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VECTOR SPACE DECOMPOSITIONS AND THE ABSTRACT IMITATION PROBLEM

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Let $\mathscr{S}$ be a Hilbert space, $\mathscr{P}$ a closed subspace of $\mathscr{S}$, L an orthogonal projection operator on $\mathscr{S}$. The "imitation problem" consists of finding the solutions $p \in \mathscr{P}$ of the equation

$$p - s = L(p - s)$$

for given $s \in \mathscr{S}$. If $\bar{W}$ is a compact bordered Riemann surface, A a boundary neighborhood, s a "singularity differential" defined on $\bar{A}$, p will be a harmonic exact differential which imitates s on $\bar{A}$ in a sense precised by L (hence the name "imitation problem"). Existence and uniqueness theorems are given for the solution. Some concrete applications are described. The paper ends with a constructive method of solution in the case of L^2-normal operators.

0. Introduction.

The "imitation problem" has originally been formulated by L. Sario (see for instance [1]). It is fundamental in the construction of harmonic functions on a Riemann surface with given singularities and given boundary behavior. It can be formulated as follows: given a "singularity function" s defined in a boundary neighborhood, and a "normal operator L", construct a harmonic function p defined on the whole Riemann surface and satisfying in the given boundary neighborhood the equation

$$p - s = L(p - s) .$$

Sario's original solution uses the sup norm. For problems involving harmonic differentials, the L^2 norm is introduced somewhat more naturally and progress has been made in various directions. (see [5]). In §1 we study the abstract "imitation problem" for an arbitrary Hilbert space and give an existence and uniqueness theorem for the solution. In §2 we consider some decompositions of the vector space $\mathfrak{H}(\bar{A})$ of harmonic exact differentials defined on a boundary neighborhood A of a compact bordered Riemann surface $\bar{R}$ and continuous in $\bar{A}$, and study some corresponding "imitation problems". In §3 we return to the L^2 case and give a constructive method of solution when the operator L is L^2-normal. The method may be applied to the case of harmonic differentials on a Riemannian manifold of dimension > 2, and also to open manifolds.

1. The abstract imitation problem in a Hilbert space.

Let

$\mathscr{S}$ be a Hilbert space, $\mathscr{B}(\mathscr{S})$ the algebra of bounded linear operators on $\mathscr{S}$. We are given a closed subspace $\mathscr{P} \subset \mathscr{S}$ corresponding to the orthogonal projection F. Given the orthogonal projection L on $\mathscr{S}$ we want to solve the equation

$$(*) \qquad p - s = L(p - s)$$

for $p \in \mathscr{P}$ given the "singularity" $s \in \mathscr{S}$. We assume moreover that $p = Ts$ where $T \in \mathscr{B}(\mathscr{S})$.

We are going to prove the theorem:

THEOREM. *Let $\mathscr{P}$ be a closed subspace of the Hilbert space $\mathscr{S}$ and let F denote orthogonal projection on $\mathscr{P}$. Let L be an arbitrary orthogonal projection operator on a subspace of $\mathscr{S}$. Then the imitation problem*

$$p - s = L(p - s)$$

admits a unique solution in $\mathscr{P}$ of the form

$$p = Ts$$

(where T is a bounded linear operator on $\mathscr{S}$) *if and only if*

$$\operatorname{Im} L \dotplus \operatorname{Im} F = \mathscr{S}.$$

Proof. Observe that $(*)$ may be written as:

$$(I - L)(I - T)s = 0$$

which is true for each $s \in \mathscr{S}$.

It follows that $I - T$ belongs to the right annihilator of $I - L$. Now $\mathscr{B}(\mathscr{S})$ being a Baer ring [4] it follows that there exists $X \in \mathscr{B}(\mathscr{S})$ such that

$$(**) \qquad I - T = LX .$$

Moreover, since $p \in \operatorname{Im} F$ we have $Ts \in \operatorname{Im} F$ hence

$$(I - F)Ts = 0 .$$

We conclude that T belongs to the right annihilator of $I - F$ hence there exists $Y \in \mathscr{B}(\mathscr{S})$ such that

$$(***) \qquad T = FY .$$

Adding up $(**)$ and $(***)$ we get the equation

$$(\dagger) \qquad I = LX + FY$$

where, we recall L, F are given orthogonal projection operators and X,

Y are unknown elements of $\mathscr{B}(\mathscr{S})$. Clearly, if $\operatorname{Im} L + \operatorname{Im} F \subsetneqq \mathscr{S}$, the last equation has no solution. We show conversely that if $\operatorname{Im} L + \operatorname{Im} F = \mathscr{S}$ the problem has always a solution. We need the:

