PACIFIC JOURNAL OF MATHEMATICS
Vol. 38, No. 3, 1971

A CONTINUOUS FORM OF SCHWARZ'S LEMMA IN NORMED LINEAR SPACES

LAWRENCE A. HARRIS

Our main result is an inequality which shows that a holomorphic function mapping the open unit ball of one normed linear space into the closed unit ball of another is close to being a linear map when the Fréchet derivative of the function at 0 is close to being a surjective isometry. We deduce this result as a corollary of a kind of uniform rotundity at the identity of the sup norm on bounded holomorphic functions mapping the open unit ball of a normed linear space into the same space.

Let Δ be the open unit disc of the complex plane, and let f: $\Delta \to \bar{\Delta}$ be a holomorphic function with $f(0) = 0$. It is easy to show that the inequality

$$(1) \qquad |f(z) - f'(0)z| \leqq \frac{2\,|z|^2}{1 - |z|}(1 - |f'(0)|)$$

holds for all $z \in \Delta$. (For example, apply the lemma given in [5] to the function $z^{-1}f(z)$. See also [3, §292].) Qualitatively, inequality (1) means that if $f'(0)$ is close to the unit circle then $f(z)$ is close to being a linear function of z as long as z remains a fixed positive distance away from the exterior of the unit disc. Our purpose is to prove a version of (1) which applies to vector-valued holomorphic functions of vectors. We deduce this result from an extremal inequality for holomorphic functions, which reduces to a theorem of G. Lumer in the linear case. It should be pointed out that the inequalities we obtain cannot be proved simply by composing with linear functionals and applying the 1-dimensional case, as for instance the generalized Cauchy inequalities can.

1. Main results. In the following, a function h defined on an open subset of a complex normed linear space with range in another is called *holomorphic* if the Fréchet derivative of h at x (denoted by $Dh(x)$) exists as a bounded complex-linear map for each x in the domain of definition of h. (See [7, Def. 3.16.4].) Denote the open (resp., closed) unit ball of a normed linear space X by X_0 (resp., X_1). Throughout, X and Y denote arbitrary complex normed linear spaces. Our main result is

THEOREM 1. *Let h: $X_0 \to Y_1$ be a holomorphic function with*

$h(0) = 0$. *Put* $L = Dh(0)$ *and let* $\mathscr{U}$ *be the set of all linear isometries of* X *onto* Y. *Suppose* $\mathscr{U}$ *is nonempty and let* $d(L, \mathscr{U})$ *denote the distance of* L *from* $\mathscr{U}$ *in the operator norm.*
Then

$$\| h(x) - L(x) \| \leqq \frac{8 \| x \|^2}{(1 - \| x \|)^2} d(L, \mathscr{U}), \ (x \in X_0) .$$

Clearly Theorem 1 contains the main result of [5], i.e., $h = L$ when L is in $\mathscr{U}$. In fact, it is a consequence of Theorem 1 that any sequence of holomorphic functions $h_n: X_0 \to Y_1$ converges uniformly to a linear map L in $\mathscr{U}$ on closed subballs of X_0 whenever the sequence of derivatives $Dh_n(0)$ converges to L in the operator norm. This may be proved by showing as in [5] that $h_n(0) \to 0$, and then applying Theorem 1 to the function $(1 + \| h_n(0) \|)^{-1} [h_n(x) - h_n(0)]$.

Let I be the identity map on X and let the symbol $\| \ \|$, when applied to functions, denote the supremum over X_0. We deduce Theorem 1 from

THEOREM 2. *Let* $\delta \geqq 0$ *and suppose* $h: X_0 \to X$ *is a holomorphic function satisfying*

$$\| I + \lambda h \| \leqq 1 + \delta \tag{2}$$

for all $\lambda \in \bar{\Delta}$. *Let* P_m *be the mth term of the Taylor series expansion for* h *about* 0. *Then*

$$\| P_m \| \leqq K_m \ \delta , \tag{3}$$

where $K_0 = 1$, $K_1 = e$ *and* $K_m = m^{m/(m-1)}$, $m \geqq 2$. *If inequality* (2) *holds when the values of* λ *are restricted to* ± 1, *then* (3) *still holds but with* δ *replaced by* $\sqrt{\delta(2 + \delta)}$.

Recall that by definition

$$P_m(x) = \frac{1}{m!} \left[\frac{d^m}{d\lambda^m} h(\lambda x) \right]_{\lambda = 0} , \ P_0(x) = h(0) . \tag{4}$$

Hence $P_1 = Dh(0)$. Moreover [7, Th. 26.3.6], P_m is of the form $P_m(x) = F_m(x, \cdots, x)$, where F_m is a continuous symmetric m-linear map. It should be noted that in general P_m is a mapping of X into the completion of X.

