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ON THE RATIO ERGODIC THEOREM FOR SEMI-GROUPS

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For a semi-group Γ of positive linear *contractions* on L_1 of a σ-finite measure space $(X, \mathscr{A}, \mu)$, strongly continuous on $(0, \infty)$, there are two ratio ergodic theorems: one, due to Chacon and Ornstein, describes the behavior at infinity; the other one, due to Krengel-Ornstein-Akcoglu-Chacon, describes the "local" behavior. In the present paper we attempt to generalize these results to the case when the semi-group is only uniformly bounded. Then the space X decomposes into two parts, Y and Z, called the *remaining* and the *disappearing* part, and both ratio theorems are shown to hold on Y. The ratio theorem at infinity fails on Z.

This generalizes the situation described in the discrete case by the second-named author, and by A. Ionescn Tulcea and M. Moretz. We have not studied the "local" behavior of the ratio on Z.

1. Definitions. Let $\Gamma = \{T_t : t \geqq 0\}$ be a semi-group of positive linear operators in L_1 of a σ-finite measure space $(X, \mathscr{A}, \mu)$. We assume that Γ is *bounded*: $\sup_{t>0} | T_t |_1 < \infty$; and that Γ is *strongly continuous* on $(0, \infty)$: i.e., for each $f \in L_1$ and each $s > 0$, we have $\lim_{t\to s} | T_t f - T_s f |_1 = 0$. It is then known (cf. [5], p. 616) that Γ is strongly integrable on every interval $[\alpha, \beta], 0 \leqq \alpha < \beta < \infty$; more precisely, for each $f \in L_1$ and $0 \leqq \alpha < \beta < \infty$, the integral $\int_\alpha^\beta T_t f dt$ is defined and is an element of $L_1(X, \mathscr{A}, \mu)$. Hence for each $f \in L_1$ there is a scalar function $T_t f(x)$, measurable with respect to the product of Lebesgue measure and μ, such that for almost all t, $T_t f(x)$, as a function of x, belongs to the equivalence class $T_t f$([5], p. 686). Moreover, there is a set $E(f)$, $\mu(E(f)) = 0$, dependent on f but independent of t, such that if $x \notin E(f)$ then $T_t f(x)$ is integrable on every finite interval $[\alpha, \beta]$ and the integral $\int_\alpha^\beta T_t f(x) dt$, as a function of x, belongs to the equivalence class $\int_\alpha^\beta T_t f dt$. Thus for each $u > 0$ and each $f \in L_1$, the integral $\int_0^u T_t f(x) dt$, denoted $S_u f(x)$, is defined for every $x \notin E(f)$.

All sets introduced in this paper are assumed measurable; all functions are measurable and extended real-valued. All relations are assumed to hold modulo sets of μ-measure zero. The indicator function of a set A is written 1_A. We write supp f for the set of points at which the function f is different from zero. For a set $A \subset X$, $L_1(A)$ denotes the class of functions f in $L_1(X)$ with $\operatorname{supp} f \subset A$; A is said to be *closed* (under T) if $T\{L_1(A)\} \subset L_1(A)$.

2. Behavior at infinity. The following Theorem 2.1 is a continuous parameter version of the Chacon-Ornstein theorem; Theorem 2.1 is included in a result of Berk [3], and was also recently obtained by Akcoglu and Cunsolo [2]. The following proof shows that the result is in fact contained in that of [4].

THEOREM 2.1. *Let $\Gamma = \{T_t : t \geqq 0\}$ be a semi-group of positive linear contractions in L_1 such that Γ is strongly continuous on $(0, \infty)$. Let $f, g \in L_1$, $g \geqq 0$. Then, as $u \to \infty$, the ratio*

$$D_u(f, g)(x) \overset{\text{def}}{=} S_u f(x)/S_u g(x) \tag{2.1}$$

converges to a finite limit a.e. on the set

$$A(g) \overset{\text{def}}{=} \{x : \sup_{u>0} S_u g(x) > 0\} .$$

Proof. For $f \in L_1$, let $\bar{f}(x) = S_1 f(x)$. For each $u > 0$, write $u = n + r$, where $n = [u]$, $0 \leqq r < 1$. Writing T for T_1, we have