LEMMA. *Let A, B be closed subspaces of a Hilbert space $\mathscr{S}$ such that $A + B = \mathscr{S}$* (vector sum). *Then, there exist closed subspaces $A_m \subset A$, $B_m \subset B$ such that*

$$A_m \dotplus B_m = \mathscr{S} \quad \text{(direct sum)} .$$

Proof. Let $\{e_\alpha\}$ be a basis for A, $\{e_\beta\}$ a basis for B. Then, $\{e_\alpha, e_\beta\}$ is a set of generators for $\mathscr{S}$. It contains a basis $\{e_{\alpha_i}, e_{\beta_j}\}$ where $\{e_{\alpha_i}\} \subset \{e_\alpha\}$ and $\{e_{\beta_j}\} \subset \{e_\beta\}$. Let then A_m be the closed span of $\{e_{\alpha_i}\}$, B_m be the closed span of $|e_{\alpha_i}|$. Then $A_m + B_m = \mathscr{S}$ and $A_m \cap B_m = \{0\}$.

We apply the lemma to $A = \operatorname{Im} L$, $B = \operatorname{Im} F$. There exist subspaces $A_m \subset \operatorname{Im} L$, $B_m \subset \operatorname{Im} F$ such that

$$A_m \dotplus B_m = \mathscr{S} .$$

Let X_0 and Y_0 be orthogonal projections on A_m and B_m respectively. Then

$$I = LX_0 + FY_0$$

and (X_0, Y_0) is a solution of (†). To study uniqueness, let (X, Y) be another solution of (†). One must have:

$$L(X - X_0) = F(Y_0 - Y) .$$

So if $\operatorname{Im} F \cap \operatorname{Im} L = \{0\}$ then necessarily $X = X_0$ $Y = Y_0$. If $\operatorname{Im} F \cap \operatorname{Im} L \neq \{0\}$ then, the operators of the form $L(X - X_0) = F(Y_0 - Y)$ are the elements of the right annihilator of the set $\{I - L, I - F\}$ hence of the form $G\mathscr{B}(\mathscr{S})$ for some orthogonal projection G. The T's we are looking for are of the form $FY = FY_0 - F(Y_0 - Y) = FY_0 - G\mathscr{B}(\mathscr{S})$. $G\mathscr{B}(\mathscr{S})$ is non void: if $\operatorname{Im} F \cap \operatorname{Im} L = \operatorname{Im} M$ where M is a projection, M satisfies $LM = FM$. In that case uniqueness is lost and we have proved the theorem.

Notes. (1) there is actually no restriction when dealing with operators L which are projections: if L denotes *any* element of $\mathscr{B}(\mathscr{S})$, $(*)$ becomes $(I - L)(I - T) = 0$. So $I - T$ belongs to the right annihilator of $I - L$ and therefore $I - F = \Lambda U$ where Λ is the orthogonal projection generating the right annihilator of $I - L$.

(2) The preceding proof can be applied to the Baer ring of linear endomorphisms of a vector space. Orthogonal projections should be replaced by projection operators.

As an example we apply the previous theory to the construction of harmonic differentials on a Riemann surface which "imitate" some singularity differential in the neighborhood of the ideal boundary (whence the name "imitation problem").

2. Vector space decompositions and the corresponding "imitation problems". Let $\bar{R}$ be any compact bordered Riemann surface. We consider the space $\mathfrak{H}(\bar{R})$ consisting of harmonic exact differentials on Int $(\bar{R})$, which are continuous on $\bar{R}$. Let γ be a cycle on $\bar{R}$, $[\gamma]$ the corresponding homology class. We introduce the space

$$H_{[\gamma]}(\bar{R}) = \left\{\omega \in \mathfrak{H}(\bar{R}) : \int_{\gamma} *\omega = 0\right\}$$

(see [1]).

