that $\beta = -\gamma$, i.e. $bc = \beta(b - ab)$. If $a(ab) = \gamma b + \delta ab$ then set $c' = c + ab - (\delta - 1)b$ and conclude from $ac' = c' + (\gamma + \delta - 1)b$ that $\delta = 1 - \gamma$.

Set $J = F(b - ab)$. Then $a(b - ab) = -\gamma(b - ab)$ so that $aJ \subset J$. Next, $(ab)c = -\beta(1 + \gamma)(b - ab)$ so that $cJ \subset J$. $J(a, b, ab) = 0$ implies $b(ab) = \delta' ab$ because $a(ab)$ and $b(ab)$ are in $Fb \dotplus Fab$. Then

$$(b + c)(a(b + c)) = b(ab) + (b - ab)c$$

and must be in $F(b + c) \dotplus F\mathrm{a} \dotplus F(ab + c)$ to avoid genus $\langle b + c, a\rangle > 1$. Hence $\delta' = 0$. Therefore $bJ \subset J$ and J is an ideal of L.

For c, d in C, cd is either an eigenvector for D_a for the eigenvalue 2 or $cd = 0$. Note a is a candidate if char $F = 2$. If $cd = b'$, b'

in B and not in $Fb \dotplus Fab$, then every term of the derived series of L would contain $C + Fa$ modulo $Fb \dotplus Fab$, contradicting L solvable by Theorem 4. If $\gamma \neq -1$, $C \dotplus Fa \dotplus F(ab + \gamma b) \dotplus F(b - ab)$ is an eigenspace decomposition of L for D_a and cd must be in J since $a(ab + \gamma b) = ab + \gamma b$. If $\gamma = -1$ then $Fb \dotplus F(ab - b)$ is an indecomposable eigenspace for D_a $(ab = b + (ab - b))$. Again cd must be in J. Thus genus $L/J = 0$. (2) Clearly C can be taken so that ac and bc are in $Fa \dotplus Fb$ for all c in C. Hence to avoid a genus 2 subalgebra, $ac = \alpha a$ and $bc = \alpha b$. If $\alpha \neq 0$ replace $\langle a, b\rangle$ by $\langle a + c, b\rangle$ reducing this case to (1). To avoid a genus 2 subalgebra from $(a + c)((a + c)b)$ and $a(a(b + c))$ conclude $a(ab)$ is in Fab and from $b((a + c)b)$ and $(b + c)(a(b + c))$ that $b(ab)$ is in Fab. Thus $L^2 = Fab$. Hence if $\dim C = 1$ the proof is complete.

Suppose c, d are linearly independent elements of C with $cd = e + x$, e in C, x in $\langle a, b\rangle$. Note from $(a + c)d$ one may suppose e is in Fc, and from $c(a + d)$ that e is in Fd, otherwise this case is reduced to (1). Hence it may be supposed that $e = 0$. If $x = \alpha a + \beta b + \gamma ab$ then $(c - \gamma a)(d + b) = \alpha a + \beta b$ so that $\alpha \neq 0$ would imply genus $\langle c - \gamma a, d + b, b\rangle > 1$ while $\beta \neq 0$ implies genus $\langle c - \gamma a, d + b, a\rangle > 1$. Thus $J = L^2 = Fab$.

The proof of Theorem 6 provides an excellent model for the proof of the next theorem. It should be remarked that a simpler proof would be possible for the next theorem if fields of char 2 and $GF(3)$ were excluded. We were unable to handle the case when $F = GF(2)$.

THEOREM 7. *Let L be a nonminimal finite dimensional genus 2 Lie algebra over any field F, except $GF(2)$, the field of two elements. Then L has a 2-dimensional ideal J with genus $L/J = 0$.*

Proof. I. Suppose there exists a subalgebra B of L with $\dim B = 4$ and genus $B = 2$. Let C be a subspace complement of B in L. Theorem 2, [2] can be applied. There are three cases to be considered. There exists a basis $a, b, ab, a(ab)$ of B such that for every c in C:

(1) ac and $bc - c$ are in B, which $b(ab) = \delta_0 a + \delta_1 ab$ and $a(a(ab)) = \gamma_0 a + \gamma_1 ab + \gamma_2 a(ab)$.

(2) $ac - c$ and bc are in B, $B^2 \subset Fb \dotplus Fab \dotplus Fa(ab)$.

(3) ac and bc are in B.

(1) Let $bc = c + \alpha' a + \gamma' a(ab)$ and $ac = \alpha a + \beta ab + \gamma a(ab)$, where c has been taken so that bc does not involve b or ab, while if ac involved b then genus $\langle a, c - \beta b - \gamma ab\rangle > 2$.

Set $c_\lambda = c - (\gamma + \lambda\gamma')ab - (\beta + \lambda(\gamma + \lambda\gamma')(1 - \delta_1))b$. Then $(a + \lambda b)c_\lambda$

can be expressed as a linear combination of c_λ, $a + \lambda b$, and b. The coefficient of b must be zero, otherwise genus $\langle a + \lambda b, c_\lambda \rangle > 2$. Therefore $(*)$ $\lambda^2\gamma'(1 + \delta_0 - \delta_1) + \lambda(\alpha' + \gamma(1 + \delta_0 - \delta_1)) + (\beta - \alpha) = 0$ for all nonzero λ in F.

The candidate for J is $F(a + ab) \dotplus Fa(ab)$. To show that J is an ideal with genus $L/J = 0$ it suffices to show (1) $\delta_1 = 1 + \delta_0$ and (2) $\gamma_0 = \gamma_1$. This is because (1) implies $b(a + ab) = \delta_0(a + ab)$ and always $b(a(ab)) = \delta_1 a(ab)$ by $J(a, b, ab) = 0$, while (2) implies $a(a(ab))$ is in J. Hence J is an ideal in B. Also, (1) implies $\alpha' = 0$ and $\alpha = \beta$ from $(*)$ because $F \neq GF(2)$. Then $bc - c$ and ac are in J. The Jacobi identities $J(a, b, c) = 0$, and $J(a, ab, c) = 0$ then imply $(ab)c$ and $(a(ab))c$ are in J. Therefore J is an ideal in L. If $cd = e + \lambda a + \mu b + J$, c, d, e in C, j in J, then conclude from $J(b, c, d) = 0$ that $cd = j + \mu b$ (calculate modulo J). Similarly $J(a, c, d) = 0$ implies $\mu = 0$. Thus genus $L/J = 0$.

$J(a, b, a(ab)) = 0$ implies $(ab)(a(ab)) = (\gamma_0\delta_1 - \gamma_1\delta_0)a + \gamma_0 ab$. From the coefficient of $a(ab)$ in $J(a, ab, a(ab)) = 0$ conclude $2\gamma_0 = 0$. From $J(b, ab, a(ab)) = 0$ conclude $\gamma_2\delta_0 = 0$ and $2\gamma_1\delta_0\delta_1 = 0$. Note, also $J(a, b, c) = 0$ implies $(ab)c = (\alpha - \beta\delta_0 + \gamma_0\gamma')a + (\alpha + \beta - \beta\delta_1 + \gamma_1\gamma')ab + (\gamma - \gamma\delta_1 + \gamma_2\gamma')a(ab)$.

