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A STUDY OF CERTAIN SEQUENCE SPACES OF MADDOX AND A GENERALIZATION OF A THEOREM OF IYER

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In this paper we examine the Köthe-Toeplitz reflexivity of certain sequence spaces and we characterize some classes of matrix transformations defined on them. The results are used to prove a generalization of a theorem by V. G. Iyer, concerning the equivalence of the notions of strong and weak convergence on the space of all integral functions, and also to generalize some theorems by Ch. Rao.

Let X, Y be two nonempty subsets of the space s of all complex sequences and $A = (a_{nk})$ an infinite matrix of complex numbers $a_{nk}(n, k = 1, 2, \cdots)$. For every $x = (x_k) \in X$ and every integer n we write

$$A_n(x) = \sum_k a_{nk} x_k ,$$

where the sum without limits is always taken from $k = 1$ to $k = \infty$. The sequence $Ax = (A_n(x))$, if it exists, is called the transformation of x by the matrix A. We say that $A \in (X, Y)$ if and only if $Ax \in Y$ whenever $x \in X$.

Throughout the paper, unless otherwise indicated, $p = (p_k)$ will denote a sequence of strictly positive numbers (not necessarily bounded in general). The following classes of sequences were defined by Maddox [4] (see also Simons [10], Nakano [8]):

$$l(p) = \left\{x\colon \sum_k |x_k|^{p_k} < \infty\right\} ,$$

$$l_\infty(p) = \left\{x\colon \sup_k |x_k|^{p_k} < \infty\right\} ,$$

$$c(p) = \left\{x\colon |x_k - l|^{p_k} \longrightarrow 0 \text{ for some } l\right\} ,$$

$$c_0(p) = \left\{x\colon |x_k|^{p_k} \longrightarrow 0\right\} .$$

When all the terms of (p_k) are constant and all equal to $p > 0$ we have $l(p) = l_p$, $l_\infty(p) = l_\infty$, $c(p) = c$, and $c_0(p) = c_0$, where l_p, l_∞, c, c_0, are respectively the spaces of p-summable, bounded, convergent and null sequences. It is easy to see that $l_\infty(p) = l_\infty$ if and only if $0 < \inf p_k \leqq \sup p_k < \infty$ and similarly for $c_0(p) = c_0$, $c(p) = c$ (see [4]). It was shown in [4], [5], [6], that the sets $l(p)$, $l_\infty(p)$, $c(p)$ and $c_0(p)$ are linear spaces under coordinatewise addition and scalar multiplication if and only if $p \in l_\infty$. The special linear space $c_0(1/k)$ was studied by Iyer [1], who identified it with the space of all integral functions.

Whenever $p \in l_\infty$ we shall write $H = \sup p_k$ and $M = \max(1, H)$.

Now let E be a nonempty subset of s. Then we shall denote by $E^\dagger$ the generalized Köthe-Toeplitz dual of E, i.e.,

$$E^\dagger = \left\{a: \sum_k a_k x_k \text{ converges for every } x \in E\right\}.$$

The following lemma collects some simple and well-known properties of dual spaces:

LEMMA 1. *The Köthe-Toeplitz duality has the following properties*:
(i) $E^\dagger$ *is a linear subspace of* s *for every* $E \subset s$.
(ii) $E \subset F$ *implies* $E^\dagger \supset F^\dagger$ *for every* $E, F \subset s$.
(iii) $E^{\dagger\dagger} \equiv (E^\dagger)^\dagger \supset E$ *for every* $E \subset s$.
(iv) $(\cup E_j)^\dagger = \cap E_j^\dagger$ *for every family* $\{E_j\}$ *with* $E_j \subset s$.

A nonempty subset E of s is said to be *perfect* or *Köthe-Toeplitz reflexive* if and only if $E^{\dagger\dagger} = E$. It is well-known that $E^\dagger$ is perfect for every E. It is also obvious that if E is perfect then it is a linear space. The converse is not always true, e.g., c is a linear space with Köthe-Toeplitz dual l_1 and therefore not perfect.

Let $E(p)$ be any one of the sets $l(p), l_\infty(p), c(p), c_0(p)$. Then to contract notation we shall put $E(p; 1) = E^\dagger(p)$, $E(p: 2) = E^{\dagger\dagger}(p)$ etc. It is obvious that $E(p; 1) = E(p; 2n+1)$ for every $n \geqq 0$. We now give the Köthe-Toeplitz duals of the above classes of sequences.

LEMMA 2. (i) *If* $0 < p_k \leqq 1$ *for every* k, *then* $l(p; 1) = l_\infty(p)$ (see Theorem 7 in [10]).

