PACIFIC JOURNAL OF MATHEMATICS
Vol. 40, No. 1, 1972

COMPLEX CHEBYSHEV ALTERATIONS

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P. Chebyshev's famous Alternation Theorem for best uniform approximation to continuous real valued functions on an interval is generalized to include best approximation to a class of continuous complex valued functions on an ellipse.

1. Preliminary remarks and definitions. For a continuous complex valued function f defined on a compact set E in the plane and, for $n \in Z^+$, let $p_n(f, E)$ denote the polynomial of degree n, of best uniform appoximation to f on E and *let*;

$$\rho_n(f, E) = \max_{z \in E} |f(z) - p_n(f, E)(z)| .$$

Chebyshev's Alternation Theorem [1, p. 29] states that if f is a continuous real valued function on an interval $[a, b]$, and p_n is a polynomial of degree n, $n \in Z^+$, then $p_n = p_n(f, [a, b])$ if and only if, there exists $n + 2$ points,

$\{x_i\}_{i=1}^{n+2}, a \leqq x_1 < x_2 < \cdots < x_{n+2} \leqq b$, with the property that $|f(x) - p_n(x)|$ attains its maximum on $[a, b]$ at these points and $f(x_i) - p_n(x_i) = -[f(x_{i+1}) - p_n(x_{i+1})]$ for $i = 1, 2, \cdots, n + 1$.

The sets we consider here are ellipses which are of course a generalization of intervals. So, for $a \geqq 0$, let $E_a = \{z + a/z: |z| = 1\}$. Now let $\mathscr{F}_n(E_n)$ denote those complex valued functions f, not themselves polynomials of degree n, continuous on E_a, having the property that there exists $n + 2$ points $\{\xi_k\}_{k=1}^{n+2}$ in E_a, such that $p_n(f, E_n) = p_n(f, \{\xi_k\}_{k=1}^{n+2})$. It is known [1, p. 22] that there always exists a set $D \subset E_a$, consisting of $n + k$ points, $2 \leqq k \leqq n + 3$, such that $p_n(f, E_a) = p_n(f, D)$. Furthermore, to this author's knowledge, every example of best uniform approximation to rational functions on infinite sets in the plane (e.g., [3], [4] and [5]) is one in which such a set consisting of $n + 2$ points exists or, can be shown equivalent to such an example.

2. Main theorem. Given $n + 2$ points $\{\xi_k\}_{k=1}^{n+2}$ in E_a let z_k be such that $\xi_k = z_k + a/z_k$, $|z_k| = 1$ and if $a = 1$, $0 \leqq \operatorname{Arg} z_k \leqq \pi$ for $k = 1, 2, \cdots, n + 2$. The z_k's are uniquely determined. Now let

$$\Phi_k = z_k^{-n/2} \prod_{\substack{j=1 \\ j \neq k}}^{n+2} [(z_k z_j - a)/|z_k z_j - a|] \text{ for}$$

$k = 1, 2, \cdots, n + 2$ where $0 \leqq \arg z^{1/2} < \pi$.

THEOREM 1. *If f is continuous on E_a and p_n is a polynomial of degree n, $n \in Z^+$, then $f \in \mathscr{F}_n(E_a)$ and $p_n = p_n(f, E_a)$ if and only if there exists $n+2$ points $\{\xi_k\}_{k=1}^{n+2}$ in E_a, with $0 \leqq \operatorname{Arg} \xi_1 < \operatorname{Arg} \xi_2 < \cdots < \operatorname{Arg} \xi_{n+2} < 2\pi$ if $a \neq 1$ or $-2 \leqq \xi_1 < \xi_2 < \cdots < \xi_{n+2} \leqq 2$ if $a = 1$, where $|f(\xi) - p_n(\xi)|$ attains its maximum on E_a and, $[f(\xi_i) - p_n(\xi_i)]/\Phi_i = -[f(\xi_{i+1}) - p_n(\xi_{i+1})]/\Phi_{i+1}$ for $i = 1, 2, \cdots, n+1$ where the Φ_i's are defined in terms of the ξ_i's as above.*

Proof. In order to prove our theorem we make use of a lemma which is a reformulation of a result [2] due to T. S. Motzkin and J. L. Walsh.

LEMMA. *A necessary and sufficient condition that the given numbers $\{\sigma_k\}_{k=1}^{n+2}$ be the deviations of some function f defined on the $n+2$ points $\{\xi_k\}_{k=1}^{n+2}$ and its polynomial of degree n of best uniform approximation to f on these points is that for some $\rho \geqq 0$;*

(1) $|\sigma_k| = \rho$ *for* $k = 1, 2, \cdots, n+2$ *and,*

(2) $\arg \sigma_k = \arg \omega'(\xi_k) + \theta_0$ *for* $k = 1, 2, \cdots, n+2$ *if* $\rho > 0$ *where*

$$\omega(\xi) = \prod_{k-1}^{n+2} (\xi - \xi_k) \textit{ and } \theta_0 = \arg \Big[\sum_{k=1}^{n+2} f(\xi_k)/\omega'(\xi_k)\Big] .$$

The necessary portion of our theorem will then follow if it is shown that;

$$\arg\{[\omega'(\xi_i)/\Phi_i]/[\omega'(\xi_{i+1})/\Phi_{i+1}]) = \pi \text{ for} \tag{2.1}$$

$i = 1, 2, \cdots, n+1$. Now substituting $z_j + a/z_j$ for ξ_j and using the definition of the Φ_j's we can show the (2.1) is equivalent to;

$$\arg\{(z_{i+1}^{n/2}/z_i^{n/2}) \prod_{\substack{j \neq i, i+1 \\ j=1}}^{n+2} [(z_i - z_j)/(z_{i+1} - z_j)]\} = 0 . \tag{2.2}$$

But, (2.2) follows since z_i and z_{i+1} are by virtue of their definition adjacent on the unit circle U(i.e., z_i and z_{i+1} are on a connected arc in U containing none of the other $z_{j's}$) and since; $\arg (z_{i+1}/z_i) = -2 \arg [z_i - z_j)/(z_{i+1} - z_j)]$ for $j \neq i, i+1$.

In order to prove the converse of our theorem we simply work backwards and show that; $\arg [f(\xi_k) - P_n(\xi_k)] = \arg \omega'(\xi_k) + \theta_0$ for some θ_0 and $k = 1, 2, \cdots, n+2$ and apply the aforementioned result of Motzkin and Walsh.

3. Special cases and applications. Chebyshev's Alternation Theorem follows as a special case of Theorem 1, when $a = 1$, since it is known [1, p. 22] that all real functions, not themselves polynomials of degree n, continuous on $[-2, 2]$ are in the class $\mathscr{F}_n([-2, 2])$.

Also of interest because of its simple form is the case where $a = 0$ or $E_a = U$ is the unit circle and where n is even. In this case our main theorem appears to provide us with a valuable tool in determining if a given function f is in $\mathscr{F}_{2m}(U)$ and if it is, in finding $p_{2m}(f, U)$.

COROLLARY 1. *If f is continuous on U and p_{2m} is a polynomial of degree $2m$, $m \in Z^+$, then $f \in \mathscr{F}_{2m}(U)$ and $p_{2m} = p_{2m}(f, U)$ if and only if there exists $2m+2$ points, $\{z_k\}_{k=1}^{2m+2}$, with $0 \leqq \operatorname{Arg} z_1 < \cdots < \operatorname{Arg} z_{2m+2} < 2\pi$ where $|f(z) - p_{2m}(z)|$ attains its maximum on U and where $[f(z_k) - p_{2m}(z_k)]/z_k^m = -[f(z_{k+1}) - p_{2m}(z_{k+1})]/z_{k+1}^m$, for $k = 1, 2, \cdots, 2m+1$.*

Corollary 1 can be used to obtain a recently discovered example of best approximation [3], namely, if $f(z) = (\alpha z + \beta)/(z - a)(1 - \bar{a}z)$, $|a| > 1$, then;

$$p_{2m}(f, U)(z) = [\alpha z + \beta - K_1 z^{2m}(1 - \bar{a}z)^2 - K_2(z - a)^2]/(z - a)(1 - \bar{a}z),$$

where

$$K_1 = (\alpha a + \beta)/a^{2m}(1 - |a|^2)^2$$

and,

$$K_2 = \bar{a}(\alpha + \beta\bar{a})/(1 - |a|^2)^2 .$$

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Received December 4, 1970. This work represents part of the author's doctoral dissertation that was directed by Professor Mishael Zedek at the University of Maryland.

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