2. *Proof of Theorem* 1 *assuming Theorem* 2. Let $h: X_0 \to X_1$ be a holomorphic function with $h(0) = 0$ and put $L = Dh(0)$. It suffices to prove that h satisfies the inequality

$$\| h(x) - L(x) \| \leqq \frac{8 \| x \|^2}{(1 - \| x \|)^2} \| I - L \|, \ (x \in X_0) ; \tag{5}$$

for Theorem 1 can then be deduced by composing the given function with inverses of linear maps in $\mathscr{U}$ and applying (5). Thus to prove (5), let

$$h(x) = P_1(x) + P_2(x) + \cdots, (P_1 = L) ,$$

be the Taylor series expansion for h about 0. This series converges to $h(x)$ for every x in X_0. (See [7, pp. 109-113].) Let $x \in X_1$ and let ℓ be a linear functional on the completion of X with $\|\ell\| \leqq 1$. Define $f(\lambda) = \ell(\lambda^{-1}h(\lambda x))$. Then $f\colon \varDelta \rightarrow \bar{\varDelta}$ is holomorphic and

$$f(\lambda) = \sum_{m=0}^{\infty} a_m \lambda^m, \; a_m = \ell(P_{m+1}(x)) .$$

By [9, p. 172], we have $|a_{m-1}| \leqq 1 - |a_0|^2 \leqq 2(1 - |a_0|)$ for $m \geqq 2$, and hence

$$\left| \ell\left(L(x) + \frac{1}{2}\lambda P_m(x)\right) \right| \leqq 1$$

for all $\lambda \in \varDelta$. It follows from the Hahn-Banach Theorem that $\|L + 1/2\ \lambda P_m\| \leqq 1$, and therefore

$$\left\| I + \frac{1}{2}\lambda P_m \right\| \leqq 1 + \delta, \; \delta = \|I - L\| ,$$

for all $\lambda \in \varDelta$. Since P_m extends to the completion of X, Theorem 2 applies to show that

$$\|P_m\| \leqq 2K_m\delta \leqq 8(m-1)\delta ,$$

where the last inequality follows from the inequalities $m/(m-1) \leqq 2$ and $m \leqq 2^{m-1}$. Hence if $x \in X_0$,

$$\|h(x) - L(x)\| \leqq \sum_{m=2}^{\infty} \|P_m(x)\| \leqq \frac{8\|x\|^2\delta}{(1 - \|x\|)^2} ,$$

which is (5).

3. *Proof of Theorem* 2. Our proof is an elaboration of an iteration argument due to H. Cartan. (See [1, pp. 13-14].) Clearly we may suppose that $\delta > 0$ and that inequality (2) is strict. Let N be any positive integer satisfying $N \geqq 1/\delta$ and put $r = 1/(N\delta)$. Then by the triangle inequality,

$$\|I + \lambda r h\| = \|(1-r)I + r(I + \lambda h)\| < 1 + 1/N \tag{6}$$

for all $\lambda \in \varDelta$. Take $\alpha = (1 + 1/N)^{-1}$. Our strategy is to compute the derivatives with respect to λ of the nth iterate of the function $\alpha I + \lambda\alpha r h$ and then apply the generalized Cauchy inequalities [7, p. 97]. The number n of iterations we take will depend on N.

Let $x \in X_0$ and define

$$f_n(\lambda) = (\alpha I + \lambda \alpha r h)^n(x) \ .$$

By (6), $f_n\colon \Delta \to X$ is a well-defined holomorphic function satisfying

$$\|f_n(\lambda)\| < 1, \ (\lambda \in \Delta) \ . \tag{7}$$

Clearly $f_1'(0) = \alpha r h(x)$, and differentiating the identity

$$f_{n+1}(\lambda) = \alpha f_n(\lambda) + \lambda \alpha r h(f_n(\lambda)) \ ,$$

we have

$$f'_{n+1}(0) = \alpha f'_n(0) + \alpha r h(\alpha^n x) \ .$$

Therefore, by induction

$$f'_n(0) = \sum_{k=0}^{n-1} \alpha^{n-k} r h(\alpha^k x) \ . \tag{8}$$

By (7) and Cauchy's inequality,

$$\|f'_n(0)\| \leqq 1 \ . \tag{9}$$

Let $\Phi_n(x)$ be the right hand side of (8). Clearly each Φ_n is holomorphic in X_0 and by (9), $\|\Phi_n\| \leqq 1$. Applying the Cauchy inequalities, we have

$$\left\| \frac{1}{m!} \left[\frac{d^m}{d\lambda^m} \Phi_n(\lambda x) \right]_{\lambda=0} \right\| \leqq 1 \ , \qquad (x \in X_0) \ .$$