(ii) *If* $p_k > 1$ *for every* k, *then* $l(p; 1) = M(p)$, (see Theorem 1 in [6]), *where*

$$M(p) = \bigcup_{N>1} \left\{a: \sum_k |a_k|^{q_k} N^{-q_k} < \infty\right\}$$

with $p_k^{-1} + q_k^{-1} = 1$. *For convenience we shall, in the future, write* $r_k = p_k^{-1}$, $s_k = q_k^{-1}$.

(iii) *For every* $p = (p_k)$, *we have* $l_\infty(p; 1) = M_\infty(p)$, (see Theorem 2 in [3]), *where*

$$M_\infty(p) = \bigcap_{N>1} \left\{a: \sum_k |a_k| N^{r_k} < \infty\right\}.$$

(iv) *Also for every* $p = (p_k)$, $c_0(p; 1) = M_0(p)$, (see Theorem 6 in [6]), *where*

$$M_0(p) = \bigcup_{N>1} \left\{a: \sum_k |a_k| N^{-r_k} < \infty\right\}.$$

We determine the Köthe-Toeplitz dual of $c(p)$ in

THEOREM 1. *For every $p=(p_k)$ we have $c(p; 1)=c_0(p; 1) \cap \gamma$ where γ is th space of all convergent series.*

Proof. Let $a \in c_0(p; 1) \cap \gamma$ and $|x_k - l|^{p_k} \to 0$. Then $l \sum_k a_k$ and $\sum_k a_k(x_k - l)$ are well defined and therefore $\sum_k a_k x_k$ converges. Whence $c_0(p; 1) \cap \gamma \subset c(p; 1)$. On the other hand, let $a \in c(p; 1)$. Then $e = (1, 1, \cdots) \in c(p)$ implies $a \in \gamma$, and $c_0(p) \subset c(p)$ implies $c(p; 1) \supset c_0(p; 1)$ by Lemma 1 (ii).

We continue by characterizing the Köthe-Toeplitz second duals and discussing the reflexivity of the sets $c_0(p)$, $l(p)$ and $l_\infty(p)$.

We have the following theorems:

THEOREM 2. *For every $p = (p_k)$ we have $c_0(p; 2) = \lambda_1$, where*

$$\lambda_1 = \bigcap_{N>1} \left\{y: \sup_k |y_k| N^{r_k} < \infty\right\}.$$

Proof. Let $E_N = \{a: \sum_k |a_k| N^{-r_k} < \infty\}$. Then $c_0(p; 1) = M_0(p) = \bigcup_{N>1} E_N$ and therefore by Lemma 1 (iv), we have $c_0(p; 2) = M_0(p; 1) = \bigcap_{N>1} E_N^\dagger$. It is easy to check now that $E_N^\dagger = \{y: \sup_k |y_k| N^{r_k} < \infty\}$ for every $N \geqq 1$, whence $c_0(p; 2) = \lambda_1$.

THEOREM 3. *For every $p = (p_k)$ we have $l_\infty(p; 2) = \lambda_2$, where*

$$\lambda_2 = \bigcup_{N>1} \left\{a: \sup_k |a_k| N^{-r_k} < \infty\right\}.$$

Proof. It is easy to see that $\lambda_2 \subset l_\infty(p; 2)$. On the other hand if $a \in l_\infty(p; 2) - \lambda_2$ then there exists a strictly increasing sequence $(k(N))$ positive integers such that for $k = k(N)$, $|a_k| N^{-r_k} > N^2$ and if we define a sequence $x = (x_k)$ by $x_k = 0$ $(k \neq k(N))$, $x_k = N^{-(2+r_k)} \operatorname{sgn} a_k$ $(k = k(N))$, $(N = 2, 3, \cdots)$ then for any integer $R > 1$ we have, for every $N \geqq R$, $|x_k| R^{r_k} \leqq N^{-2}$, when $k = k(N)$. Hence $x \in l_\infty(p; 1)$. However, for $k = k(N)$, $a_k x_k > 1$, contrary to $a \in l_\infty(p; 2)$.

THEOREM 4. (i) *Let $p_k > 1$ for every k. Then $l(p)$ is perfect if and only if $p \in l_\infty$.*

(ii) *Let $0 < p_k \leqq 1$ for every k. Then $l(p)$ is perfect if and only if $l(p) = l_1$.*

Proof. (i) Let $1 < p_k \leqq H$. Then we have

$$q_k = (1 - r_k)^{-1} \geqq H(H-1)^{-1} > 1$$

for every k. It is obvious now that, by Lemma 2 (ii), $l(q) \subset l(p; 1)$ and therefore that $l(q; 1) \supset l(p; 2)$, by Lemma 1 (ii). On the other hand $\inf q_k > 1$ implies $l(q; 1) = l(p)$ (see [6] p. 432) and therefore

$$l(p) \subset l(p; 2) \subset l(q; 1) = l(p) .$$

Whence $l(p; 2) = l(p)$, i.e., $l(p)$ is perfect. Conversely if $l(p)$ is perfect then it is a linear space and therefore $p \in l_\infty$.