Suppose $bc = c + \alpha' a + \gamma' a(ab)$, $\gamma' \neq 0$, and c cannot be chosen so that $\gamma' = 0$. Then $\delta_1 = 1$, otherwise take $c' = c + (1 - \delta_1)^{-1}\gamma' a(ab)$. Note (1) holds unless $F = GF(3)$ because of $(*)$ and because $F = GF(2)$ has been excluded by hypothesis. Suppose for the moment that $\delta_0 \neq 0$. Then $\gamma_0 = 0$, $\gamma_1 = 0$, and $\gamma_2 = 0$. From the coefficient of ab in $J(a, ab, c) = 0$ conclude $\alpha = 0$, and from the coefficient of a that $\beta = 0$, contradicting $(*)$. To establish (2) calculate $J(b, ab, c) = 0$ and find that $2\alpha + \gamma_1\gamma' = 0$ and $2\alpha + \gamma_0\gamma' = 0$. Multiplication of the last relation by 2 implies $2\alpha = 0$ since $2\gamma_0 = 0$. Therefore $\gamma_0 = \gamma_1 = 0$.

Suppose $bc = c$. The $(*)$ implies $\alpha = \beta$. To avoid c in $\langle b, a + c\rangle$ conclude from $b((a+c)b)$ that (1) holds. Then $(ab)c = (\alpha - \alpha\delta_0)(a + ab) - \gamma\delta_0 a(ab)$. $J(b, ab, c) = 0$ implies $2\alpha - \alpha\delta_0 = 0$. Use $\gamma_2\delta_0 = 0$, $\alpha\delta_0 = 2\alpha$, and $2\gamma_0 = 0$ to find because of $J(a, ab, c) = 0$ that $a(ab)c = -\gamma(\gamma_0 - \gamma_1\delta_0)a - \gamma(\gamma_0 + \gamma_1\delta_0)ab - \alpha a(ab)$. Set $a' = a + \lambda c$. Then $a'b = ab - \lambda c$, $a'(a'b) = \lambda' a(ab)$, where $\lambda' = 1 - \lambda(\gamma - \gamma\delta_0)$, and $a'(a'(a'b) = \lambda'(\gamma_0 + \lambda\gamma(\gamma_0 - \gamma_1\delta_0))a + \lambda'(\gamma_1 + \lambda\gamma(\gamma_0 + \gamma_1\delta_0))ab + \lambda'(\gamma_2 + \lambda\alpha)a(ab)$. Finally, the discriminant of $a', b, a'b, a'(a'b)$, $a'(a'(a'b)))$ with respect to $a, b, ab, a(ab), c$ is $\lambda'^2\lambda^2[(\gamma_1 - \gamma_2) + 2\lambda\gamma_1\delta_0]$, which must be zero for all λ in F to avoid genus $\langle a', b\rangle = 3$. This establishes (2) with the possible exception of $F = GF(3)$. Here $\gamma_0 = 0$ and $\gamma_1\delta_0\delta_1 = 0$. Done unless $\delta_1 = 0$, in which case $\delta_0 = -1$. Then $\lambda' = 1 - 2\alpha$ and $\gamma_1 - \gamma_0 + 2\lambda\gamma_1\delta_0 = \gamma_1(1 - 2\lambda)$ so that there exists a choice of λ so that $\gamma_0 = \gamma_1$.

The remaining cases are relatively easy.

(2) Take C so that $ac=c$ and suppose $bc=\alpha_c a+\beta_c b+\gamma_c ab+\delta_c a(ab)$ for all c in C. Note if $b(ab)=\delta_0 b+\delta_1 ab+\delta_2 a(ab)$, with $\delta_2 \neq 0$, setting $b'=b-\delta_2 a$ reduces this case to (1). Hence suppose $\delta_2=0$. Let $a(a(ab))=\gamma_0 b+\gamma_1 ab+\gamma_2 a(ab)$.

Set $c'=c+a(ab)-(\gamma_2-1)ab-(\gamma_1+\gamma_2-1)b$. Then $ac'=c'+(\gamma_0+\gamma_1+\gamma_2-1)b$. Therefore (1) $\gamma_2=1-\gamma_0-\gamma_1$.

Set $c_\lambda=c-\lambda\delta_c ab-\lambda(\delta_c+\gamma_c-\lambda\delta_c\delta_1)b$. Conclude from $(a+\lambda b)c_\lambda$ that (2) $\beta_c+\gamma_c+\delta_c=0$ and (3) $\alpha_c+\delta_c(\delta_0+\delta_1)=0$.

Consider $(b+\lambda c)(a(b+\lambda c))$. Note to avoid c in $\langle a, b+\lambda c\rangle$ the sum of the coefficients of b, ab, and $a(ab)$ in the linear expression of the product under consideration must be zero. This implies (4) $\delta_0+\delta_1=0$. Hence (5) $\alpha_c=0$ follows from (3).

Set $J=F(b-ab) \dotplus F(ab-a(ab))$. Then (1) implies $aJ\subset J$ and (4) implies $b(b-ab)$ is in J, from which it follows $b(ab-a(ab))=b(ab)-a(b(ab))$ is in J. Next, bc is in J because of (2) and (5). The Jacobi identities $J(a,b,c)=0$ and $J(a,ab,c)=0$ then imply $cJ\subset J$. Thus J is an ideal in L.

Suppose $cd=e+\gamma a+\mu b+j$, c, d, e in C, j in J. The $J(a,c,d)=0$ implies $e=0$ and $\mu=0$, while $J(b,c,d)=0$ implies $\gamma=0$. Therefore genus $L/J=0$.

(3) Suppose $b(ab)=\delta_0 a+\delta_1 b+\delta_2 ab$ and $a(a(ab))=\gamma_0 a+\gamma_1 b+\gamma_2 ab+\gamma_3 a(ab)$. Take C any complement of B so that $ac=\alpha_c a$ and $bc=\beta_c a+\gamma_c b+\delta_c(ab)$ for all c in C.

There exists a nonzero λ in F so that $a'=a+\lambda c$, b, $a'b$, and $a'(a'b)$ are linearly independent because $F\neq GF(2)$. Then $\alpha_c=0$ because $a'c=\alpha_c a'-\lambda\alpha_c d$, otherwise this case would be reduced to (1) or (2). Repeating this argument for $c'=c+a(ab)-\gamma_3 ab-\gamma_2 b$ from ac' conclude $\gamma_0=0$ and $\gamma_1=0$.

If $\gamma_c=0$ replace b by $b+c$ to reduce this case to either (1) or (2). From $(b+c)(a(b+c))$ it follows that $\delta_1=0$. From the coefficient of $a(ab)$ in $J(b,ab,c)=0$ conclude $\beta_c=0$.

Note $Fa'b \dotplus Fa'(a'b)=Fab \dotplus Fa(ab)$, a' chosen as above. Then from $b(a'b)$ conclude $\delta_0=0$.

Set $J=Fab \dotplus Fa(ab)$. If $\dim C=1$ the proof of this case is complete. If not, suppose c and d are linearly independent elements of C, with $cd=e+b'$, e in C, b' in B. Note $\dim\langle b+c, a\rangle$ and $\dim\langle b+d, a\rangle=4$. Then in order to avoid a reduction to a previous case conclude from $(b+c)d$ that e is in Fc and from $(b+d)c$ that e is in Fd. Hence e is zero. If $bc=\gamma_c a(ab)$ then $(a(ab))c=\gamma_c a(a(a(ab)))$. Therefore $b(cd)=(bc)d-c(bd)=\gamma_c(a(ab))d-\gamma_d(a(ab))c=0$. Clearly $a(cd)=0$. Thus cd is in the center of B.