Let now $\bar{W}$ be a compact bordered Riemann surface, $\bar{A}$ the complement of a regularly embedded domain Ω. We use the standard notation

$$\alpha = Bd\bar{\Omega}$$
$$\beta = Bd\bar{W}\ .$$

In the vector space $\mathfrak{H}(\bar{A})$ we consider the subspaces

$$H_{0\beta}(\bar{A}) = \{\omega \in \mathfrak{H}(\bar{A}),\ \omega = df,\ df|_{\beta} = 0\}$$
$$H_{0\alpha}(\bar{A}) = \{\omega \in \mathfrak{H}(\bar{A}),\ \omega = df,\ df|_{\alpha} = 0\}$$
$$H^{\star}_{0\beta}(\bar{A}) = \left\{\omega \in \mathfrak{H}(\bar{A});\ \omega = df,\ *df|_{\beta} = 0,\ \int_{\alpha_i} *df = 0, \text{ for each component } \alpha_i \text{ of } \alpha\right\}$$
$$H^{\star}_{0\alpha}(\bar{A}) = \left\{\omega \in \mathfrak{H}(\bar{A});\ \omega = df,\ *df|_{\alpha} = 0,\ \int_{\beta_i} *df = 0, \text{ for each component } \beta_i \text{ of } \beta\right\}$$
$$H'_{0\beta}(\bar{A}) = H_{0\beta}(\bar{A}) \cap H_{[\beta]}(\bar{A})$$
$$H'_{0\alpha}(\bar{A}) = H_{0\alpha}(\bar{A}) \cap H_{[\beta]}(\bar{A})\ .$$

Observe that:

$$H^{\star\prime}_{0\beta}(\bar{A}) = H^{\star}_{0\beta}(\bar{A}) \cap H_{[\beta]}(\bar{A}) = H^{\star}_{0\beta}(\bar{A})\ .$$

Another important subspace will be

$$H_{ext}(\bar{A}) = \{\omega \in \mathfrak{H}(\bar{A}) : \omega = \hat{\omega}|_{\bar{A}} \quad \text{where} \quad \hat{\omega} \in \mathfrak{H}(\bar{W})\}\ .$$

Clearly $H_{ext}(\bar{A}) \subset H_{[\beta]}(\bar{A})$.

Let now $\Gamma(\bar{A})$ be the space of square integrable harmonic differentials on $\bar{A}$. We denote

$$h_{[\beta]}(\bar{A}) = \text{closure in } \Gamma(\bar{A}) \text{ of } H_{[\beta]}(\bar{A})$$
$$h'_{0\gamma}(\bar{A}) = \text{closure in } \Gamma(\bar{A}) \text{ of } H'_{0\gamma}(\bar{A})$$
$$h^{\star}_{0\gamma}(\bar{A}) = \text{closure in } \Gamma(\bar{A}) \text{ of } H^{\star}_{0\gamma}(\bar{A})$$

where γ stands for α or β. We have the following vector space decompositions:

PROPOSITION.

$$h_{[\beta]}(\bar{A}) = h'_{0\alpha}(\bar{A}) \oplus h^{\star}_{0\beta}(\bar{A})$$
$$h_{[\beta]}(\bar{A}) = h^{\star}_{0\alpha}(\bar{A}) \oplus h'_{0\beta}(\bar{A}) .$$

Proof. We prove the first equality. The second is obtained by symmetry. First, we show

$$h_{[\beta]}(\bar{A}) = H'_{0\alpha}(\bar{A}) \oplus H^{\star}_{0\beta}(\bar{A}) .$$

Observe that $H'_{0\alpha}(\bar{A})$ is orthogonal to $h^{\star}_{0\beta}(\bar{A})$: let $df \in H'_{0\alpha}(\bar{A})$, $dg \in H^{\star}_{0\beta}(\bar{A})$. The innner product on the Hilbert space $\Gamma(\bar{A})$ induces an inner product on $H_{[\beta]}(\bar{A})$. So,

$$(df, dg)_{\bar{A}} = \int_{\beta-\alpha} f * \overline{dg} = \int_{\beta} f * \overline{dg} = 0 .$$