(ii) The sufficiency is trivial. For the necessity we observe that $l(p; 2) = l(p)$ implies $m = \inf p_k > 0$. For suppose not. Then there exists a strictly increasing sequence (k_i) of positive integers such that $p_{k_i} < i^{-1}$. We put $a_k = 0$ $(k \neq k_i)$, $a_k = i^{-r_k}$ $(k = k_i)$. Then for every integer $N > 1$ we have, for $i \geqq 2N$, $|a_k| N^{r_k} \leqq 2^{-i}$ and $|a_k|^{p_k} = i^{-1}$, where $k = k_i$. Whence $a \in l(p; 2) - l(p)$, contrary to the assumption that $l(p)$ is perfect. Now from $0 < m \leqq p_k \leqq 1$ we have $l_m \subset l(p) \subset l_1$, i.e., $l(p) = l(p; 2) = l_1$ (since $l_m^\dagger = l_\infty$ by Theorem 7 in [8]).

THEOREM 5. *$l_\infty(p)$ is perfect if and only if $p \in l_\infty$.*

Proof. Sufficiency. Let $p \in l_\infty$ and $a \in l_\infty(p; 2)$. Then there exists $N > 1$ such that $\sup_k |a_k| N^{-r_k} = K < \infty$. Hence $|a_k| N^{-r_k} K^{-1} < 1$ for every k and therefore $|a_k|^{p_k} \leqq N \max(1, K^H)$ for every k, i.e., $a \in l_\infty(p)$. Whence $l_\infty(p)$ is perfect.

Necessity. Let $l_\infty(p; 2) = l_\infty(p)$ and suppose that there exists a strictly increasing sequence (k_i) of positive integers such that $p_{k_i} > i$. Then the sequence a defined by $a_k = 0$, $k \neq k_i$, $a_{k_i} = 2$, $i = 1, 2, 3 \cdots$, belongs to $l_\infty(p; 2) - l_\infty(p)$ and this contradicts the assumption that $l_\infty(p)$ is perfect.

THEOREM 6. *The following statements are equivalent:*
(i) $\inf p_k > 0$;
(ii) $l_\infty(p; 1) = l_1$;
(iii) $l_\infty(p; 2) = l_\infty$.

Proof. The proof is trivial.

THEOREM 7. *The following statements are equivalent:*
(i) $c_0(p; 2) = l_\infty$;
(ii) $\inf p_k > 0$;
(iii) $c_0 \subset c_0(p)$.

Proof. (i) implies (ii). For we have $l_1 = c_0(p; 3) = c_0(p; 1) = M_0(p)$ and therefore $\inf p_k > 0$ (see [6] p. 434). (ii) implies (iii) by Lemma 1 in [4]). Finally from (iii) and Theorem 2 we have $l_\infty = c_0^{\dagger\dagger} \subset c_0(p; 2) \subset l_\infty$, i.e., $c_0(p; 2) = l_\infty$. Whence (iii) implies (i).

THEOREM 8. *$c_0(p)$ is perfect if and only if $p \in c_0$.*

Proof. For the sufficiency let $p \in c_0$ and take $x \in c_0(p; 2)$. Then by Theorem 2 we have $C_N \equiv \sup_k |x_k| N^{r_k} < \infty$ for every integer $N > 1$. Suppose that $x \notin c_0(p)$. Then there exists a strictly increasing sequence $(k(s))$ os positive integers and a positive number l such that $|x_k|^{p_k} \geqq l$ for $k = k(s)$, $s = 1, 2, \cdots$. Therefore, for every integer $N > 1$, we have

$$(Nl)^{r_k} \leqq N^{r_k} |x_k| \leqq \sup_s N^{r_k} |x_k| \leqq C_N < \infty, \ (k = k(s)) \ .$$

Let now N_0 be an arbitrary integer bigger than 1 and choose N such that $Nl > N_0$. Then we have $N_0^{r_k} < (Nl)^{r_k} < C_N$, $(k = k(s))$, i.e., $\lim \sup_s N_0^{r_k} < \infty$ $(k = k(s))$ contrary to the fact that $r_{k(s)} = p_{k(s)}^{-1} \to \infty$ $(s \to \infty)$. Whence $x \in c_0(p)$ and this proves the sufficiency. For the necessity let us suppose that $c_0(p; 2) = c_0(p)$ and that $p \notin c_0$. Then there exists a strictly increasing sequence of positive integers k_j and a positive number l such that $p_{k_j} \geqq l$ $(j = 1, 2, \cdots)$. We define a sequence x as follows: $x_k = 0$ $(k \neq k_j)$, $x_k = 1$ $(k = k_j)$, $(j = 1, 2, \cdots)$. Then it is easy to see that $x \in c_0(p; 2) - c_0(p)$ contrary to the assumption that $c_0(p)$ is perfect. Whence $p \in c_0$ and this completes the proof of the theorem.

In the second part of this paper we characterize certain classes of matrix transformations and we show that certain theorems proved by K. Ch. Rao (see [9]) are particular cases of our theorems.

We start by characterizing the class $(c(p), c)$ of matrix transformations. We have the following theorem.

THEOREM 9. *Let $p \in l_\infty$. Then $A \in (c(p), c)$ if and only if*

(i) *There exists an absolute constant $B > 1$ such that*

$$C = \sup_n \sum_k |a_{nk}| B^{-r_k} < \infty \ .$$

(ii) *$\lim a_{nk} = \alpha_k$ $(n \to \infty)$ exists for every fixed k.*

(iii) *$\lim_n \sum_k a_{nk} = \alpha$ exists.*

Proof. Sufficiency. Let $|x_k - l|^{p_k} \to 0$. It is easy to check that $(\alpha_k) \in c_0(p; 1)$. Given $\varepsilon > 0$ there exists $k_0 = k_0(\varepsilon, x)$ such that

$$|x_k - l|^{p_k/M} < \min(1, \varepsilon)B^{-1}(2C + 1)^{-1} < 1$$

for every $k > k_0$. Therefore, we have

$$B^{r_k}|x_k - l| < B^{Mr_k}|x_k - l| < (\min(1, \varepsilon)(2C + 1)^{-1})^{Mr_k} < \varepsilon(2C + 1)^{-1}$$

for every $k > k_0$. Putting $b_{nk} = a_{nk} - \alpha_k$ we have $b_{nk} \to 0$ ($n \to \infty$, k fixed) and $\sum_k |b_{nk}| B^{-r_k} \leqq 2C$. Whence

$$\left|\sum_k (a_{nk} - \alpha_k)(x_k - l)\right| \leqq \left|\sum_{k \leqq k_0} b_{nk}(x_k - l)\right| + \varepsilon$$

and therefore

$$\lim_n \sum_k x_{nk}x_k = l\alpha + \sum_k \alpha_k(x_k - l) ,$$

i.e., $A \in (c(p), c)$.

Necessity. Then necessity of (ii) and (iii) is obvious. For the necessity of (i) we observe that $A \in (c_0(p), c)$ whenever $A \in (c(p), c)$ (since $(c(p), c) \subset (c_0(p), c)$). Therefore each A_n, defined by $A_n(x) = \sum_k a_{nk}x_k$ for every $x \in c_0(p)$, is a continuous linear functional on $c_0(p)$ (see Theorem 6 in [6]) which is a complete linear topological space since $p \in l_\infty$. The proof of the necessity of (i) is now a simple application of the uniform boundedness principle.

COROLLARY 1. *Let $p \in l_\infty$ and denote by $(c(p), c; P)$ the class of matrix transformations which transform every sequence in $c(p)$ to a sequence in c with the same limit. Then $A \in (c(p), c; P)$ if and only if*

(i)′ *Condition* (i) *Theorem* 9 *holds.*

(ii)′ $\lim_n a_{nk} = 0$ *for every fixed k.*

(iii)′ $\lim_n \sum_k a_{nk} = 1$.

COROLLARY 2. *Let $p \in l_\infty$. Then $A \in (c_0(p), c)$ if and only if conditions* (i) *and* (ii) *of Theorem* 9 *hold.*

COROLLARY 3. (See C. Rao's Theorem (III) in [9]). $A \in (c_0(1/k), c)$ *if and only if*