Hence by (4),

$$\left\| \sum_{k=0}^{n-1} \alpha^{n+(m-1)k} r P_m(x) \right\| \leqq 1 \ , \qquad (x \in X_0) \ , \tag{10}$$

so

$$\|P_m\| \leqq \frac{1 - \alpha^{m-1}}{r\alpha^n[1 - \alpha^{n(m-1)}]} \ ,$$

assuming $m \geqq 2$. Since $1 - \alpha^{m-1} \leqq (m-1)(1-\alpha)$, $1/r = N\delta$ and $N(1/\alpha - 1) = 1$, it follows that

$$\|P_m\| \leqq \frac{(m-1)\delta}{\alpha^{n-1}[1 - \alpha^{n(m-1)}]} \ . \tag{11}$$

Finally, letting n be the greatest integer in $N(m-1)^{-1} \log m$ and taking the limit in (11) as $N \to \infty$, we obtain inequality (3) for $m \geqq 2$. When $m = 1$, inequality (3) follows from (10) with $n = N$. When $m = 0$, we may obtain (3) from (9) by letting $x = 0$ and taking the limit as $n \to \infty$.

The proof of the second part of Theorem 2 follows from quite general considerations. Suppose $\|I \pm h\| \leqq 1 + \delta$. By the first part of Theorem 2, it suffices to prove that the inequality

$$\|I + \lambda t h\| \leqq 1 + t^2 , \qquad t = \sqrt{\delta(2+\delta)} , \tag{12}$$

holds for all $\lambda \in \Delta$. To do this, let $x \in X_0$ and $\ell \in (X^*)_1$ be given. Then $|\ell(x) \pm \ell(h(x))| \leqq 1 + \delta$, and consequently $|\ell(x)|^2 + |\ell(h(x))|^2 \leqq (1+\delta)^2$. Hence if $\lambda \in \Delta$, $|\ell(x + \lambda t h(x))| \leqq |\ell(x)| + t\,|\ell(h(x))|$

$$\leqq (1 + t^2)^{1/2}(1 + \delta) = 1 + t^2 ,$$

where the last inequality follows from the Cauchy-Schwarz inequality. This in conjunction with the Hahn-Banach Theorem proves (12).

4. Further remarks. Note that by Theorem 2 (or by [2, §§2, 3]) if $\delta \geqq 0$ and L: $X \to X$ is a linear map satisfying $\|I \pm L\| \leqq 1 + \delta$, then $\|L\| \leqq e\sqrt{\delta(2+\delta)}$. This readily implies Theorem 18 of [8]. Note also that in the case $\delta = 0$, Theorem 2 shows that I is an extreme point of $H^\infty(X_0, X)_1$, where $H^\infty(X_0, X)$ denotes the space of all bounded holomorphic functions h: $X_0 \to X$ with the sup norm. A simpler proof of this fact has already been given in [6]. It would be interesting to know whether or not K_m is the best possible constant in (3) which is independent of δ and h. See [4] for a related result.

Note added in proof. The author has recently shown that the answer to the above is affirmative.

References

1. S. Bochner and W. T. Martin, *Several complex variables*, Princeton University Press, Princeton, New Jersey, 1948.
2. H. T. Bohnenblust and S. Karlin, *Geometrical properties of the unit sphere of Banach algebras*, Ann. of Math., **62** (1955), 217-229.
3. C. Caratheodory, *Theory of functions of a complex variable*, Chelsea, New York, 1954.
4. B. W. Glickfeld, *On an inequality of Banach algebra geometry and semi-inner product space theory*, Illinois J. Math., **14** (1970), 76-81.
5. L. A. Harris, *Schwarz's lemma in normed linear spaces*, Proc. Natl. Acad. Sci. (U.S.A.), **62** (1969), 1014-1017.
6. ———, *Schwarz's lemma and the maximum principle in infinite dimensional spaces*, Doctoral Dissertation, Cornell University, Ithaca, New York, 1969.
7. E. Hille and R. S. Phillips, *Functional analysis and semi-groups*, Amer. Math. Soc. Colloq. Publ. 31, Providence, 1957.
8. G. Lumer, *Semi-inner-product spaces*, Trans. Amer. Math. Soc., **100** (1961), 29-43.
9. Z. Nehari, *Conformal mapping*, McGraw-Hill, New York, 1952.

Received October 16, 1970.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY