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PRÜFER AND VALUATION RINGS WITH ZERO DIVISORS

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Manis has developed a valuation theory on commutative rings with unity producing valuation rings which are not integral domains. Griffin has used the valuation theory of Manis to extend the notion of Prüfer domains to rings with zero divisors, obtaining what Griffin calls Prüfer rings. In this paper, properties of overrings of Prüfer and valuation rings are discussed. An example is given to show that valuation rings need not be Prüfer rings. It is shown that every overring of a Prüfer valuation ring is a valuation ring.

1. Introduction. All rings considered will be commutative and have unity. An element will be called *regular* if it is not a zero divisor and an ideal will be called *regular* if it contains a regular element. An *overring* of a ring R is a subring of K, the total quotient ring of R, containing R. The complement of P in R will be denoted by $R\backslash P$. Proper containment will be denoted by $\subset$. If R is a ring and A is an ideal of R, the pair (S, Q) is said to *dominate* (R, A) if S is an overring of R and Q is an ideal of S such that $Q \cap R = A$.

Let R be a ring with total quotient ring K and regular prime ideal P. Then (R, P) is said to be a *valuation pair* if any of the following equivalent conditions are satisfied.

(1) For each $x \in K\backslash R$, there exists $y \in P$ such that $xy \in R\backslash P$.

(2) There is a mapping v from K onto a totally ordered additive abelian group with a symbol ∞ adjoined such that for all $x, y \in K$, $v(xy) = v(x) + v(y)$ and $v(x + y) \geqq \min \{v(x), v(y)\}$ and, moreover, $R = \{x \in K \mid v(x) \geqq 0\}$ and $P = \{x \in K \mid v(x) > 0\}$.

(3) If (S, Q) dominates (R, P), where Q is a prime ideal of S, then $S = R$.

(4) There is an algebraically closed field L and a homomorphism from R into L which cannot be extended to any overring of R.

The proof of the equivalence of conditions (1) through (3) is due to Manis [4], and the proof that (4) is equivalent to these is due to Kelly and Larsen [2]. When (R, P) is a valuation pair or R is a total quotient ring, R is said to be a *valuation ring*.

If P is a prime ideal of R, Griffin [1] defines the *large quotient ring* $R_{[P]}$ *with respect to* P to be the set of all $x \in K$ for which there exists $s \in R\backslash P$ such that $xs \in R$. If (R, P) is a valuation pair, then $R = R_{[P]}$. If A is an ideal of R, $[A]R_{[P]}$ is the set of all $x \in K$

for which there exists $s \in R\backslash P$ such that $xs \in A$. Griffin defines a *Prüfer ring* to be a ring in which each finitely generated regular ideal is invertible. Equivalently, a ring R is a Prüfer ring if $(R_{[P]}, [P]R_{[P]})$ is a valuation pair for each regular maximal ideal P.

2. Prüfer and valuation rings.

The example in § 3 will serve to show that some of the results of this section cannot be generalized.

THEOREM 2.1. *Let R be a Prüfer ring and let M and N be regular prime ideals of R. Then $R_{[M]} \subseteqq R_{[N]}$ if and only if $N \subseteqq M$.*

Proof. The "if" direction is clear. To prove the "only if" part, let $a \in N\backslash M$ and let b be any regular element of N. Hence $(a, b) \subseteqq N$, $(a, b) \nsubseteq M$. Then by [3; Lemma 2.1], $T((a, b)) \subseteqq R_{[M]}$, but $T((a, b)) \nsubseteq R_{[N]}$, where $T((a, b))$ denotes the transform of the ideal (a, b). Hence $R_{[M]} \nsubseteq R_{[N]}$. This contradiction implies that $N \subseteqq M$.

THEOREM 2.2. *Let (V, M) be a valuation pair, and let N be a regular prime ideal of R contained in M. Then $(V_{[N]}, [N]V_{[N]})$ is a valuation pair.*

Proof. Let $x \in K\backslash V_{[N]}$, where K is the total quotient ring of V. There exists $y \in M$ such that $xy \in V\backslash M$ since (V, M) is a valuation pair. If $y \in M\backslash N$, then $x \in V_{[N]}$, which is a contradiction. Hence $y \in N = [N]V_{[N]} \cap V$ and the theorem follows.

THEOREM 2.3. *The following are equivalent.*

(1) *(R, P) is a Prüfer valuation pair.*

(2) *R is a Prüfer ring with a unique regular maximal ideal P.*

(3) *(R, P) is a valuation pair, where P is the unique regular maximal ideal of R.*

Proof. (1) $\Rightarrow$ (2): Let N be a regular prime ideal of R. Since $R_{[P]} = R \subseteqq R_{[N]}$, by Theorem 2.1, $N \subseteqq P$. Hence P is a unique regular maximal ideal.

(2) $\Rightarrow$ (3): Since P is the unique regular maximal ideal, $R = R_{[P]}$. Hence, by the definition of a Prüfer ring, (R, P) is a valuation pair.

(3) $\Rightarrow$ (1): This is clear since $R = R_{[P]}$ and $P = [P]R_{[P]}$.

In § 3 we will prove the existence of a valuation pair (S, Q) such that Q is not maximal. This example, taken together with Theorem 2.3, will show that a valuation ring need not be a Prüfer ring.

LEMMA 2.4. *Let (V, P) be a Prüfer valuation pair. Let W be an overring of V with a regular prime ideal M. Then $M \subseteqq P$.*

Proof. By Theorem 2.3, P is the unique regular maximal ideal of V. Since all of the regular elements of M are in V, $M \cap V$ is a regular ideal and hence $M \cap V \subseteqq P$. Let $x \in M \backslash V$, then there exists $y \in P$ such that $xy \in V \backslash P$. But then $xy \notin M$ which is a contradiction. Therefore $M \subseteqq P$.

THEOREM 2.5. *Let (V, P) be a Prüfer valuation pair. Then every overring of V is a Prüfer valuation ring.*

Proof. Let W be an overring of V. Let K be the total quotient ring of V. If $W = K$, the proof is trivial. If $W \neq K$, let M be a regular maximal ideal of W. Let N be any proper regular prime ideal of W. By Lemma 2.4, both M and N are contained in P. Since V is a Prüfer ring either $M \subseteqq N$ or $N \subseteqq M$. But M was assumed to be a maximal ideal of W which implies that $N \subseteqq M$. Therefore, M is a unique regular maximal ideal of W. Since W is an overring of a Prüfer ring, W is a Prüfer ring. Hence Theorem 2.3 implies that W is a Prüfer valuation ring.

3. Example. In this section an example will be presented that will be used to dispose of some past conjectures, possibly the most important of which is that all valuation rings are Prüfer rings. Our example is related to an example due to Nagata [5, p. 131].

Let K be a field and consider $K[X, Y]$. Let

$$F = \{f(X, Y) \in K[X, Y] \mid f(X, Y) \text{ is irreducible, } f(0, 0) = 0, f(X, 0) \neq 0\}.$$

For each $f \in F$ let Z_f be an indeterminate and define $R^* = K[X, Y, \{Z_f\}_{f \in F}]$. Let I be the ideal of R^* generated by the set of all elements of the form fZ_f or $Z_f Z_g$ for all $f, g \in F$. Set $R = R^*/I$ and make the obvious identification of elements. Let P be the ideal of R generated by $\{Y, \{Z_f\}_{f \in F}\}$ and let A be the ideal of R generated by $\{X, Y, \{Z_f\}_{f \in F}\}$. It is straightforward to verify that A is a maximal ideal that properly contains the prime ideal P and that $A \backslash P$ consists of zero divisors.

Zorn's Lemma guarantees the existence of a valuation pair (S, Q) which dominates (R, P). Let C be the ideal generated by $\{Z_f \mid f \in F\}$ and let T be the total quotient ring of R. We will show that Q is not a maximal ideal of S, but first we prove two lemmas.

LEMMA 3.1. *Let $f(X, Y) \in R$ be such that $f(X, Y) \notin K[Y]$. Let n be minimal such that $k_1 Y^n X^p$ is a term of $f(X, Y)$ where $p \geqq 1$, $0 \neq k_1 \in K$. Then $f(X, Y)$ is regular if and only if there exists a term of $f(X, Y)$ of the form $k_2 Y^m$ where $m \leqq n$ and $0 \neq k_2 \in K$.*

Proof. To prove the "if" part, assume there is such a $k_2 Y^m$. Let t be such that $Y^t(k + g(X, Y)) = f(X, Y)$ where $g(X, Y) \in K[X, Y]$ has zero constant term and $0 \neq k \in K$. Since Y and $k + g(X, Y)$ are regular, we have shown that $f(X, Y)$ is regular.

Now suppose that there does not exist such a $k_2 Y^m$. Then $f(X, Y) = Y^n(h(X, Y))$ where $h(X, Y) \in A \backslash P$. Hence $f(X, Y)$ is a zero divisor.

LEMMA 3.2. *Let $f(X, Y) \in K[X, Y]$ and let $Z \in C$. Then $f(X, Y) + Z$ is a regular element if and only if $f(X, Y)$ is a regular element.*

Proof. Since C is a prime ideal, if $b \in R \backslash C$ is a zero divisor, then there exists a $Z' \in C$ such that $bZ' = 0$. But $ZZ' = 0$. Hence $f(X, Y) + Z$ is a zero divisor if and only if $f(X, Y)$ is a zero divisor.

THEOREM 3.3. *The ideal Q is not a maximal ideal of S.*

Proof. Let N be the ideal of S generated by Q and X. We will show that $Q \subset N \subset S$. Clearly $Q \subset N$. Suppose $N = S$. Then there exists $s \in S$ and $q \in Q$ such that $1 = sX + q$. Let $q = a/b$ with $a, b \in R$. Then a and b can be written as $a = f(Y) + Xg(X, Y) + Z$ and $b = f'(Y) + Xg'(X, Y) + Z'$ where $Z, Z' \in C$, $g(X, Y), g'(X, Y) \in K[X, Y]$ and $f(Y), f'(Y) \in K[Y]$. By Lemma 3.2, since b is regular, $f'(Y) + Xg'(X, Y)$ is regular and $f'(Y) \neq 0$. Hence, by Lemma 3.2 again, $d = f'(Y) + Xg'(X, Y) - Z'$ is regular. Then ad/bd is such that the denominator has no term involving an element in C. Therefore, without loss of generality we may assume that $Z' = 0$.

Let Z_X be the indeterminate such that $Z_X \cdot X = 0$. By multiplying both sides of $1 = sX + q$ by bZ_X, we have $bZ_X = aZ_X$. Hence $f'(Y)Z_X = f(Y)Z_X$. But a polynomial in Y is a zero divisor if and only if it is the zero polynomial, hence $f'(Y) = f(Y) \neq 0$. Thus,

$$\frac{a}{b} = \frac{f(Y) + Xg(X, Y) + Z}{f(Y) + Xg'(X, Y)}. \tag{1}$$

Write $f(Y) = Y^m(k + h(Y))$ where $0 \neq k \in K$ and $h(Y) \in YK[Y]$. By Lemma 3.1 $g'(X, Y) = Y^m(g^*(X, Y))$ where $g^*(X, Y) \in K[X, Y]$. Hence (1) can be rewritten as

(2) $(k + h(Y) + Xg^*(X, Y))a/b = k + h(Y) + (Xg(X, Y) + Z)Y^{-m} \in Q.$

We now show that $ZY^{-m} \in Q$. Let $\{Z_\alpha\}$ be the set of indeterminates which appear in Z and let $\{f_\alpha\}$ be a set of elements in F such that $Z_\alpha f_\alpha = 0$. Since $f_\alpha \in A \backslash P$, $\prod_\alpha f_\alpha \in A \backslash P$. But $Z(\prod_\alpha f_\alpha) Y^{-m} = 0 \in Q$. Since (S, Q) is a valuation pair, $T \backslash Q$ is multiplicatively closed. Hence $ZY^{-m} \in Q$.