Let now dk be an element of $H_{[\beta]}(\bar{A})$. We want to find $df \in H'_{0\alpha}$ and $dg \in H^{\star}_{0\beta}(\bar{A})$ such that

$$dk = df + dg .$$

We must have $dg|_{\alpha} = dk|_{\alpha}$, $*dg|_{\beta} = 0$, $\int_{\alpha_i} *dg = 0$ for each component α_i of α. Also $df|_{\alpha} = 0$, $*df|_{\beta} = *dk|_{\beta}$ and $\int_{\alpha_i} *df = \int_{\alpha_i} *dh$ for each component α_i of α. Such a problem has a unique solution.

We now take closures in $\Gamma(\bar{A})$. Observe that $h'_{0\alpha}(\bar{A})$ and $h^{\star}_{0\beta}(\bar{A})$ are orthogonal since $H'_{0\alpha}(\bar{A})$ and $H^{\star}_{0\beta}(\bar{A})$ are dense and orthogonal. It follows that

$$h_{[\beta]}(\bar{A}) = h'_{0\alpha}(\bar{A}) + h^{\star}_{0\beta}(\bar{A}) .$$

We now consider some orthogonal projections in the space $h_{[\beta]}(\bar{A})$, which may be used as operators L of §1.

(1) Let Λ_0 be orthogonal projection on $h^{\star}_{0\beta}(\bar{A})$. We have

$$\ker \Lambda_0 = h'_{0\alpha}(\bar{A}) .$$

In particular $*\Lambda_0\, df \in h_{0\beta}(\bar{A})$ and hence $\Lambda_0\, df$ has "vanishing normal derivative" on β. Moreover $(I - \Lambda_0)\, df \in h'_{0\alpha}$. So $\Lambda_0\, df|_{\alpha} = df|_{\alpha}$ and Λ_0 has the property of Sario's "L_0 operator".

(2) Let Λ_1 be orthogonal projection on $h'_{0\beta}(\bar{A})$. We have

$$\ker \Lambda_1 = h_{0\alpha}(\bar{A}) \text{ .}$$

So $\Lambda_1 df|_\beta \in h'_{0\beta}$ and "$\Lambda_1 df$ vanishes on β" However $*(I - \Lambda_1)df|_\alpha \in h'_{0\alpha}$ hence

$$*\Lambda_1 df|_\alpha = *df|_\alpha$$

and Λ_1 differs from Sario's "L_1 operator" by its behavior on α. Some other vector space decompositions will be of interest:

PROPOSITION. $H_{[\beta]}(\bar{A}) = H_{ext}(\bar{A}) \oplus H'_{0\beta}(\bar{A})$.

Proof. Observe that $H_{ext}(\bar{A}) \cap H'_{0\beta}(A) = \{0\}$. This is a consequence of the fact that on $\bar{W}$, $\Gamma^*_{hse} \cap \Gamma_{he}$ is orthogonal to $\Gamma_{he} \cap \Gamma_{ho}$.

Now consider any $df \in H_{[\beta]}(\bar{A})$. Let $\widehat{df}$ be the unique harmonic exact differential on $\bar{W}$ which has same boundary values as df. Now:

$$df = \widehat{df}|_{\bar{A}} + (df - \widehat{df})|_{\bar{A}}$$

and

$$\widehat{df}|_{\bar{A}} \in H_{ext}(\bar{A}),\ (df - \widehat{df})|_{\bar{A}} \in H'_{0\beta} \text{ .}$$

which proves the validity of the direct sum decomposition.

We shall denote by K_1 the corresponding projection on $H_{ext}(\bar{A})$ and by L_1 the corresponding projection on $H'_{0\beta}(\bar{A})$.

PROPOSITION. $H_{[\beta]}(\bar{A}) = H_{ext}(\bar{A}) \oplus H^{\star}_{0\beta}(\bar{A})$.