(i)* $|a_{nk}|^{1/k} \leqq D$ *for every n, k.*

(ii)* $\lim_n a_{nk} = \alpha_k$ *exists for every fixed k.*

Proof. It is enough to prove that in the case $p_k = 1/k$ for every k, condition (i)* and condition (i) of Theorem 9 are equivalent. If condition (i) of Theorem 9 holds then $\sum_k |a_{nk}| B^{-k} \leqq C$ for an absolute constant $B > 1$ and therefore $|a_{nk}| B^{-k} \leqq C$ for every n, k. Whence, for every n, k we have

$$|a_{nk}|^{1/k} \leqq C^{1/k} B \leqq B \max(1, C) \equiv D < \infty ,$$

i.e., condition (i)*. On the other hand if condition (i)* holds, then for an integer $T \geqq \max(1, 2D)$ we have

$$|a_{nk}| \leqq D^k \leqq T^k 2^{-k}, \text{ for every } n, k .$$

Whence

$$\sum_k |a_{nk}| T^{-k} \leqq \sum_k 2^{-k} < \infty \text{ for all } n ,$$

i.e., condition (i) of Theorem 9 holds.

THEOREM 10. *Let $p \in l_\infty$. Then $A \in (c_0(p), l_\infty(p))$ if and only if there exists an absolute constant $B > 1$ such that*

$$(1) \qquad T = \sup_n \left\{\sum_k |a_{nk}| B^{-r_k}\right\}^{p_n} < \infty .$$

Proof. Sufficiency. Let $x \in c_0(p)$. Then there exists k_0 such that $|x_k| < B^{-r_k}$ for every $k > k_0$. Therefore for every n we have

$$\left|\sum_k a_{nk}x_k\right|^{p_n} \leqq K(S_1 + S_2) ,$$

where,

$$K = \max(1, 2^{H-1}),\ S_1 = \left|\sum_{k \leqq k_0} a_{nk}x_k\right|^{p_n},\ S_2 = \left|\sum_{k > k_0} a_{nk}x_k\right|^{p_n} .$$

We observe now that (1) implies

$$|a_{nk}| T^{-r_n} < B^{r_k} \leqq \max_{k \leqq k_0} B^{r_k} \equiv R < \infty$$

for every n and for every $k \in [1, k_0]$. Whence

$$S_1 \leqq \left(\sum_{k \leqq k_0} RT^{r_n} |x_k|\right)^{p_n} = T\left(R \sum_{k \leqq k_0} |x_k|\right)^{p_n} \leqq T \max(1, Q) < \infty$$

where $Q = (R \sum_{k \leqq k_0} |x_k|)^M$. For the term S_2 we have

$$S_2^{r_n} = \left|\sum_{k > k_0} a_{nk}x_k\right| \leqq \sum_{k > k_0} |a_{nk}| B^{-r_k} \leqq T^{r_n} ,$$

i.e., $S_2 \leqq T$. Whence $A \in (c_0(p), l_\infty(p))$.

Necessity. Let $A \in (c_0(p), l_\infty(p))$. Then we have

$$N(x) = \sup_n |A_n(x)|^{p_n} < \infty$$

for every $x \in c_0(p)$. Put $f_n(x) = |A_n(p)|^{p_n}$. Then for every n, f_n is a continuous function on $c_0(p)$ (see Theorem 6 in [6]). Since $c_0(p)$ is a complete metric space (see Theorem 1 in [5] and p. 318 in [7]) and $f_n(x) \leqq N(x)$ for every n, we have, by the uniform boundedness principle, that there exists a sphere $S[\theta, \delta] \subset c_0(p)$ with $\delta < 1, \theta = (0, 0, \cdots)$ and an absolute constant K such that

$$\left| \sum_k a_{nk} x_k \right|^{p_n} \leqq K \tag{2}$$

for every n and for every $x \in S[\theta, \delta]$. For every integer $m > 0$ we define a sequence $(x^{(m)})$ of elements of $c_0(p)$ as follows:

$$x_k^{(m)} = \delta^{Mr_k} \operatorname{sgn} a_{nk} \ (1 \leqq k \leqq m), x_k^{(m)} = 0 \ (k > m) .$$

Then $x^{(m)} \in S[\theta, \delta]$ for every m and by (2)

$$\left\{ \sum_{k \leqq m} |a_{nk}| B^{-r_k} \right\}^{p_n} \leqq K ,$$

for every m, n, where $B = \delta^{-M}$. Whence $T < \infty$.

REMARK. It is easy to check that if $p \in l_\infty$, $p' \in l_\infty$, then we have $A \in (c_0(p), l_\infty(p'))$ if and only if there exists an absolute constant $B > 1$ such that

$$S = \sup_n \left\{ \sum_k |a_{nk}| B^{-r_k} \right\}^{p'_n} < \infty .$$

Let now Q be the set of all $p = (p_k)$ for which there exists $N > 1$ such that $\sum_k N^{-r_k} < \infty$. Then we have the following

THEOREM 11. *Let $p \in Q$. Then $A \in (c_0(p), l_\infty(p))$ if and only if*

$$D = \sup_{n,k} |a_{nk}|^{(r_k + r_n)^{-1}} < \infty .$$

Proof. Let condition (i) of Theorem 10 holds. Then we have

$$|a_{nk}| B^{-r_k} \leqq T^{r_n}$$

for all n, k. Hence $|a_{nk}| B^{r_k} T^{r_n}$ and if we put $C = \max(T, B)$ we have $D \leqq C < \infty$. Obviously in this part of the proof we do not require that $p \in Q$. If now $p \in Q$, then there exists $N > 1$ such that $\sum_k N^{-r_k}$ and therefore if $D < \infty$ we have for an integer $B > ND$ that

$$\textstyle\sum_k |a_{nk}| B^{-r_k} \leqq (B/N)^{r_n} \sum_k N^{-r_k} .$$

Whence condition (1) of Theorem 10 holds and therefore $A \in (c_0(p), l_\infty(p))$.

COROLLARY 1. (See C. Rao's Theorem (v) in [9]). $A \in (c_0(1/k),$

$l_\infty(1/k))$ *if and only if* $\sup_{n,k} |a_{nk}|^{(n+k)^{-1}} < \infty$.

THEOREM 12. *For every* $p = (p_k)$ *we have* $A \in (M_0(p), c_0)$ *if and only if*

(i) $C_N \equiv \sup_{n,k} |a_{nk}| N^{r_k} < \infty$ *for every* $N > 1$.

(ii) $\lim_n a_{nk} = 0$ *for every fixed* k.

Proof. The sufficiency is trivial and so is the necessity of (ii). The necessity of (i) is proved by an argument similar to the one that was used to prove Theorem 3 in [3].

Before we proceed to discuss weak convergence in $c_0(p)$ we prove a theorem concerning the relation between the classes $c_0(p)$ and l_1.

THEOREM 13. *The following statements are true*:

(1) $l_1 \subset c_0(p)$ *if and only if* $\inf p_k > 0$.

(2) $c_0(p) \subset l_1$ *if and only if* $p \in Q$.

(3) $c_0(p) \neq l_1$ *for every* $p = (p_k)$.

Proof. (1) If $\inf p_k > 0$ then $c_0(p) = c_0 \supset l_1$. On the other hand if $l_1 \subset c_0(p)$ but we suppose that $\inf p_k = 0$ then we can find a strictly increasing sequence (k_i) of positive integers such that $p_{k_i} < i^{-1}$ $(i = 1, 2, \cdots)$. It is easy to see now that the sequence $x_k = 0$ $(k \neq k_i)$, $x_k = 2^{-r_k}$ $(k = k_i)$, $(i = 1, 2, \cdots)$ belong to $l_1 - c_0(p)$ contrary to the assumption that $l_1 \subset c_0(p)$. Hence (1) is true.

(2) Let $p \in Q$, $x \in c_0(p)$. Then there exist $N > 1$ and $k_0 = k_0(N, x)$ such that $\sum_k N^{-r_k} < \infty$ and $|x_k| \leqq N^{-r_k}$ for every $k > k_0$. Whence $x \in l_1$. On the other hand if $c_1(p) \subset l_1$ but $\sum_k N^{-r_k} = \infty$ for every $N > 1$ then there exists strictly increasing sequence (k_N) of positive integers such that

$$\sum_{k_{N-1} < k \leqq k_N} (N + 1)^{-r_k} > N$$

and if we put $x_k = (N + 1)^{-r_k}$ for $k_{N-1} < k \leqq k_N$ then we have $x \in c_0(p) - l_1$ contrary to the assumption that $c_0(p) \subset l_1$. Hence (2) is true.

(3) If $p \in Q$ then there exists $N > 1$ such that $\sum_k N^{-r_k} < \infty$ and $(N^{-r_k}) \in l_1 - c_0(p)$, i.e., $c_0(p)$ is in this case a proper subset of l_1. On the other hand if $p \notin Q$ then by (2) $c_0(p)$ contains an element which is not in l_1 and hence (3) is true.

In the proof of our next theorem we will make use of the following

LEMMA 3. $l_\infty(1/k) = M_0(1/k)$.

Proof. Let $x \in l_\infty(1/k)$. Then by choosing an integer $N > 2(D+1)$, where $D = \sup_k |x_k|^{1/k} < \infty$, we have

$$\sum_k |x_k| N^{-k} < \sum_k 2^{-k} < \infty, \text{ i.e., } x \in M_0(1/k) .$$

Conversely for $x \in M_0(1/k)$ we have

$$|x_k|^{1/k} \leqq N C^{1/k} \leqq N \max(1, C) < \infty$$

for some $N > 1$ such that $C = \sum_k |x_k| < \infty$. Whence $x \in l_\infty(1/k)$ and therefore the lemma is true.