By equation (2), we have $k + Xg(X, Y)Y^{-m} \in Q$. Let $g(X, Y) = Y^p(k' + Yr(X, Y))$ where $r(X, Y) \in K[X, Y]$ and $0 \neq k' \in K[X]$. Therefore, $k + Y^{p-m}X(k' + Yr(X, Y)) \in Q$. If $p - m \geqq 0$, then $Q \cap R = P$ contains a polynomial with nonzero constant term which is impossible. Thus $p - m < 0$. Since $Y^{m-p} k \in Q$, $X(k' + Yr(X, Y)) \in P$ which is also a contradiction. These contradictions prove that $N \subset S$.

For the remainder of this section let M be a maximal ideal of S that contains Q. Since (S, Q) is a valuation pair, $S_{[Q]} = S = S_{[M]}$. Hence Theorem 3.3 shows that Theorem 2.1 is not true if the ring is assumed to be a valuation ring instead of a Prüfer ring. Since $M \nsubseteq Q$, this also shows that the condition that V is Prüfer cannot be dropped from Lemma 2.4.

THEOREM 3.4. *The ring S is not a Prüfer ring.*

Proof. Clear by Theorem 3.3 and the remark following Theorem 2.3.

COROLLARY 3.5. *There exists a prime ideal N of S such that $(S_{[N]}, [N]S_{[N]})$ is not a valuation pair.*

Proof. This is clear from the definition of a Prüfer ring and Theorem 3.4.

THEOREM 3.6. *If (V, L) and (V, N) are valuation pairs, then $L = N$.*

Proof. Let $x \in L \backslash N$. Then there exists $y \in W \backslash V$, where W is the total quotient ring of V, such that $xy \in V \backslash L$ and there exists $z \in N$ such that $yz \in V \backslash N$. But then $(xy)z \in N$ and $x(yz) \notin N$, a contradiction.

REMARK. Patrick Kelly has independently proved a more general version of Theorem 3.6.

COROLLARY 3.7. *Let (S', M') be a valuation pair that dominates (S, M). Then $S'_{[M']} \neq S_{[M]}$.*

Proof. Since (S, Q) is a valuation pair and $Q \neq M$, Theorem 3.6 implies that $S' \neq S$. But $S' = S'_{[M']}$ and $S = S_{[M]}$.

We can also observe that since $(S_{[M]}, [M]S_{[M]})$ is the same as (S, M), which by Theorem 3.6 is not a valuation ring, the condition in Theorem 2.2 that $N \subsetneqq M$ cannot be deleted. Also, we see that M is an ideal that satisfies Corollary 3.5.

Let W be a ring and let N be a prime ideal of W. Then Griffin [1; page 57] has defined the *core of* N, $C(N)$, to be the set of all $x \in W$ such that for all regular $r \in W$, there exists $s \in W \backslash N$ such that $xs \in (r)$. It has been conjectured that if (W, N) is a valuation pair, then the ideal $C(N)$ is a maximal ideal in the total quotient ring of W. By the following theorem we see that this conjecture is false.

THEOREM 3.8. *The ideal* $C(Q)$ *is not a maximal ideal of* T.

Proof. If $C(Q)$ were a maximal ideal of T, then $S/C(Q)$ would be a valuation ring in the field $T/C(Q)$. However, since $Q/C(Q)$ is not a maximal ideal of $S/C(Q)$, this is impossible.

Added in proof. Malcolm Griffin has also given examples of valuation rings which are not Prüfer rings in Queen's University Preprint #1970-37.

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