$$\begin{aligned} S_u f &= \int_0^u T_t f dt = \sum_{k=0}^{n-1} \int_k^{k+1} T_t f dt + \int_n^{n+r} T_t f dt \\ &= \sum_{k=0}^{n-1} T^k \int_0^1 T_t f dt + T^n \int_0^r T_t f dt \end{aligned}$$

and hence

$$S_u f(x) = \sum_{k=0}^{n-1} T^k \bar{f}(x) + T^n (S_r f)(x) . \tag{2.2}$$

We may assume that f is nonnegative; then $0 \leqq S_r f(x) \leqq \bar{f}(x)$ and $0 \leqq S_r g(x) \leqq \bar{g}(x)$ a.e., $0 \leqq r \leqq 1$. Thus, for u sufficiently large, we have on $A(g)$

$$\frac{\sum_{k=0}^{n-1} T^k \bar{f}(x)}{\sum_{k=0}^{n} T^k \bar{g}(x)} \leqq D_u(f, g)(x) \leqq \frac{\sum_{k=0}^{n} T^k \bar{f}(x)}{\sum_{k=0}^{n-1} T^k \bar{g}(x)} . \tag{2.3}$$

This completes the proof, since the Chacon-Ornstein theorem and Lemma 2 [4] imply that the first and last terms in (2.3) converge to the same finite limit on the set $\{x : \sum_{k=0}^{\infty} T^k \bar{g}(x) > 0\} = A(g)$.

For a bounded semi-group Γ, we have the following decomposition of the space X.

PROPOSITION 2.1. *Let $\Gamma = \{T_t: t \geqq 0\}$ be a bounded semi-group of positive linear operators in L_1. Then the space X decomposes into Y and Z with the following properties: Z is T_t-closed for $t \geqq 0$;*

$$(2.4)\qquad \begin{cases} 0 \not\equiv f \in L_1^+(Y) \text{ implies } \liminf_{t\to\infty} \int T_t f d\mu > 0 ; \\ f \in L_1(Z) \text{ implies } \lim_{t\to\infty} \int T_t \mid f \mid d\mu = 0 . \end{cases}$$

Proof. This result in the discrete parameter case was obtained by the second author in [11]. To prove the proposition, we apply the discrete case result to $T = T_1$, obtaining the decomposition $X = Y + Z$ with the properties

$$(2.5)\qquad \begin{cases} 0 \not\equiv f \in L_1^+(Y) \text{ implies } \liminf_{n\to\infty} \int T_n f d\mu > 0 ; \\ f \in L_1(Z) \text{ implies } \lim_{n\to\infty} \int T_n \mid f \mid d\mu = 0 . \end{cases}$$

Suppose that $f \in L_1^+$ and $\liminf_{t\to\infty} \int T_t f d\mu = 0$; then given $\varepsilon \in 0$, there is an $s > 0$ such that $0 \leqq \int T_s f d\mu < \varepsilon$. For $t > s$, we have

$$\begin{aligned} 0 \leqq \int T_t f d\mu &= \int T_{t-s}(T_s f) d\mu \\ &\leqq \mid T_{t-s} \mid_1 \cdot \mid T_s f \mid_1 \leqq \varepsilon \cdot \sup_{t \geqq 0} \mid T_t \mid_1 , \end{aligned}$$

which shows that $\lim_{t\to\infty} \int T_t f d\mu = 0$; in view of (2.5), (2.4) is now proved. That Z is T_t-closed for each $t \geqq 0$ is an easy consequence of (2.4).

The next proposition permits us to construct a semi-group Γ' of positive linear *contractions* related to Γ; the ratio ergodic properties of Γ are then studied via Γ'.