Suppose x is a nonzero element of the center of B. Then $x=$

$\gamma a + \mu b + \kappa ab + \eta a(ab)$, $\eta \neq 0$, otherwise $ax = 0$ and $bx = 0$ imply $x = 0$. From $bx = 0$ conclude $\delta_2 = 0$ and hence $\gamma = 0$. If $cd = x$, $\mu \neq 0$ then $(b + c)d = \mu c + b''$, b'' in $\langle b + c, a\rangle$, contradicting genus $L = 2$. Therefore $L^2 = J$.

II. Let $L = B \dot{+} D$, D a subspace, B a subalgebra with genus $B = 2$ and $\dim B = 5$. Suppose L has no genus 2 4-dimensional subalgebras. Apply Theorem 2, [2] to conclude (1) for some generator of B, say, b, $bd - d$ is in B for all d in D, and for the remaining generators a and c of B ad and cd are always in B or (2) $BD \subset D$. In (1) a further normalization is desirable Either a, b, and ab or b, c, and cb are linearly independent, say a, b and ab are linearly independent. If cb depends on a, b, ab, c replace b by $b + c$ and obtain with a change in notation that a, b, c, ab, cb span B. Note $B^2 \subset Fa \dot{+} Fc \dot{+} Fab \dot{+} Fcb$. (1) D can be taken so that $bd = d$ for all d in D. Suppose $b(ab) = \gamma_0 a + \gamma_1 ab$. Conclude from $b(d + ab - (1 - \gamma_1)a)$ that $\gamma_1 = 1 + \gamma_0$. An exactly analogous argument leads to $b(cb) = \gamma_0' c + (1 + \gamma_0')cb$. Then $\gamma_0 = \gamma_0' = \gamma$ because $b((a + c)b)$ must be a linear combination of $a + c$, b, and $ab + cb$. Next $a(ab) = \delta_1 ab$ because $a(ab)$ and $b(ab)$ are in $Fa \dot{+} Fab$ by $J(a, b, ab) = 0$. Note $(b + \lambda a)(a(b + \lambda a) = b(ab) + \lambda a(ab) = \gamma a + (1 + \gamma + \lambda\delta_1)ab$. Therefore $\delta_1 = 0$ since there exists λ in F such that a, $b + \lambda a$, c, ab, $c(b + \lambda a)$ are linearly independent. Analogously, $c(cb) = 0$.

Suppose $ac = \beta a + \gamma c + \delta ab + \mu cb$. For any nonzero λ in $F(a + \lambda d)c$ must be a linear combination of $a + \lambda d$, $ab - \lambda d$, b, c, and bc. Therefore $\alpha = \delta$. Analogously, $\gamma = \mu$. The same argument applies to $c((a + \lambda d)b)$. This implies $c(ab) = \alpha'(a + ab) + \gamma' c + \mu' cb$. Analogously, $a(cb) = \alpha'' a + \beta'' ab + \gamma''(c + cb)$. Set $J = F(a + ab) \dot{+} F(c + cb)$. Then $bJ \subset J$. Then $J(a, b, c) = 0$ implies $\gamma' = \mu'$ and $\alpha'' = \beta''$, i.e. $c(ab)$ and $a(cb)$ are in J. It is immediate that J is an ideal in B.

Suppose $ad = \alpha_1 a + \beta_1 b + \gamma_1 c + \delta_1 ab + \mu_1 cb$ and $cd = \alpha_2 a + \beta_2 b + \gamma_2 c + \delta_2 ab + \mu_2 cb$. For any nonzero λ in F such that $1 - \lambda\gamma \neq 0$ set $d_\lambda = d + (\lambda/(1 - \lambda\gamma))\mu_1 c + (\lambda(\delta_1 + \lambda/(1 - \lambda\gamma))\mu_1\alpha)a$. Then $(b + \lambda a)d_\lambda = d_\lambda + \lambda\beta_1 b + \lambda(\alpha_1 - \delta_1)a + \lambda(\gamma_1 - \mu_1)c$. Therefore $\gamma_1 = \mu_1$ and $\lambda\beta_1 = \alpha_1 - \delta_1$. Analogously, $\alpha_2 = \delta_2$ and $\lambda\beta_2 = \gamma_2 - \mu_2$. Hence ad and cd are in J with the possible exception $F = GF(3)$, $F = GF(2)$ being excluded by hypothesis. To handle this special case note first $(ab)d = ad + (ad)b$ by $J(a, b, d) = 0$. Then $J(a, ab, d) = 0$ can be rewritten as the relation $2(1 + \gamma)ad + (2 + \gamma)(ad)b = b((ab)d)$. Here a b term can occur only in $2(1 + \gamma)ad$. Done, unless $2(1 + \gamma) = 0$. The above relation now simplifies to $(ad)b = b((ab)d)$. Note $b(ab) = -a$, $b(cb) = -c$. Calculate both sides of this last relation and find $\alpha_1 = \gamma_1 = \delta_1 = \mu_1 = 0$. Then $\beta_1 = 0$, otherwise genus $\langle a, d, c\rangle > 2$. A similar argument implies cd is always in J. It follows that $dJ \subset J$. Hence J is an ideal in L.

If $de = f + \alpha a + \beta b + \gamma c + j$, d, e, f in D, j in J, then $J(b, d, e) = 0$ implies $de = \beta b + j$. If $\beta \neq 0$ then L/J would not be solvable for every term of its derived series would contain $(D \dotplus Fb)/J$. Hence de is in J. Therefore genus $L/J = 0$.

(2) Suppose a and c chosen so that ac is in $Fa \dotplus Fc$. Then take D so that $bd = \lambda_d b$. If $\lambda_d \neq 0$ this case reduces to last one since there exists λ in F so that genus $\langle b + \lambda d, a, c\rangle = 2$. Suppose $bd = 0$ for all d in D. If $b(ab) = \delta_0 a + \delta_1 b + \delta_2 ab$ then repeat this argument for b and $d + ab + \delta_2 a$. Conclude $b(ab) = \delta_2 ab$. Likewise $b(cb)$ is in Fcb. Suppose $ad = \alpha a + \beta b + \gamma c + \delta ab + \mu cb$. Repeat the argument for $b + \lambda a$ and $d + \lambda\delta a + \lambda\mu c$ and conclude ad is in $J = Fab \dotplus Fcb$. Similarly, cd is in J. It now follows immediately that $ac = 0$ from $(a + d)c$ a linear combination of $a + d, b, ab, c, cb$ and $a(c + d)$ a linear combination of $a, b, ab, c + d, cb$. Thus $(ab)c = a(bc)$ by $J(a, b, c) = 0$. Clearly genus $\langle a, b, c + d\rangle > 2$ if $a(bc)$ involved c, while genus $\langle a, b, c + d\rangle > 2$ if $a(bc)$ involved c, while genus $\langle a + d, b, c\rangle > 2$ if $(ab)c$ involved a. Therefore $a(bc)$ and $(ab)c$ are both in J. $J(a, b, ab) = 0$ implies $a(ab) = \gamma_1 b + \gamma_2 ab$. Consider $a(a(b + c))$ to conclude $\gamma_1 = 0$. Likewise $J(c, b, cb) = 0$ and $c(c(b + a))$ forces $c(cb)$ to be in J. Suppose $de = f + \alpha a + \beta b + \gamma c + j$, d, e, f in D, j in J. Consider $(a + \lambda d)e$. To avoid a previous case or genus $L > 2$ conclude $f = 0$ and $\alpha = 0$. It has been shown that ad is in B^2, hence $(a + \lambda d)e$ is in $\langle a + \lambda d, b, c\rangle^2 = J$. Therefore $\beta = 0$ and $\gamma = 0$. Thus $J = L^2$.

4. The classification of genus 1 and 2 Lie algebras.

PROPOSITION 7. *If L is nilpotent then genus $L = \dim L^2$.*

Proof. The result follows from the fact that the span of any minimal set of generators intersects the derived algebra in zero.

PROPOSITION 8. *Let L be nilpotent with $L^2 = Fz$. Let μ be the alternating bilinear form defined by $xy = \mu(x, y)z$ for all x and y in L. Then the isomorphism class of L is determined by the integers* $\dim L$ *and rank μ.*

Proof. Suppose ψ is an isomorphism of L onto $\psi(L)$. Then if $\psi(\mu)$ is defined by $\psi(x)\psi(y) = \psi(\mu)(\psi(x), \psi(y))\psi(z)$ from $\psi(xy) = \psi(\mu(x, y)z) = \mu(x, y)\psi(z)$ it follows that μ and $\psi(\mu)$ are equivalent.

Conversely, suppose that $V = W \dotplus Fz$ is a vector space over F and that μ and $\hat{\mu}$ are two equivalent bilinear forms on $V \times V$, i.e.

$\mu(x, y) = \hat{\mu}(T(x), T(y))$ for a linear transformation T of V. Turn V into an algebra $(V, \cdot)$ over F by $xy = \mu(x, y)z$ and an algebra $(V, *)$ by $T(x)*T(y) = \hat{\mu}(T(x), T(y))T(z)$. Then T is an isomorphism of $(V, \cdot)$ onto $(V, *)$ because $T(xy) = T(\mu(x, y)z) = \mu(x, y)T(z) = \hat{\mu}(T(x), T(y))T(z) = T(x)*T(y)$.

PROPOSITION 9. *A nonnilpotent Lie algebra with* $\dim L^2 = 1$ *is the direct sum of an abelian ideal and a 2-dimensional non-abelian ideal.*

Proof. Let $L^2 = Fz$. If $Lz = 0$ then L is nilpotent. Take x so that $xz = z$. If $xy = \gamma z$ then $x(y - \gamma z) = 0$. Thus the inner derivation D_x decomposes L into 1-dimensional eigenspaces all with eigenvalue 0 except for Fz corresponding to eigenvalue 1. The proposition follows from the fact that the product of two eigenvectors is again an eigenvector.

PROPOSITION 10. *Let L be a genus 1 Lie algebra with a 1-dimensional ideal J having nonabelian genus 0 quotient. Then*

(1) *L^2 is abelian and either*

(a) $L = L^2 \dotplus Fx$, $L^2 = B \dotplus Fa \dotplus Fz$ *with* $xz = z$, $xa = a + z$, $xb = b$ *for all b in B. or*

(b) $L = A \dotplus Fz \dotplus Fx$ *with* $xz = \lambda z$, $\lambda \neq 1$, $az = 0$ *and* $xa = a$ *for all a in A. or*

(2) *L^2 is nilpotent genus one,* $L^2 = A \dotplus Fz$, $L = L^2 \dotplus Fx$, *with* $xz = 2z$ *and* $xa = a$ *for all a in A.*

In (b) x is characterized so that λ determines the isomorphism class of L, while in (2) L^2 determines the isomorphism class of L. The algebras described in (a) and (b) are never isomorphic because D_x has an indecomposable 2-dimensional eigenspace in (a) and not in (b).

Proof. Let $L = A \dotplus Fx \dotplus Fz$, where $xz = \lambda z$, $az = \beta_a z$, $xa = a + \alpha_a z$, and $ab = \mu_{a,b} z$ for all a, b in A. For any a in A, $\beta_a = 0$, because a is in L^2 modulo Fz and therefore the trace of D_a on Fz is zero. From $J(x, a, b) = 0$ conclude $(2 - \lambda)\mu_{a,b} = 0$

If $\lambda = 1$ some $\alpha_a \neq 0$ otherwise L has genus 0. Take a so that $\alpha_a = 1$. Then $Fa \dotplus F(xa - a)$ is an indecomposable eigenspace for D_x.

If $\lambda \neq 1$ replace a by $a + (1 - \lambda)^{-1}\alpha_a z$ and find that $xa = a$.

The algebra (1) (a) is obtained when $\lambda = 1$; (1) (b) when $\lambda \neq 1$ and $\mu_{a,b} = 0$ for all a and b in A, and (2) when $\lambda = 2$ and some $\mu_{a,b} \neq 0$.

The nonminimal genus 1 Lie algebras have been classified. A minimal genus 1 Lie algebra has dimension 3 and the classification of all such algebras is discussed in [4], pages 11 through 14.

The next two propositions furnish the key to the classification of the genus 2 Lie algebras.

PROPOSITION 11. *Let M be a commutative associative algebra of 2 by 2 matrices over a field F. Then $\dim M \leqq 2$. If $\dim M = 2$ then M has an identity.*

Proof. Let E_{ij} denote the 2 by 2 matrix with a 1 in the ith jth spot and zeros elsewhere. The only two distinct E_{ij} to commute are E_{11} and E_{22}. If $\dim M > 2$ no spot can always be zero. Hence M contains $A = \begin{pmatrix} 1\beta \\ 0\gamma \end{pmatrix}$, $B = \begin{pmatrix} 0\beta' \\ 1\gamma' \end{pmatrix}$, and $C = \begin{pmatrix} 0\beta'' \\ 0\gamma'' \end{pmatrix}$. If $\gamma'' \neq 0$ take $\gamma = 0$ and $\gamma' = 0$ and it follows $AB \neq BA$, while if $\lambda'' = 0$, $\beta'' \neq 0$, it follows by taking $\beta = 0$ and $\beta' = 0$ that $BC \neq CB$.

If $\dim M = 2$ then $\dim \langle M, I \rangle = 2$, since $\langle M, I \rangle$ is commutative. Hence I is in M.

PROPOSITION 12. *Let M be a 2-dimensional commutative associative algebra of 2 by 2 matrices over any field F.*

If $\operatorname{char} F \neq 2$ then M has a basis I and T with $T^2 = \beta I$, β in F. If $\operatorname{char} F = 2$ then if M has a basis I and T with $T^2 = \delta T + \beta I$, $\delta \neq 0$, then M contains no S with $S^2 = \beta' I$, and furthermore T may be taken so that $\delta = 1$.

If M_i is such an algebra with basis I and T_i, $T_i^2 = \beta_i I$, $i = 1, 2$. Then M_1 and M_2 are isomorphic if and only if $\beta_1 = \lambda^2\beta_2$, λ a nonzero element of F. If $\operatorname{char} F = 2$ and $T_i^2 = T_i + \beta_i I$, $i = 1, 2$, then M_1 and M_2 are isomorphic if and only if there exists μ in F with $\mu^2 + \mu + \beta_1 + \beta_2 = 0$.

Proof. If $\varphi: M_1 \to M_2$ is an onto isomorphism then I and T a basis of M_1 go into I and $\varphi(T)$ a basis for M_2. Therefore, $T^2 = \delta T + \beta I$ implies $\varphi(T)^2 = \delta\varphi(T) + \beta I$. If also $\varphi(T)^2 = \delta_1\varphi(T) + \beta_1 I$. Then $(\delta - \delta_1)\varphi(T) = (\beta_1 - \beta)I$ implies $\delta = \delta_1$ and $\beta = \beta_1$. Hence to prove the proposition it is only necessary to find a canonical choice for T within M.

Suppose $T^2 = \delta T + \beta I$. Set $S = \lambda T + \mu I$, $\lambda \neq 0$. Then $S^2 = (\lambda\delta + 2\mu)S + (\beta\lambda^2 - \mu\lambda\delta - \mu^2)I$.

If $\operatorname{char} F = 2$ and $\delta = 0$ then β can be adjusted by squares as in the $\operatorname{char} F \neq 2$ case. If $\delta \neq 0$ no allowable choice of λ and μ gives a zero coefficient of S; hence choose $\lambda = \delta^{-1}$. Then the coefficient of I becomes $\mu^2 + \mu + \beta\delta^{-2}$. Here if I and T_1 and I and T_2 were basis for M with δ_1 and δ_2 nonzero then there would exist μ_1 and μ_2 such that $\mu_1^2 + \mu_1 + \beta_1 = \mu_2^2 + \mu_2 + \beta_2$, i.e. $\mu^2 + \mu + \beta_1 + \beta_2 = 0$ for $\mu = \mu_1 + \mu_2$.

A nilpotent algebra A is said to have index n if $A^n \neq 0$ while $A^{n+1} = 0$.

A genus 2 index 2 nilpotent Lie algebra can only be decomposed into two nonzero genus 1 Lie algebras. The only decomposition of a genus 1 Lie algebra is the decomposition into its center and an indecomposable genus 1 nilpotent algebra.

The classification and isomorphism of problems for genus 1 nilpotent Lie algebras are both solved by considering the bilinear form introduced in Proposition 8. A generalization of this approach to the genus 2 index 2 nilpotent Lie algebras leads to a solution of the isomorphism problem over some fields, but never to a solution of the classification problem. This point is clarified by introducing the definition: $L = L_1 \overset{*}{+} L_2$, L_1 and L_2 subalgebras of L, $L_1 \cap L_2 \subset L^2$ and $[L_1, L_2] = 0$. If $L = L \overset{*}{+} L_2$ implies $L \leqq L^2$ or $L_2 \leqq L^2$ then L is said to be (*) indecomposable. The only (*) indecomposable genus 1 nilpotent Lie algebras are Fx with x in $L - L^2$ and a 3-dimensional genus 1 nilpotent Lie algebra ($L = Fx + Fy + Fz$ with $[x, y] = z$). The number of 3-dimensional summands in L is counted by one-half the rank of μ (as introduced in Proposition 8). The situation for the genus 2 index 2 case appears to be quite different. We are unable to describe the (*) indecomposable Lie algebras in this case.

Let $\mu_1 + L^2, \cdots, \mu_n + L^2$ be a basis for L/L^2. If $L^3 = 0$ then the multiplication in L can be specified by an n by n skew symmetric matrix $M = (u_{ij})$, u_{ij} in L^2 and defined by $[\mu_i, \mu_j] = u_{ij}$. A change in basis for L/L^2 leads to the replacing of M by DMD^t, D a nonsingular matrix with entries in F.

Ordinary decomposition of M is equivalent to (*) decomposition of L. The classification problem for the genus 2 index 2 nilpotent Lie algebras reduces to finding a canonical form for the matrix M. An

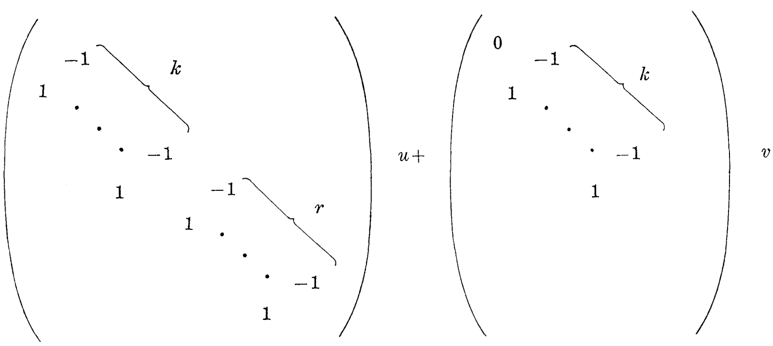

FIGURE 1

infinite number of reasonable candidates for (*) indecomposable algebras are described by matrices of Figure 1 with $Fu \dotplus Fv = L^2$, $k \geqq 1$, $r \geqq 1$.

We have not found a calculable property of M invariant under $M \rightarrow DMD^t$, D a nonsingular matrix with elements in F, which would distinguish indecomposable from decomposable M.

The classical theory of regular pencils of skew symmetric matrices leads to a partial solution of the isomorphism problem for genus 2 index 2 nilpotent Lie algebras. We associate to M the regular pencil of symmetric matrices $\Lambda = A + \lambda B$, where $M = v_1A + v_2B$ and the notation and terminology is that of Gantmacher [3]. Theorem 6 and its corollary, p. 91 and p. 92, vol. II [3], when applicable to F, give a criterion in terms of the elementary divisors of Λ. A change in the choice of a basis v_1, v_2 of L^2 is equivalent to substituting $\alpha x + \beta y$ for x and $\gamma x + \delta y$ for y, with $\alpha\delta - \beta\gamma \neq 0$, $\alpha, \beta, \gamma, \delta$ in F. Once the elementary divisors associated to L by writing $M = v_1A + v_2B$ and the elementary divisors associated to L' by writing $M' = v_1'A + v_2'B$ have been calculated, the isomorphism question for L reduces to calculating whether there exists a substitution replacing x by $x + y$ and y by $x + y$ carrying the elementary divisors associated with L into those associated with L'.

PROPOSITION 13. *Let L be a nilpotent genus 2 Lie algebra over any field F. Then either the index of L is 2 or $L = C \dotplus Fa \dotplus Fb \dotplus L^2$, with $L^2 = Fu \dotplus Fv$, $C \dotplus Fu$ a genus 1 nilpotent subalgebra, $C(Fa \dotplus Fb \dotplus L^2) = 0$, $au = 0$, $av = u$, $ba = v$, $bu = 0$, and $bv = 0$. The isomorphism class of L is determined by $\dim L$ and that of $C \dotplus Fu$.*

Proof. Let $L^2 = Fu \dotplus Fv$, which is possible by Proposition 7. Let $M\left\{\begin{pmatrix}\alpha\beta\\ \gamma\delta\end{pmatrix},\ xu = \alpha u + \gamma v,\ xv = \beta u + \delta v \text{ for some } x \text{ in } L\right\}$. Then M is a commutative associative algebra which must consist of nilpotent transformations and hence $\dim M \leqq 1$ (Proposition 12). Therefore $M = (0)$ and index $L = 2$ or $M = F\begin{pmatrix}01\\ 00\end{pmatrix}$ for u, v a suitably chosen basis of L^2.

