PACIFIC JOURNAL OF MATHEMATICS
Vol. 42, No. 1, 1972

A NOTE ON H-EQUIVALENCES

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If X is a space, with base point, the set of homotopy classes of based self-equivalent maps, from X to itself, forms a group, which has been studied by many authors. In this note, we study a related group, in the case where X is an H-space. The main result is that all such groups are finitely-presented. The methods combine results from algebraic topology with combinatorial group theory.

If X is an H-space with multiplication $\mu: X \times X \to X$, a self-map $f: X \to X$ is called an H-map if

$$\begin{array}{ccc} X \times X & \xrightarrow{\mu} & X \\ \big\downarrow{\scriptstyle f \times f} & & \big\downarrow{\scriptstyle f} \\ X \times X & \xrightarrow{\mu} & X \end{array}$$

is homotopy commutative. Such maps were first studied in [6], and later in [1]. Arkowitz and Curjel [1] showed that if X is a connected complex, which is an H-space, X has finite-dimensional, commutative, rational Pontrjagin algebra, and the total homotopy groups of X are finitely-generated, then the group of homotopy classes of self-maps, which are H-maps, is finitely-generated. We denote this group by $A(X)$, and remark that it is known to be frequently a complicated, non-Abelian group. Observe that this theorem of [1] suffices to handle the case when X is a finite, connected complex, which is an H-space. The purpose of this note is to show how this result can be strengthened. We shall prove

THEOREM. *If X satisfies the assumptions of the theorem of Arkowitz and Curjel, then $A(X)$ is finitely-presented* (see [3] for a definition).

The class of finitely-presented groups is countable, while it is known that there are uncountably many groups with 2 generators. (This result about uncountability, due to B. H. Neumann, may be found in [3]). Hence, our theorem narrows down the possibilities for $A(X)$ appreciably.

To prove this Theorem, we need several propositions.

PROPOSITION 1. *Let $N \subset G$ be a normal subgroup of the group G.*

Set $K = G/N$. If K and N are finitely presented, so is G.

Proof. See p. 130 in [2]. I believe that this is the first place where this proposition, which is not difficult, has appeared in the literature.

REMARK. On the contrary, if G and K are finitely-presented, N need not even be finitely-generated.

PROPOSITION 2. *Let $H \subset G$ be a subgroup of finite index. If G is finitely-presented, so is H.*

Proof. See p. 93 of [4].

As a converse of Proposition 2, we have the following proposition which we shall deduce briefly from Proposition 1.

PROPOSITION 3. *If $H \subset G$ is a finitely-presented subgroup of finite index, then G is finitely-presented.*

Proof. Let H_0 be the intersection of all conjugates of H in G. H_0 is a normal subgroup of finite-index, as there are only finitely-many conjugates. By Proposition 2, H_0 is finitely-presented. G/H_0 is finite, and hence, finitely-presented. The result follows immediately from Proposition 1.

PROPOSITION 4. *If $G_1, \cdots, G_k$ are finitely-presented, so is the group $\prod_{i=1}^{k} G_i$.*

Proof. For lack of a reference, we indicate the proof. As generators, we select the elements

$$\begin{array}{ll} (x_1, 1, \cdots, 1), (x_2, 1, \cdots, 1), \cdots, (x_k, 1, \cdots, 1) \\ (1, y_1, 1, \cdots, 1), \cdots \qquad\qquad , (1, y_l, 1, \cdots 1) \\ \cdots \end{array}$$

where the x_i generate G_1, the y_j generate G_2, etc. A defining set of relations is then given by the relations among the x_i, the relations among the y_j, etc. plus the commutativity relations

$$(x_i, 1, \cdots, 1)\cdot(1, y_j, 1, \cdots, 1) = (1, y_j, 1, \cdots, 1)\cdot(x_i, 1, \cdots, 1) \quad \text{etc.}$$

We now prove our Theorem.

(a) Let k be the maximal dimension for which $H_i(X, Q) \neq 0$. Let $F \subset \pi'_*(X) = \sum_{i=1}^{k} \oplus \pi_i(X)$ be the (graded) free subgroup. We shall denote, by $\mathrm{Aut}_1(G)$, the group of graded automorphism of the graded group G, reserving the symbol Aut for the usual group of automorphisms. According to [5.], if F_0 is a finitely-generated, free, Abelian group, Aut (F_0) is finitely-presented. It is clear that $\mathrm{Aut}_1(F)$ is a direct product of such groups, and hence by Proposition 4, it is finitely-presented. Because $\mathrm{Aut}_1(F) \subset \mathrm{Aut}_1(\pi'_*(X))$ is clearly a subgroup of finite index, we conclude from Proposition 3 that the group $\mathrm{Aut}_1(\pi'_*(X))$ is finitely-presented.

(b) It is shown in [1] that the natural map

$$P\colon A(X) \to \mathrm{Aut}_1(\pi'_*(x))$$

has finite kernel, and that the image of p (see p. 146 of [1]) is a subgroup of finite index. It is here that the assumptions on X are used.

By (a) above, and Proposition 2, we see that Im (p) is finitely-presented. ker (p) being trivially finitely-presented, our theorem follows immediately from Proposition 1.

In conclusion, we would like to make some remarks about the full group of homotopy equivalences, $G(x)$, for such a space X. Clearly, we have a similar homomorphism p_1 and Im (p_1) is of finite-index. However, ker p_1 is no longer finite. For consider the space

$$X = K(Z, 2n) \times K(Z, 4n) \quad n > 0$$

with the usual H-space structure. A self-map is determined up to homotopy by 2-cohomology classes, the classes $f^*(i_{2n})$ and $f^*(i_{4n})$, these being the images of the fundamental classes. We set, for any integer k,

$$f_k^*(i_{2n}) = i_{2n} \,.$$
$$f_k^*(i_{4n}) = i_{4n} + k(i_{2n} \cup i_{2n}) \,.$$

It is easy to check that such a map f_k induces the identity automorphism on homotopy groups, but that all the different f_k represent distinct homotopy classes. Hence, the kernel of p_1 is infinite. An easy cohomology calculation shows that when $k \neq 0, f_k$ is not an H-map. One also see quickly that $A(X)$ does not have finite index in $G(X)$ in this case.

Nevertheless, one can prove that $G(X)$ is finitely-presented, by considering the kernel of p_1. This will be studied in the forthcoming thesis of Mr. Daniel Sunday.

References

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Received February 5, 1971 and in revised form June 24, 1971. Partially supported by NSF contract GP-8691.

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