We now prove the following.

THEOREM 14. *The following statements are equivalent*:
(1) $A \in (l_\infty(1/k), c_0)$;
(2) $\sum_k |a_{nk}| N^k \rightarrow 0$ $(n \rightarrow \infty)$ *for every* $N > 1$;
(3) $\sup_k |a_{nk}|^{1/k} \rightarrow 0$ $(n \rightarrow \infty)$;
(4) (i) $\sup_{n,k} |a_{nk}| N^k < \infty$ *for every* $N > 1$,
(ii) $\lim_n a_{nk} = 0$ *for every fixed* k.

Proof. By Theorem 3 in [3], (1) implies (2). Now given $\varepsilon > 0$ we choose $N > 1$ such that $N^{-1} < \varepsilon$. By (2) there exists n_0 such that $\sum_k |a_{nk}| N^k < 1$, $n > n_0$, which implies $|a_{nk}| N^k < 1$ for all $k \geqq 1$, $n > n_0$. Therefore

$$\sup_k |a_{nk}|^{1/k} < N^{-1} < \varepsilon ,$$

for every $n > n_0$. Hence (2) implies (3). It is easy to see that (3) implies (4) and finally that (4) implies (1) by Theorem 12 and Lemma 3.

This completes the proof of Theorem 14.

We conclude this paper with the study of weak convergence in $c_0(p)$.

In 1948, V. G. Iyer [1] proved an interesting theorem concerning the equivalence of the notions of strong and weak convergence in $c_0(1/k)$. Namely

THEOREM I. *The notions of strong and weak convergence in* $c_0(1/k)$ *are equivalent.*

In the present paper we show that Iyer's theorem is true for a more general class of $c_0(p)$ spaces, namely for the spaces $c_0(p)$ for which $p \in c_0$.

Before proceeding any further we make some remarks concerning the $c_0(p)$ spaces and give some definitions.

It has been shown by Maddox (see Theorem 6 in [6]) that for

$p \in l_\infty$ the space $c_0^*(p)$ of all continuous linear functionals on $c_0(p)$ is isomorphic to the space $M_1(p)$ in the sense that every continuous linear functional on $c_0(p)$ can be expressed in the form $f(x) = \sum_k a_k x_k$ with $a \in M_0(p)$ and vice versa. Therefore in what follows we can talk of expressions $\sum_k a_k x_k$, $a \in M_0(p)$, $x \in c_0(p)$ instead of continuous linear functionals on $c_0(p)$.

The following definitions are well-known.

DEFINITION 1. Let X be a linear topological space. Then we say that a sequence $(x^{(n)})$ of elements of X converges weakly to an element x of X if and only if $\lim_n f(x^{(n)}) = f(x)$ for every $f \in X^*$ (i.e., for every continuous linear functional on X).

DEFINITION 2. A linear metric space is said to have the Schur property if and only if every weakly convergent sequence of its elements is necessarily convergent in the metric of the space. If the above space is a B-space then the definition coincides with the one given in [2].

Note that convergence in metric (strong convergence) implies weak convergence to the same limit.

We now examine the conditions under which the space $c_0(p)$ has the Schur property. It has been remarked that $p \in l_\infty$ is necessary and sufficient for the linearity of $c_0(p)$. Furthermore if $p \in l_\infty$ then $c_0(p)$ is a complete linear topological space under the topology induced by the paranorm g defined by $g(x) = \sup_k |x_k|^{p_k/M}$ for every $x = c_0(p)$ (see Theorem 1 in [5]).

The following result gives the exact condition for $c_0(p)$ to have the Schur property and consequently includes Iyer's theorem as a special case.

THEOREM 15. *The linear topological space $c_0(p)$ has the Schur property if and only if $p \in c_0$.*

Proof. Sufficiency: Let $p \in c_0$, and $(x^{(n)}) \subset c_0(p)$ be convergent weakly to θ, i.e., $\lim_n f(x^{(n)}) = 0$ for every $f \in c_0^*(p)$. Then

$$\lim_n \sum_k a_k x_k^{(n)} = 0$$

for every $a \in M_0(p)$. Whence $X = (x_{nk}) \in (M_0(p), c_0)$ (where for convenience we put x_{nk} instead of $x_k^{(n)}$) and therefore by Theorem 12 we have

(i) $C_N = \sup_{n,k} |x_{nk}| N^{r_k} < \infty$ for every $N > 1$,

(ii) $\lim_n x_{nk} = 0$ for every fixed k.