PROPOSITION 2.2. *Let $\Gamma = \{T_t: t \geqq 0\}$ be a bounded semi-group of positive linear operators in L_1, strongly continuous on $(0, \infty)$. Then there is a function e such that*

$$(2.6)\qquad e \in L_\infty^+,\ \operatorname{supp} e = Y,\ T_t^* e = e \text{ for } t > 0 .$$

Proof. We may assume that $Y \neq \phi$ for otherwise the proposition is obviously true. Let

$$H = \{h \in L_\infty : T_t^* h = h,\ t > 0\} ;$$
$$D = \{1/2^n : n = 0, 1, 2, \cdots\} ;$$
$$G = \{g : g = f - T_r f, f \in L_1, r \in D\} .$$

Let $sp(G)$ denote the linear span of G. We first show that $H \neq \{0\}$. Let $h \in L_\infty$ be such that $\int g \cdot h d\mu = 0$ for every $g \in sp(G)$. It follows from $\int (f - T_r f) \cdot h = 0$, holding for each $f \in L_1$, $r \in D$, that

$$T_r^* h = h, r \in D . \tag{2.7}$$

The strong continuity of Γ on $(0, \infty)$ now implies that (2.7) holds for any $r > 0$. Assume *ab contrario* that $H = \{0\}$; then $h = 0$, and $sp(G)$ is dense in L_1. Thus given $f \in L_1^+(Y)$, and $\varepsilon > 0$, there is a function $g \in sp(G)$ such that $|f - g|_1 < \varepsilon$. We note that g is a linear combination of functions of the form $f_j - T_{r_j} f_j$, where $f_j \in L_1$, $r_j \in D$, $1 \leqq j \leqq m$; hence letting $r = \min\{r_1, r_2, \cdots, r_m\}$, we have

$$\lim_n n^{-1} \cdot \left| \sum_{i=0}^{n-1} T_r^i g \right|_1 = 0 . \tag{2.8}$$

Thus

$$\begin{aligned} \liminf_n | T_r^n f |_1 &\leqq \limsup_n n^{-1} \cdot \left| \sum_{i=0}^{n-1} T_r^i f \right|_1 \\ &\leqq \lim_n n^{-1} \left| \sum_{i=0}^{n-1} T_r^i g \right|_1 + \varepsilon \cdot \sup_t | T_t |_1 \\ &= \varepsilon \cdot \sup | T_t |_1 . \end{aligned}$$

This contradicts relation (2.4) and the assumption $Y \neq \phi$, since $\varepsilon > 0$ is arbitrary and Γ is bounded. Now let $0 \neq h \in H$ and write $h = h^+ - h^-$, where $h^+ = \max(h, 0)$, $h^- = -\min(h, 0)$. We may assume $h^+ \neq 0$; otherwise we replace h by $-h$. We have $T_t^* h^+ \geqq h^+$ for $t > 0$. Let $h' = \lim_n T_n^* h^+$; clearly, $0 \neq h' \in L_\infty^+$ and, by the monotone continuity of T_r^* (cf. [9], p. 187), we have $T_r^* h' = h'$ for $r \in D$. It now follows from the strong continuity of Γ that $T_t^* h' = h'$ for $t > 0$. Let π be a probability measure equivalent with μ, and let s be the supremum of numbers $\pi(\operatorname{supp} h)$ where h ranges over H^+, the class of nonnegative functions in H. There exists a sequence of functions $h_n \in H^+$ with $\pi(\operatorname{supp} h_n) \to s$. If $e \in L_\infty^+$ is a proper linear combination of the h_n's, and $E = \operatorname{supp} e$, then $e \in H^+$, $E \subset Y$ and $\pi(E) = s$. We next show that $E = Y$. We note that E is T_t^*-closed, $t > 0$. Indeed, there are functions $f_n \uparrow 1_E$ and constants $c_n > 0$ such that $c_n f_n \leqq e$. Hence $(\operatorname{supp} T_t^* f_n) \subset E$, and by the monotone continuity of T_t^*, $(\operatorname{supp} T_t^* 1_E) \subset E$. Applying the duality relation we can now see that $E^c = X - E$ is T_t-closed, $t > 0$. If T'_t is the restriction of T_t to $L_1(E^c)$, then $\Gamma' = \{T'_t: t \geqq 0\}$ is a semi-group of positive linear operators in L_1, strongly continuous on $(0, \infty)$. Under Γ', E^c decomposes into sets Y' and Z' according to Proposition 2.1. Since E^c is closed under T_t for $t > 0$, we have $\int T'_t f d\mu = \int T_t f d\mu$ for $f \in L_1(E^c)$ and $t > 0$. Hence $f \in L_1(Z)$ implies $\lim_t \int T'_t |f| d\mu = \lim_t \int T_t |f| d\mu = 0$, and $0 \not\equiv f \in L_1^+(Y - E)$ implies $\liminf_t \int T'_t f d\mu = \liminf_t \int T_t f d\mu > 0$. Consequently, $Y' = Y - E$ and $Z' = Z$. Thus if $E \neq Y$, then Y' is nonnull, and hence the first part of the proof, with Γ' replacing Γ, shows that there is a function e_1, $0 \not\equiv e_1 \in L_\infty^+(E^c)$, and $T'^*_t e_1 \geqq e_1$, $t > 0$. Since $T'^*_t e_1 = 1_{E^c} \cdot T^*_t e_1$, we have $T^*_t e_1 \geqq e_1$, $t > 0$. Let $e' = \lim_n T^*_n e_1$; then $e' \in H^+$ and $(\operatorname{supp} e') \cap E^c$ is nonnull. Thus $e + e' \in H^+$ and $\pi(\operatorname{supp}(e + e')) > s$, which contradicts the definition of s. Hence $\operatorname{supp} e = Y$ and the proposition is proved.

Assume that $\Gamma = \{T_t: t \geqq 0\}$ satisfies the hypothesis of Proposition 2.2. Let e be a solution of (2.6); we may assume that $0 < e \leqq 1$ on Y. T_t may be extended to a positive linear map on $\mathscr{M}^+$, the cone of nonnegative measurable functions on $(X, \mathscr{A})$: for each fixed $t \geqq 0$, if $f \in \mathscr{M}^+$, $T_t f$ is defined as $\lim_n T_t f_n$ where $f_n \in L_1^+$, and $f_n \uparrow f$ a.e. The extended operators T_t also satisfy the semi-group property on $\mathscr{M}^+$; i.e.,

$$T_{t+s} f = T_t(T_s f), f \in \mathscr{M}^+, t, s \geqq 0 . \tag{2.9}$$

For each $t \geqq 0$, we define an operator V_t on L_1^+ by the relation

$$V_t f = e \cdot T_t(f/(e + 1_Z)) , \tag{2.10}$$

and extend V_t by linearity to L_1. One shows, as in [11], that $\Gamma' = \{V_t: t \geqq 0\}$ is a family of positive linear contractions in L_1. That Γ' is a semi-group is a consequence of (2.9), (2.10), and the fact that Z is T_t-closed, $t \geqq 0$. Let $K = \{g: g = f \cdot e, f \in L_1\}$. For a fixed $s > 0$ and $g = f \cdot e \in K, f \in L_1$, we have

$$\begin{aligned} | V_t g - V_s g |_1 &= \left| e \cdot T_t\Big(\frac{g}{e + 1_Z}\Big) - e \cdot T_s\Big(\frac{g}{e + 1_Z}\Big)\right|_1 \\ &\leqq | e |_\infty \cdot | T_t(f \cdot 1_Y) - T_s(f \cdot 1_Y) |_1 \end{aligned} \tag{2.11}$$

which, by the strong continuity of Γ, tends to zero as $t \to s$. The case of a general $g \in L_1(Y)$ follows by approximation, since K is a dense subspace of $L_1(Y)$ and $| V_t |_1 \leqq 1$. Finally, because $V_t g = V_t(g \cdot 1_Y)$ for $g \in L_1$, we conclude that Γ' is strongly continuous on $(0, \infty)$.

Theorem 2.1 may now be applied to Γ': if $f' \in L_1^+$, $g' \in L_1^+$, then

$$\lim_{u\to\infty} \int_0^u V_t f'(x)dt \Big/ \int_0^u V_t g'(x)\,dt$$

exists a.e. on the set $\{x: \sup_{u>0} \int_0^u V_t g'(x)dt > 0\}$. For arbitrary measurable nonnegative functions f and g, we write $f' = f\cdot e$, $g' = g\cdot e$. If $f' \in L_1^+$, $g' \in L_1^+$, then for sufficiently large u,

$$\frac{\int_0^u V_t f'(x)dt}{\int_0^u V_t g'(x)dt} = \frac{\int_0^u e(x)\cdot T_t f(x)dt}{\int_0^u e(x)\cdot T_t g(x)dt} = D_u(f, g)(x) \tag{2.12}$$

on $Y \cap A(g)$, where $A(g) \stackrel{\text{def}}{=} \{x: \sup_{u>0} S_u g(x) > 0\}$. Thus the last ratio in (2.12) converges to a finite limit a.e. on the set $Y \cap A(g)$. The above discussion is now summarized in the following theorem:

THEOREM 2.2. *Let $\Gamma = \{T_t: t \geqq 0\}$ be a bounded semi-group of positive linear operators in L_1, storongly continuous on $(0, \infty)$. If f, g are measurable functions such that $f\cdot e$, $g\cdot e \in L_1^+$, then* $\lim_{u\to\infty}(D_u(f, g)(x)$ *exists* a.e. *on the set $Y \cap A(g)$.*

We say that the ratio theorem *holds* (for Γ) on a subset B of X if whenever $f \in L_1$, $g \in L_1^+$, $\lim_{u\to\infty} D_u(f, g)(x)$ exists a.e. on the set $B \cap A(g)$; otherwise we say that the ratio theorem *fails* on B. We showed that the ratio theorem holds on Y. We now show

THEOREM 2.3. *Let $\Gamma = \{T_t: t \geqq 0\}$ be bounded semi-group of positive linear operators in L_1, strongly continuous on $(0, \infty)$. If there is a function $g \in L_1^+(Z)$ such that the set $C(g) \stackrel{\text{def}}{=} \{x: \sup_{u>0} S_u g(x) = \infty\}$ is nonnull, then the ratio theorem fails on every nonnull subset of $C(g)$.*

Proof. Theorem 2.3 in the discrete parameter case was given in [7]; (see also [11] and [6]). The method of proof in [7] extends to the continuous case. Assume that the ratio theorem holds on a nonnull subset A of $C(g)$, where $g \in L_1^+(Z)$. In particular, $\lim_{u\to\infty} D_u(f, g)(x)$ exists a.e., on A for every $f \in L_1$. Let R be the operator from L_1 into $\mathscr{M}$, the space of real-valued measurable functions on $(X, \mathscr{A})$, defined by $Rf(x) = 1_A(x)\cdot\lim_{u\to\infty} D_u(f, g)(x)$. Since $S_u g(x) \to \infty$ on A, we have for each $t > 0$

$$R(T_t g)(x) = \lim_{u\to\infty} \frac{\int_0^u T_{s+t} g(x)ds}{\int_0^u T_s g(x)ds}$$

$$= \lim_{u\to\infty} \left[\frac{\int_0^u T_s g(x)ds}{\int_0^u T_s g(x)ds} - \frac{\int_0^t T_s g(x)ds}{\int_0^u T_s g(x)ds} + \frac{\int_u^{u+t} T_s g(x)ds}{\int_0^u T_s g(x)ds} \right] \geqq 1 \tag{2.13}$$

on A. On the other hand, since $|T_t g|_1 \to 0$ as $t \to \infty$, we may choose a subsequence $(T_{t_n} g)$ with $\sum_{n=1}^\infty T_{t_n} g \in L_1$. Then $0 \leqq \sum_{n=1}^\infty R(T_{t_n} g) \leqq R(\sum_{n=1}^\infty T_{t_n} g) < \infty$ μ-a.e.; hence $\lim_n R(T_{t_n} g) = 0$ μ-a.e., but this contradicts (2.13).

3. Local behavior. Akcoglu and Chacon [1] have shown that for a semi-group $\Gamma = \{T_t: t \geqq 0\}$ of positive linear contractions in $L_1(X, \mathscr{A}, \mu)$, there is a decomposition of the space X into an 'initially conservative part', C, and 'initially dissipative part', D. The set C may be defined as $\{x: S_u f(x) > 0$ for all $u > 0\}$, where f is any strictly positive function in $L_1(X, \mathscr{A}, \mu)$. We note that this decomposition remains valid for bounded semi-groups. The main result in [1] can be stated as follows:

THEOREM A. *Let $\Gamma = \{T_t: t \geqq 0\}$ be a semi-group of positive linear contractions in $L_1(X, \mathscr{A}, \mu)$, strongly continuous on $(0, \infty)$. If $\in L_1$, $gf \in L_1^+$, then $\lim_{u\downarrow 0} (S_u f(x))/(S_u g(x))$ exists a.e. on the set $C \cap \{g>0\}$.*

We recall from § 2 that for a bounded semi-group $\Gamma = \{T_t: t \geqq 0\}$ of positive linear operators in L_1, strongly continuous on $(0, \infty)$, we can construct a semi-group $\Gamma' = V_t: t \geqq 0\}$ of positive linear *contractions* related to Γ defined by (2.10). Theorem A is thus applicable to Γ'. Let $X = C + D = C' + D'$ be the initial decompositions corresponding to Γ and Γ' respectively.

Theorem A applied to Γ' shows that if $f' \in L_1$, $g' \in L_1^+$, then $\lim_{u\downarrow 0} \int_0^u V_s f'(x)ds \big/ \int_0^u V_s g'(x)ds$ exists a.e. on the set $C' \cap \{g' > 0\}$. For arbitrary measurable nonnegative functions f and g, we let $f' = f\cdot e$, $g' = g\cdot e$. If $f', g' \in L_1^+$, then

$$\frac{\int_0^u V_s f'(x)ds}{\int_0^u V_s g'(x)ds} = \frac{e(x)\cdot \int_0^u T_s f(x)ds}{e(x)\cdot \int_0^u T_s g(x)ds} = \frac{S_u f(x)}{S_u g(x)} \tag{3.1}$$

on the set $\left\{x: \int_0^u V_s g'(x)ds > 0 \text{ for } u > 0\right\}$, which contains the set $C' \cap$

$\{g' > 0\}$, as shown in [1], Lemma 2.3. Thus $\lim_{u\downarrow 0} (S_u f(x))/(S_u g(x))$ exists a.e., on the set $C' \cap (g' > 0\}$.

It is clear from $g' = g\cdot e$ that $\{g' > 0\} = \{g > 0\} \cap Y$. We next show that $C' = C \cap Y$. Let C be defined in terms of some fixed function $g \in L_1$, $g > 0$. For each $u > 0$,

$$\int_0^u T_s g(x)ds = \int_0^u T_s g_Y(x)ds + \int_0^u T_s g_Z(x)ds\ . \tag{3.2}$$

The last integral in (3.2) vanishes a.e. on Y since Z is T_s-closed, $s \geqq 0$. Hence $\int_0^u T_s g(x)ds = \int_0^u T_s g_Y(x)ds > 0$ on $C \cap Y$. Let $g' = g_Y\cdot e$. Then $\int_0^u V_s g'(x)ds = e(x)\cdot\int_0^u T_s g_Y(x)ds > 0$ for $u > 0$ on $C \cap Y$. This shows that $C' \supset C \cap Y$. Next, since $V_s g(x) = 0$ a.e. on Z for any $g \in L_1$, C' may be obtained as the set $\left\{x\colon \int_0^u V_s g(x)ds > 0 \text{ for } u > 0\right\}$ for any $g \in L_1^+$ such that $g > 0$ on Y. Let $g' = g\cdot e$. Then $g' > 0$ on Y and hence $\int_0^u V_s g'(x)ds > 0$ on C', $u > 0$. Since $\int_0^u V_s g'(x)ds = \int_0^u e(x)\cdot T_s g(x)ds$, we conclude that $\int_0^u T_s g(x)ds > 0$ a.e. on C', $u > 0$. Hence $C' \subset C \cap Y$. We have proved:

THEOREM 3.1. *Let $\Gamma = \{T_t\colon t \geqq 0\}$ be a bounded semi-group of positive linear operators in L_1, strongly continuous on $(0, \infty)$. If f, g are nonnegative measurable functions such that $f\cdot e$, $g\cdot e \in L_1^+$, then $\lim_{u\downarrow 0} (S_u f(x))/S_u g(x))$ exists a.e. on the set $\{g > 0\} \cap C \cap Y$.*

Of course, the restriction of the above statement to C is not a loss of generality, since on D the ratio D_u is of the form 0/0. The local behavior of D_u on Z does not seem to be easy to ascertain by the methods of the present paper.

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