Proof. $H_{ext}(\bar{A}) \cap H^{\star}_{0\beta}(\bar{A}) = \{0\}$. Thus assume $\omega = df \in H_{ext}(\bar{A})$ and $*df|\beta = 0$. By the uniqueness of the solution to the Neumann problem $df = 0$. Consider now any $df \in H_{[\beta]}(\bar{A})$. Let df be the harmonic exact differential on $\bar{W}$ such that $*(\widehat{df})|_\beta = *df|_\beta$ and $\int_{\alpha_i} *\widehat{df} = \int_{\alpha_i} *df$ for each component α_i of α. We can write

$$df = \widehat{df}|_{\bar{A}} + (df - (\widehat{df}))|_{\bar{A}}$$

where

$$\widehat{df}|_{\bar{A}} \in H_{ext}(\bar{A}) \text{ ,} \qquad (df - (\widehat{df})|_{\bar{A}} \in H^{\star}_{0\beta}(A) \text{ ,}$$

which proves the validity of the direct sum decomposition. We denote by K_0 the projection on $H_{ext}(\bar{A})$ and by L_0 the projection on $H^{\star}_{0\beta}(\bar{A})$.

Application. Solution to the "imitation problem" for harmonic differentials in $H_{[\beta]}(\bar{A})$. (cf. §1. note 2).

Assume that we have a decomposition:

$$H_{[\beta]}(\bar{A}) = H_{ext}(\bar{A}) \oplus \breve{H}(\bar{A}) . \qquad \text{(direct sum)} .$$

We denote by L the corresponding projection onto $\breve{H}(\bar{A})$ and by K the corresponding projection onto $H_{ext}(\bar{A})$.

The "imitation problem" consists in studying the solutions $\omega \in H_{ext}(\bar{A})$ of the equation:

$$(*) \qquad \omega - s = L(\omega - s) , \qquad s \in H_{[\beta]}(\bar{A}) .$$

One can apply the existence and uniqueness theorem of §1, or check directly.

Uniqueness of the solution: let ω_1, ω_2 be two solutions of $(*)$ then

$$\omega_1 - \omega_2 = L(\omega_1 - \omega_2)$$

or

$$(I - L)(\omega_1 - \omega_2) = 0$$
$$\text{i.e.} \quad \omega_1 - \omega_2 \in \operatorname{Ker}(I - L) = \operatorname{Im} L .$$

Now $\omega_1 - \omega_2 \in \operatorname{Im} K$ and $\operatorname{Im} K \cap \operatorname{Im} L = \{0\}$. It follows that $\omega_1 - \omega_2 = 0$ and the solution is unique.

Existence of the solution. To solve $(I - L)(\omega - s) = 0$ set $\omega = Ks$. We then get:

$$(I - L)(I - K)s = 0$$

which is verified for all $s \in H_{[\beta]}(\bar{A})$ since

$$\operatorname{Im}(I - K) = \operatorname{Ker} L ;$$

from the direct sum decomposition.

EXAMPLES.

(1) L_1 and K_1. The unique solution to

$$\omega - s = L_1(\omega - s)$$

is given by $\omega = K_1 s$. Such a ω has the same boundary behavior as s.

(2) L_0 and K_0. The unique solution to

$$\omega - s = L_0(\omega - s)$$

is given by $\omega = K_0 s$ and $*\omega$ and $*s$ have same boundary behavior.

3. $-L^2$-normal operators and the "imitation problem". We

now return to the L^2 theory and show a constructive method of solution. We consider the Hilbert space $\mathfrak{H}_1$ defined as the closure in the L^2-norm on $\bar{A}$ of the space of harmonic exact differential on $\bar{A}$. We are considering operators

$$L: \mathfrak{H}_1 \longrightarrow \mathfrak{H}_1$$

such that

(i) L is an orthogonal projection operator. (in particular $L^2 = L$ and $||L|| = 1$)

(ii) $\mathrm{Im}\,(I - L) \cap H_{ext}(\bar{A}) = \{0\}$.

Such operators will be called L^2-normal.

We consider in particular the operator

$$K: \mathfrak{H}_1 \longrightarrow \mathfrak{H}_1$$

where K denotes orthogonal projection onto the subspace $\mathfrak{K}$ of exact harmonic differentials in $\mathfrak{H}_1$ which admit a harmonic extension to all of W. The next generalized q-lemma shows that $\mathfrak{K}$ is closed.

GENERALIZED q-LEMMA. *There exist numbers $q(\bar{A})$ and $q'(\bar{A})$ lying between 0 and 1 such that for each $\omega \in \Gamma_{he}(\bar{W})$. $q'(\bar{A})\,||\omega||_{\bar{W}} \leqq ||\omega||_{\bar{A}} \leqq q(\bar{A})\,||\omega||_{\bar{W}}$.*

Proof. We know that $\Gamma_{he}(\bar{W})$ has the Montel property. Consider the subset $S \subset \Gamma_{he}(\bar{W})$ defined as

$$S = \{\omega \in \Gamma_{he}(\bar{W}) \colon ||\omega||_{\bar{W}} = 1\} .$$

We first want to show that then exists $q(\bar{A})$, $0 < q(\bar{A}) < 1$ such that

$$||\omega||_{\bar{A}} \leqq q(\bar{A})$$

for each $\omega \in S$.

If this is not the case, there is a sequence (ω_n) from S such that $||\omega_n||_{\bar{A}} \longrightarrow 1$.

By the Montel property, (ω_n) has a convergent subsequence (ω_{n_i}) and $\omega_{n_i} \longrightarrow \hat{\omega} \in S$. (since S is closed). Now $||\omega_{n_i}||_{\bar{A}} \longrightarrow 1$ and hence $||\hat{\omega}||_{\bar{A}} = 1$ and so $\mathrm{supp}\ \hat{\omega} \subseteqq \bar{A}$. But no element of $\Gamma_{he}(\bar{W})$ has support contained in $\bar{A}$ a proper subset of Int $\bar{W}$. ([3] p. 186).

Hence there exists $q(\bar{A})$, $0 < q(\bar{A}) < 1$ such that

$$||\omega||_{\bar{A}} \leqq q(\bar{A})\,||\omega||_{\bar{W}} .$$

To get the second inequality, consider $\bar{\Omega}$:

$$||\omega||_{\bar{\Omega}} \leqq q(\bar{\Omega})\,||\omega||_{\bar{W}}$$

hence

$$||\omega||_W - ||\omega||_{\bar{A}} \leqq q(\bar{\Omega})||\omega||_{\bar{W}}$$

or

$$(1 - q(\bar{\Omega}))||\omega||_{\bar{W}} \leqq ||\omega||_{\bar{A}}$$

and we have $q'(\bar{A}) = 1 - q(\bar{\Omega})$. Which proves the lemma.

NOTE. We have $1 - q(\bar{\Omega}) \leqq q(\bar{A})$. So $q(\bar{A}) + q(\bar{W} - \bar{A}) \geqq 1$.

COROLLARY. *$\mathfrak{K}$ is a closed subspace of $\mathfrak{H}_1$.*

Proof. We show $\mathfrak{K}$ contains all the limits of its Cauchy sequences. Let (ω_n) be Cauchy in $\mathfrak{K}$. Let $(\hat{\omega}_n)$ be the corresponding sequence in $\Gamma_{he}(\bar{W})$ (such that $\hat{\omega}_n|_{\bar{A}} = \omega_n$). Now $(\hat{\omega}_n) \to \hat{\omega} \in \Gamma_{he}(\bar{W})$ in the L^2 norm on $\Gamma_{he}(\bar{W})$. Since the L^2-norms on $\Gamma_{he}(\bar{W})$ and $\mathfrak{K}$ are equivalent. It follows that

$$(\omega_n) \longrightarrow \hat{\omega}|_A$$

in the L^2 norm on $\mathfrak{K}$ and hence $\mathfrak{K}$ is closed. We now prove:

THEOREM. *Let L be a L^2-normal operator on $\mathfrak{H}_1$. Then the equation $\omega - s = L(\omega - s)$ admits a solution $\omega \in \mathfrak{K}$. The solution is unique provided $\mathfrak{K} \cap \operatorname{Im} L = (0)$.*