We shall prove now that $g^M(x^{(n)}) \to 0$ $(n \to \infty)$. Suppose, on the contrary, that

$$\limsup_n g^M(x^{(n)}) > 0 .$$

Then there exists a subsequence $(y^{(s)})$ of $(x^{(n)})$ and a positive number l such that

$$(1) \qquad g^M(y^{(s)}) \geqq 2l \qquad s = 1, 2, \cdots .$$

We write $s(1) = 1$ and let $k(1) \geqq 1$ be such that

$$|y_{s(1)k(1)}|^{p_{k(1)}} > l ,$$

where $y^{(s)} = (y_{sk})$ for every s. Obviously such an integer $k(1)$ exists because of (1). Now $p \in c_0$ implies that there exists $\bar{k}(1)$ such that $p_k < 2^{-1}$ for every $k > \bar{k}(1)$. By (ii) there exists $s(2) > s(1)$ such that

$$\max_{1 \leqq k \leqq k(1)+\bar{k}(1)} |y_{s(2)k}|^{p_k} < l .$$

Whence there exists $k(2) > k(1) + \bar{k}(1)$ such that

$$p_{k(2)} < 2^{-1}, \ |y_{s(2)k(2)}|^{p_{k(2)}} > l .$$

Continuing this way let $k(i-1)$, $s(i-1)$ be defined such that,

$$k(i-1) > k(i-2) > \cdots > k(1) \geqq 1, \ s(i-1) > s(i-2) > \cdots > s(1) = 1$$
$$p_{k(i-1)} < (i-1)^{-1}, \qquad |y_{s(i-1)k(i-1)}|^{p_{k(i-1)}} > l .$$

Then there exists $\bar{k}(i-1) > k(i-1)$ such that $p_k < i^{-1}$ for every $k > \bar{k}(i-1)$ and $s(i) > s(i-1)$ such that

$$\max_{1 \leqq k \leqq k(i-1)+\bar{k}(i-1)} |y_{s(i)k}|^{p_k} < l .$$

Whence there exists $k(i) > k(i-1) + \bar{k}(i-1)$ such that

$$|y_{s(i)k(i)}|^{p_{k(i)}} > l .$$

With this method we construct two strictly increasing sequences $(k(i))$, $(s(i))$ of positive integers such that

$$p_{k(i)} < i^{-1}, \qquad |y_{s(i)k(i)}|^{p_{k(i)}} > l, \ i = 1, 2, \cdots .$$

Now we choose an integer $N > 1$ such that $Nl > 2$. Then we have

$$\begin{aligned} C_N = \sup_{n,k} |x_{nk}| N^{r_k} &\geqq \sup_i |y_{s(i)k(i)}| N^{r_{k(i)}} \\ &\geqq \sup_i (Nl)^{r_{k(i)}} \\ &\geqq \sup_i 2^{r_{k(i)}} \geqq \sup_i 2^i = \infty , \end{aligned}$$

contrary to condition (i). Whence we must have

$$\limsup_n g^M(x^{(n)}) = 0 ,$$

i.e., that $x^{(n)} \to \theta$ strongly.

Necessity. Suppose now that the space $c_0(p)$ has the Schur property. If $p \notin c_0$ then there exists a positive number l and a strictly increasing sequence $(k(\mathrm{i}))$ of positive integers such that $p_{k(i)} \geqq l$ for $i = 1, 2, \cdots$. Consider the sequence

$$(x^{(i)}) = (e_{k(i)}) \subset c_0(p)$$

where $e_{k(i)}$ denotes the sequence with 1 in the $k(i)$th place and 0 everywhere else. Then for every integer $N > 1$ we have

(i) $\sup_{i,k} |x_{ik}| N^{r_k} = \sup_i N^{r_{k(i)}} < N^{1/l} < \infty$,

(ii) $\lim_i x_{ik} = 0$ for every fixed k,

i.e., $x^{(i)} \to \theta$ weakly in $c_0(p)$. On the other hand for every pair of integers m, n $(m \neq n)$ we have

$$g(x^{(n)} - x^{(m)}) = g(e_{k(n)} - e_{k(m)}) = 1 ,$$

i.e., the sequence $(x^{(i)})$ is not a Cauchy sequence in $c_0(p)$ and therefore not convergent. Whence we have a contradiction to the fact that $c_0(p)$ has the Schur property.

COROLLARY. (Theorem I). *The space* $c_0(1/k)$ *has the Schur property.*

Note that by Theorems 8 and 15 $c_0(p)$ has the Schur property if and only if it is perfect.

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