If index $L > 2$ there is an element a of L determined up to L^2 by requiring $au = 0$ and $av = u$. Note if $bL^2 = 0$ and $cL^2 = 0$ then $J(a, b, c) = 0$ implies bc is in Fu. Therefore there must exist b' in L such that $b'L^2 = 0$ and $b'a = v + \lambda u$, otherwise v would not be in L^2. Set $b = b'' + v$ and find $ba = v$. Let C be a complement of $L^2 \dotplus Fa \dotplus Fb$ in L such that $C(L^2 \dotplus Fa) = 0$. Note b is also determined up to C and C is determined up to L^2, and hence the isomorphism class of L is determined by rank μ, where μ is defined as in the statement of Proposition 8.

The following situation arises a number of times in the propositions to follow. Therefore it seems best to determine the suitable invariants once and for all.

Let A be an n-dimensional subspace of a Lie algebra L over a field F such that $A^2 \subset Fz$ for some z in L. Suppose $A = B \dotplus Fa$. Let $\mu: A \times A \to F$ denote the alternating bilinear from defined by $cd = \mu(c, d)z$ for all c and d in A. Choose any basis of B and any complementary vector of B in A and associate to μ a matrix C in the usual way. It is easy to see that a canonical form for C is:

$$\begin{pmatrix}
 & & & \underbrace{1 \cdots 1}_{s} \\
K & & & \\
 & \ddots\, r & & \\
 & & K & \\
\left.\begin{matrix}-1\\ \vdots \\ -1\end{matrix}\right\} s & & &
\end{pmatrix}$$

FIGURE 2

where $K = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Let is (B, A) denote the triplet of integers (r, s, n). Note the integer r is the rank of μ restricted to B, while s is $\dim \{d$ in the radical of μ restricted to $B \mid ad \neq 0\}$.

If $A = C \dotplus Fa \dotplus Fb$ and the allowable basis for A must be of the form a basis of C and elements of the form $a + c$, $b + d$, c and d in C then the canonical form of μ becomes:

$$\begin{pmatrix}
 & & & & & \underbrace{1 \cdots 1}_{t} \\
 & & & & \underbrace{1 \cdots 1}_{s} & \\
 & K & & & & \\
 & & \ddots\, r & & & \\
 & & & K & & \\
 & \left.\begin{matrix}-1\\ \vdots \\ -1\end{matrix}\right\} s & & & & \\
\left.\begin{matrix}-1\\ \vdots \\ -1\end{matrix}\right\} t & & & & &
\end{pmatrix}$$

FIGURE 3

In this case let is $(C, a, b) = (r, s, t, n)$.

PROPOSITION 14. *Let L be a nonnilpotent genus 2 Lie algebra with a 2-dimensional abelian derived algebra. Let ρ denote the adjoint representation of L on L^2. Then either* (1) *$\rho(L)$ contains a nonsingular transformation and the kernel of ρ is the center of L, with the isomorphism class of L determined by the isomorphism class of the associative algebra $\rho(L)$, or* (2) $\rho(L) \cong F\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$. *Here $L = B \dotplus Fa \dotplus Fu \dotplus Fv$, $au = u$, $av = 0$, $bu = 0$, $bv = 0$ for all b in B, and $B \dotplus Fa \dotplus Fv$ is nilpotent genus 1. Here $is(B, a)$ determines the isomorphism class of L.*

Proof. (1) Let $\rho(a)$ be a nonsingular. Let $B = \{b - \rho(a)^{-1}ab \mid b$ in $L\}$. Then $\rho(B) = 0$. From $J(a, b, c) = 0$, if either $\rho(b) = 0$ or $\rho(c) = 0$ it follows that $bc = 0$.

(2) Let $au = u$, $av = 0$. Note if $ab' = u + v$ then $ab = v$ for $b = b' - u$. Hence B can be taken so that $L = B \dotplus Fa \dotplus Fu \dotplus Fv$, with $B(Fu \dotplus Fv) = 0$ and $aB \subset Fv$. From $J(a, b, c) = 0$ conclude bc in Fv for b and c in B. Thus $B \dotplus Fa \dotplus Fv$ is a nilpotent genus 1 algebra. It cannot be abelian for then v would not be in L^2. Note regardless of the choice of C with $B = C \dotplus D$ as in the statement of the proposition that a can be taken so that $ac = 0$, while $\dim \{d$ in $D \mid ad \neq 0\}$ is independent of the choice of a.

PROPOSITION 15. *Let L be a genus 2 Lie algebra with a 2-dimensional nonabelian ideal J with genus 0 quotient. Then there exists an ideal A, which is non-abelian genus 0, such that $L = A \oplus J$.*

Proof. Let $L = B \dotplus J$, B a subspace of L. Suppose $uv = v$, u and v a basis for J. Since the derived algebra of an ideal is an ideal $bu = \alpha u + \gamma v$ and $bv = \beta v$, b in B. Then $J(b, u, v) = 0$ implies $\alpha = 0$. Set $b' = b - \gamma v + \beta u$ and find that $b'u = 0$ and $b'v = 0$. Let A be spanned by the b', b in B. Suppose $ab = c + k$, c in A, k in J. Then $J(u, a, b) = 0$ and $J(v, a, b) = 0$ imply $ab = c$. Hence $L = A \oplus J$, A genus 0. If A were abelian then genus $L = 0$.

LEMMA 3. *Let L be a genus 2 Lie algebra over a field F with a 2-dimensional abelian ideal J having a non-abelian genus 0 quotient. Let ρ denote the adjoint representation of L on J. If $\dim \rho(L) > 1$ then J has a basis so that*

(i) $\rho(a) = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$ *and* $\rho(x) = \begin{pmatrix}\alpha & 0\\ 0 & \alpha + 1\end{pmatrix}$, *or*

(ii) Char $F = 2$, (a) $\rho(a) = \begin{pmatrix}0 & \lambda\\ 1 & 0\end{pmatrix}$, $\lambda \neq 0$, $\rho(x) = \begin{pmatrix}\alpha & 0\\ 0 & \alpha + 1\end{pmatrix}$

(b) $\rho(a) = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ *and* $\rho(x) = \begin{pmatrix}0 & \alpha\\ 1 & 1\end{pmatrix}$.

Proof. Let $L = A \dot{+} Fx \dot{+} J$, with $xa - a$ and ab in J for all a and b in A. Then (*): $\rho(x)\rho(a) - \rho(a)\rho(x) = \rho(a)$, for all a in A. $\dim \rho(A) = 2$ implies some $\rho(a) = I$, contradicting (*).

Suppose $\rho(a) \neq 0$ and its eigenvalues are not in F. Then there exists a basis of J such that $\rho(a) = \begin{pmatrix} 0 & \lambda \\ 1 & 0 \end{pmatrix}$, since trace $\rho(a) = 0$ because of (*). If $\rho(a)$ has zero as its only eigenvalue then there exists a basis of J such that $\rho(a) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. In either case it may be supposed that $\rho(x) = \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}$. Then (*) implies (i) or (ii) a.