Proof. Assume there exists $p \in \mathfrak{H}_1$ such that

$$(^+) \qquad -Kp - s = L(p - s)\ .$$

We then have

$$L(-Kp - s) = L^2(p - s) = L(p - s) = -Kp - s\ .$$

Setting $\omega = -Kp$ we obtain an element of $\mathfrak{K}$ such that

$$\omega - s = L(\omega - s)\ .$$

It then suffices to solve $(^+)$. We rewrite it as:

$$(^{++}) \qquad [I - (I - (K + L))]p = -(I - L)s\ .$$

The latter admits a solution $p \in \mathfrak{H}_1$ (which can be written as a Neumann series) if

$$||I - (K + L)|| < 1$$

or, what is the same, if the aperture

$$\theta(\operatorname{Im}(I - K), \operatorname{Im}(K)) < 1\ .$$

(For the definition and properties of the aperture see [2] p. 69.) Now

$$\theta(\text{Im}\,(I-L), \text{Im}(K)) = \max\{\text{dist}\,[S(\text{Im}\,(I-L)), \text{Im}\,K], \text{dist}\,[S(\text{Im}\,K), \text{Im}\,(I-L)]\}$$

(where $S(V)$ denotes the unit sphere in the subspace V).

Now the unit spheres in $\text{Im}\,(I-L)$, $\text{im}\,K$ are closed and bounded hence compact since $\mathfrak{H}_1$ has the Montel property.

Assume that the max is given by the first term; let $x \in S(\text{Im}\,(I-L))$. The projection of x on $\text{Im}\,K$ lies in the unit ball of $\text{Im}\,K$ which is compact. Hence we can consider in the computation of θ the distance from $S(\text{Im}\,(I-L))$ to the unit ball of $\text{Im}\,K$ and the distance is thus attained.

Let

$$df \in S(\text{Im}\,(I-L)), \qquad dg \in \text{Im}\,K$$

be corresponding points. One has

$$\theta = \frac{|(df, dg)|}{\|df\|\,\|dg\|} .$$

If now $\theta = 1$, then $|(df, dg)| = \|df\|\,\|dg\|$ and hence $df = \lambda dg$ where λ is a constant, and also $df = (I-L)dh$.

Now dg is extendable and $df \in \text{Im}\,(I-L)$. It follows that $df = 0$, a contradiction.

(A similar reasoning is valid in case the max in the definition of θ is given by the second term.)

It follows that $\theta < 1$ and $(^{++})$ has the solution.

$$\omega = -Kp = K \sum_{n=0}^{\infty} [I - (K+L)]^n (I-L)s .$$

NOTE. Instead of $\mathfrak{H}_1$ one could work in a closed subset of $\mathfrak{H}_1$ e.g. $h_{[\beta]}(\bar{A})$.

The uniqueness is discussed as before: we get uniqueness provided

$$\text{Im}\,K \cap \text{Im}\,L = \{0\} .$$

i.e. no differential in the image of L is extendable to $\bar{W}$.

If ω_1 and ω_2 are solutions, then

$$(1-L)(\omega_1 - \omega_2) = 0 .$$

Now

$$\omega_i = -Kp_i \qquad i = 1, 2 .$$

So

$$(I - L)K(p_1 - p_2) = 0 .$$

Hence if

$$\operatorname{Im} K \cap \operatorname{Im} L = \{0\} , \qquad p_1 = p_2 \quad \text{and} \quad \omega_1 = \omega_2 .$$

Conversely, if there is a differential $\tau \in \mathfrak{H}_1$ such that

$$\tau = L\mu = K\nu$$

then if ω is a solution in $\mathfrak{R}$ of

$$\omega - s = L(\omega - s)$$

we have

$$\omega + \tau - s = \tau + L(\omega - s) = L(\mu + \omega - s) = L(\tau + \omega - s)$$

and uniqueness is lost.

As examples we could take:

(i) $L = \Lambda_0$ orthogonal projection on $h^{*}_{0\beta}(\bar{A})$.

Then

$$\operatorname{Im} (I - L) = h'_{0\alpha}(\bar{A})$$

and

$$\operatorname{Im} (I - L) \cap H_{ext}(\bar{A}) = \{0\} \quad \text{and} \quad \operatorname{Im} L \cap H_{ext}(\bar{A}) = \{0\} ;$$

(ii) $L = \Lambda_1$ orthogonal projection on $h'_{0\beta}(\bar{A})$. Similar results are valid.

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