If $\rho(a)$ has nonzero eigenvalues in F, a may be taken so that 1 and -1 are its eigenvalues. Choose a basis for J so that $\rho(a) = \begin{pmatrix} 1 & \varepsilon \\ 0 & -1 \end{pmatrix}$, $\varepsilon = 0$ or 1, $\varepsilon = 1$ only if $\operatorname{char} F = 2$. Take $\rho(x) = \begin{pmatrix} 0 & \beta \\ \alpha & \delta \end{pmatrix}$. Then (*) implies ii (b).

PROPOSITION 16. *A non-minimal genus 2 Lie algebra over a field F, with char $F \neq 2$, has a nilpotent derived algebra.*

Proof. If J and L/J are abelian then the desired result is clear. If J is abelian and L/J is non-abelian genus 0 the result follows immediately from Lemma 3. If J is non-abelian and L/J is genus 0 the result is clear from Proposition 15.

The solution of the problem of extending an abelian 2-dimensional algebra J by a non-abelian genus 0 algebra is quite complicated. In the next proposition, accordingly, the description of the different algebras which arise is less explicit than those given in Proposition 10. Nevertheless, it is not difficult to derive a similarly explicit list of Lie algebras from the information given in the next proposition.

PROPOSITION 17. *Let L be a genus 2 Lie algebra over a field F having a 2-dimensional abelian ideal J with non-abelian genus 0 quotient. Suppose L^2 is nilpotent. Then*

(1) *$L = J \dot{+} A \dot{+} Fx$, with $xa = a$ for all a in A. Here $(A \dot{+} J)^2 \subset K \subset J$, where $(\rho(x) - 2I)K = 0$. If $AJ = 0$, to avoid genus $L < 2$ it is necessary that $\rho(x) \neq \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$, λ in F, for any basis of J. The isomorphism class of L is determined when $AJ = 0$ by the equivalence class of $\rho(x)$ and the isomorphism class of the nilpotent subalgebra $A \dot{+} J$, which as of now is undetermined, and when $\rho(a) \neq 0$ by is (ker ρ, a).*

(2) *$L = J \dot{+} B \dot{+} Fa \dot{+} Fx$, with $xb = b$ for all b in B, $xa - a$ a nonzero element of J. If $(B \dot{+} Fa)J = 0$ then either $\rho(x) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $J \dot{+} B \dot{+} Fa$ is abelian or $\rho(x) = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and $J \dot{+} B \dot{+} Fa$ is nilpotent*

with isomorphism class determined by is (B, a). *If* $(B \dotplus Fa)J \neq 0$ *then* $J = Fu \dotplus Fv$, $xu = \lambda u$, $xv = v$, $\lambda \neq 1$, $xa = a + v$, *with either* $B = \ker \rho$, λ *and* $is(B, a)$ *determining the isomorphism class of* L, *or* $\ker \rho = C \dotplus Fa$ *with* $bu = v$, $bv = 0$, *and* $is(C, a, b)$ *determining the isomorphism class, or* $bu = 0$, $bv = u$, $\lambda \neq 2$ $ab = 1/(2 - \lambda)u$, *with* $aC = 0$ *and* $bC = 0$.

(3) $L = J \dotplus C \dotplus Fa \dotplus Fb \dotplus Fx$, *with* $xc = c$ *for all* c *in* C, $xa - a$ $xb - b$ *linearly independent elements of* J. *Here* $\rho(x) = I$, $AJ = 0$, *and* $C \dotplus Fa \dotplus Fb$ *is abelian.*

Proof. (1) Suppose x and A chosen so that $L = J \dotplus A \dotplus Fx$ with $xa = a$ for all a in A. From $J(x, a, b) = 0$ it follows that $x(ab) = 2ab$. Hence $(A \dotplus J)^2 \subset K \subset J$ with $(\rho(x) - 2J)K = 0$.

If $\rho(x) = I$ then genus $L = 0$ and if $\rho(x) = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$, $\lambda \neq 1$ with respect to basis u, v of J then Fv has genus 0 quotient and hence genus $L = 1$.

If $\dim \rho(c) = 1$ then choose basis $\{b_i\}$ for A and basis u, v for J and write $b_i b_j = \alpha_{ij} u + \beta_{ij} v$ with is $((\alpha_{ij}), (\beta_{ij}))$ determining multiplication in $A \dotplus J$.

If $\dim \rho(c) = 2$, the hypothesis of L^2 nilpotent implies the structure of case (3) Lemma 3. Thus $au = v$, $av = 0$ and $xu = \alpha u$. $xv = (\alpha + 1)v$. Thus $\dim K \leqq 1$ and the invariant $is(\ker \rho, a)$ determines the multiplication in $A \dotplus J$.

(2) Suppose x and A chosen so that $L = J \dotplus B \dotplus Fa \dotplus Fx$ with $xb = b$ for all b in B and $xa - a$ a nonzero element of J, and x and $B \dotplus Fa$ cannot be taken so that $xa = a$, $xb = b$ for all b in B.

If $D_x - I$ were non-singular and $xa = a + k'$, k' in J, set $a' = a + k$, where $(D_x - I)k = k'$, and find $xa' = a'$. It follows that $D_x - I$ is nonzero and singular. Thus $\rho(x) - I = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}$, $\lambda \neq 0$, or $\rho(x) - I = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

If $(B \dotplus Fa)J = 0$ then $J(x, a, b) = 0$ implies $(B \dotplus Fa \dotplus J)^2 \subset K \subset J$, K as in (1).

Suppose $(B \dotplus Fa)J \neq 0$ then by Lemma 3, (i) $\rho(x)$ has two distinct eigenvalues, hence also $\rho(x) - I$, and therefore $\rho(x) = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$.

If $B = \ker \rho$ then $au = v$, $av = 0$ or $au = 0$, $av = u$. In either case $J(x, a, b) = 0$ implies ab in K, K as in (1).

Suppose $B \neq \ker \rho$ then since a can be modified by multiples of b and $\dim (B \dotplus Fa)J = 1$, by Lemma 3 a can be taken so that $aJ = 0$ and then normalized so that $xa = a + \beta' v$, $\beta' \neq 0$. Let $bJ \neq 0$ and $C \dotplus Fa = \ker \rho$. Then $J(x, a, b) = 0$ implies $x(ab) = 2ab + \beta' bv$. Either $bu = v$, $bv = 0$ and $is(C, a, b)$ determines the multiplication of $B \dotplus Fa \dotplus J$ or $bu = 0$ and $bv = u$. In the latter case let $ab = \alpha u + \beta v$

and conclude $\beta = 0$ and $\alpha(\lambda - 2) + \beta' = 0$. Hence $\lambda \neq 2$ and it follows that $aC = 0$ and $bC = 0$ from $J(x, a, c) = 0$ and $J(x, b, c) = 0$ for c in C.

(3) Suppose $L = J \dotplus C \dotplus Fa \dotplus Fb \dotplus Fx$, with $xc = c$ for all c in C, $xa - a$, $xb - b$ linearly independent elements of J, and it is possible to choose A and x so that $xa = a$ for all elements of A. Then $\rho(x) = I$, for $\rho(x) - I$ nonsingular was ruled out in the last case, while if $\rho(x) - I$ has rank 1, then the argument there employed can be used to show that every a can be chosen so that $xa - a$ is in Fv, v any fixed element of J not in the range of $\rho(x) - I$. Lemma 3, (i) gives $(C \dotplus Fa \dotplus Fb)J = 0$ and $J(x, c, d) = 0$, for any c, d in $C \dotplus Fa \dotplus Fb$, then implies $cd = 0$.

PROPOSITION 18. *Let L be a genus 2 Lie algebra over a field F having a 2-dimensional abelian ideal J with non-abelian genus 0 quotient. Suppose L^2 is not nilpotent. Then char $F = 2$, $L = L^2 \dotplus K \dotplus Fx$, dim $\rho(L) = 2$ and either $L^2 = J \dotplus A$ with $xa = a$ for all a in A or $L^2 = J \dotplus \ker \rho \dotplus Fa$, $xb = b$ for all b in $\ker \rho$, $xa = a$ a nonzero element of J. In every case $(\ker \rho \dotplus Fa)^2$ is contained in a 1-dimensional subspace of J so that the isomorphism class of $\rho(L)$ and is $(\ker \rho, a)$ determine the isomorphism class of L.*

Proof. If L^2 is not nilpotent then $\dim \rho(L) = 2$. Then char $F = 2$ and $L^2 \subset J$ by Lemma 3. But, A modulo J is in L^2, hence $L = L^2 \dotplus Fx$.

Suppose $L = J \dotplus A \dotplus Fx$, $xa = a$ for all a in A. Here $x(ab) = 0$ and $\rho(x)$ as in Lemma 3, (ii) cannot have two identical eigenvalues. Therefore it must have rank 1, which excludes (ii) b of Lemma 3. If $\ker \rho = B$ then $(B \dotplus Fa \dotplus J)^2$ is contained in a 1-dimensional subspace of J and is (B, a) is the appropriate invariant. If $\ker \rho \neq B$ then we can take a in $\ker \rho$, since $\rho(x) - I$ has rank 1, $\alpha = 0, 1$ in (ii) a of Lemma 3. Here $J(x, a, b) = 0$ leads to a contradiction.

PROPOSITION 19. *Let L be a solvable minimal genus 2 Lie algebra without a 2-dimensional ideal with genus 0 quotient. Then L belongs to the same isomorphism class as one of the following algebras:*

(1) *$L = L^2 \dotplus Fx$, L^2 a 3-dimensional abelian ideal without a 2-dimensional invariant subspace for D_x. The equivalence class of D_x determines the isomorphism class of L.*

(2) *$L = L^2 \dotplus Fx$, L^2 3-dimensional nilpotent with $(L^2)^2 = Fx \neq 0$ and D_x irreducible on L^2/Fz. Here the equivalence class of D_x determines the isomorphism class of L.*

(3) *$L = L^2 \dotplus Fx$, L^2 4-dimensional abelian, with the characteristic polynomial of D_x on L of the form $p(y) = y(q(y))^2$, $q(y)$ irreducible.*

Two such algebras are isomorphic if and only if the corresponding $p(y)$ and $p'(y)$ are projectively equivalent, i.e. there exist nonzero element λ of F such that $p(\lambda y) = p'(y)$.

Proof. (1, 2) $\dim L = 4$ and $L = L^2 \dotplus Fx$. If L^2 is abelian then (1). If not let $L^2 = Fa \dotplus Fb \dotplus Fab$. If $(L^2)^2$ were 2-dimensional then its quotient would be genus 0. Hence $(L^2)^2 = Fab$ is a 1-dimensional ideal and therefore annihilated by L^2. The multiplication table of L is determined by D_x acting on L^2/Fab since x is specified only up to the addition of elements of L^2. This gives (2).

(3) $\dim L = 5$. L does not have a 1-dimensional ideal J. Otherwise, genus $L/J = 2$ implies J complemented by a genus 2 subalgebra by Proposition 2; genus $L/J = 0$ implies genus $L = 1$; genus $L/J = 1$ implies L/J has a 1-dimensional ideal with genus 0 quotient (hence L has a 2-dimensional ideal with genus 0 quotient).

Suppose L has a maximal abelian ideal J of dimension 2. Then genus $L/J = 1$. Let x and y be preimages of two generators of L/J under the natural map L onto L/J. Then x and y generate a 3-dimensional subalgebra S. If $\dim S^2 = 1$ take x and y so that $x(xy) = 0$; if $\dim S^2 = 2$ with x in S^2 then $J(x, y, xy) = 0$ implies $x(xy) = 0$. It follows $(x(x(y + a)) = x(xa) = 0$ for all a in I to avoid $\langle x, y + a\rangle$ containing a nonzero element of I and thus having dimension $\geqq 4$. If $D_x = 0$ then $[D_x, D_y] = 0$ and $I \dotplus S^2$ is an abelian ideal. Therefore there exist a and b spanning I with $xa = b$ and $xb = 0$. Take y so that $ya = \gamma a$. Then $(xy)a = x(ya) + (xa)y = \gamma b + by$ and therefore $(x + a)((x + a)y) = -by = 0$ to avoid $\dim\langle x + a, y\rangle \geqq 4$. Therefore Fb is an ideal in L, a contradiction.

Suppose L has a maximal abelian ideal J of dimension 3. Note $\dim L^2 \geqq 3$, for genus $L = 2$ implies $\dim L^2 \geqq 2$ and if $\dim L^2 = 2$ then genus $L/L^2 = 0$. If $J \neq L^2$ then $\dim (J \cap L^2) = 2$ to avoid $J \cap L^2$ a 1-dimensional ideal. Write $L = J \cap L^2 \dotplus Fx \dotplus Fy \dotplus Fz$, with x in L^2, y in J, and conclude from $xz = \alpha x + a$ and $yz = \beta y + b$ with a and b in $J \cap L^2$ that genus $L/(J \cap L^2) = 0$. If $J = L^2$ let $L = L^2 \dotplus Fx \dotplus Fy$. Then $xy \neq 0$ implies Fxy is a 1-dimensional ideal because $x(xy)$ and $y(xy)$ are in $L^2 \cap (Fx \dotplus Fy \dotplus Fxy) = Fxy$. Therefore $xy = 0$ for all x and y in L^2, i.e. $(x + a)y = 0$ and $x(y + a) = 0$ for all a in L^2, a contradiction. If $J \subsetneqq L^2$ let $L = J \dotplus Fx \dotplus Fy$ with x in L^2 and y chosen so that $xy = x + a$, a in J, which is possible because x is in L^2. If $a \neq 0$ then from $x(xy) = xa$ and $y(xy) = x + a + ya$ in L^2 and $Fx \dotplus Fy \dotplus Fxy$ conclude xa and ya are in Fa so that Fa is an ideal. It follows from $x(y + b)$ and $(x + b)y$ that $xb = 0$ and $yb = 0$ for all b in J, a contradiction.

Suppose L has a maximal abelian ideal J of dimension 4. Then there exist and b in J such that x, xa, xb, a, and b span L. To avoid a 2-generated 4-dimensional subalgebra $x(xa) = \alpha a + \beta xa$ and $x(xb) = \alpha' b + \beta' xb$ with $\alpha = \alpha'$ and $\beta = \beta'$ necessary for $x(x(a + b))$ to be in $F(a + b) + Fx(a + b)$. The characteristic polynomial of D_x on L^2 is $(y^2 - \beta y + \alpha)^2$. If $y^2 - \beta y + \alpha$ were not irreducible L would contain a 2-dimensional ideal with genus 0 quotient.

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