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FIXED POINT THEOREMS FOR POINT-TO-SET MAPPINGS AND THE SET OF FIXED POINTS

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Let X be a Banach space and K be a nonempty convex weakly compact subset of X. Belluce and Kirk proved that (1) If $f: K \to K$ is continuous, $\inf_{x \in K} \|x - f(x)\| = 0$ and $I - f$ is a convex mapping, then f has a fixed point in K. (2) If $f: K \to K$ is nonexpansive and $I - f$ is a convex mapping on K, then f has a fixed point in K. In this paper the concept of convex mapping has been extended to point-to-set mappings. Theorems 1 and 2 in § 2 extend the above fixed point theorems by Belluce and Kirk.

Let W stand for the set of fixed points of $f: K \to cc(K)$. The set W is called a singleton in a generalized sense if there is $x_0 \in W$ such that $W \subset f(x_0)$. In § 3 two examples are given to show that W is not necessarily a singleton in a generalized sense if f is strictly nonexpansive or if $I - f$ is convex. But one can be sure that W is a convex set if $I - f$ is a convex or a semiconvex mapping.

1. Preliminaries.

NOTATIONS AND DEFINITIONS. Let X be a topological space, define

1. 2^X = the family of all nonempty closed subsets of X.
2. $b(X) = \{A \in 2^X;\ A \text{ is bounded}\}$, where X is a metric space.
3. $k(X) = \{A \in 2^X;\ A \text{ is convex}\}$, where X is a linear topological space.
4. $cpt(X) = \{A \in 2^X;\ A \text{ is compact}\}$.
5. $cc(X) = k(X) \cap cpt(X)$, where X is a linear topological space.

In the remainder of this section we assume X to be a metric space with metric d, unless otherwise stated.

6. Let $x \in X$ and $r > 0$, define $S(x, r) = \{y \in X;\ d(y, x) < r\}$.
7. For $x \in X$, $A \in 2^X$, define $d(x, A) = \inf\{d(x, y);\ y \in A\}$.
8. Given $A \in 2^X$ and $r > 0$, define $V_r(A) = \{x \in X;\ d(x, A) < r\}$.

LEMMA 1. *Let $x, y \in X$ and let A be a nonempty subset of X. Then $d(x, A) \leqq d(x, y) + d(y, A)$.*

This is a simple consequence of the triangle inequality.

DEFINITION 1. Let X be a topological space. A mapping

$f: X \to 2^X$ is said to be upper semicontinuous (abbreviated by u.s.c.) at x_0 if for any open set U containing $f(x_0)$, there exists a neighborhood V of x_0 such that $f(y) \subset U$ for any $y \in V$. The mapping f is said to be u.s.c. in X if it is u.s.c. at any x in X.

DEFINITION 2. A map $f: X \to b(X)$ is continuous if it is continuous from the metric topolgy of X to the Hausdorff metric topology of $b(X)$.

DEFINITION 3. A mapping $f: X \to b(X)$ is nonexpansive on X if $D(f(x), f(y)) \leqq d(x, y)$ for any x, y in X, where D is the Hausdorff metric on $b(X)$.

DEFINITION 4. A mapping $f: X \to b(X)$ is a contraction mapping if there is $0 \leqq k < 1$, such that $D(f(x), f(y)) \leqq kd(x, y)$ for any $x, y \in X$.

It is clear that a nonexpansive mapping $f: X \to b(X)$ is continuous. For the relation between a continuous map and an upper semicontinuous map, we have the following:

PROPOSITION 1. *If $f: X \to cpt(X)$ is continuous, then it is upper semicontinuous.*

REMARK 1. The condition that the values of f are compact subsets is not removable in the above proposition. As a matter of fact a nonexpansive mapping f on X into 2^X may fail to be upper semicontinuous. Examples like the following seem to be in the folklore.

EXAMPLE 1. Let $X = [0, 1] \times [0, 1] - \{(0, 1)\}$ with the usual metric. Let $(x, y) \in X$, define

$$f((x, y)) = \begin{cases} \text{the segment } \{(x, z);\ z \in [0, 1]\} \text{ if } x \neq 0\,. \\ \text{the segment } \{(0, z);\ z \in [0, 1)\} \text{ if } x = 0\,. \end{cases}$$

Then $f: X \to 2^X$ is nonexpansive on X, but it is not u.s.c. at $(0, y)$ for any $y \in [0, 1)$. Because if we take

$$U = \{(x, y) \in X;\ x + y < 1\}\,,$$

then U is open and contains $f((0, y))$. However U does not contain $f((x, z))$ for $(x, z) \in X$ and $x \neq 0$. Therefore no neighborhood of $(0, y)$ exists such that U contains the image of f at every point of the neighborhood. That is, f is not u.s.c. at $(0, y)$.

DEFINITION 5. A real valued function g on X is said to be lower semicontinuous on X if for any real number a, the set

$$\{x \in X;\ g(x) > a\}$$

is open in X.

PROPOSITION 2. *If $f\colon X \to 2^X$ is upper semicontinuous, then the function g, where $g(x) = d(x, f(x))$, is lower semicontinuous.*

Proof. Let a be a real number and $x_0 \in A = \{x;\ g(x) > a\}$. We want to prove that A is an open set. Let $r = g(x_0) - a$, then $r > 0$ and the open set $V_{r/3}(f(x_0))$ contains $f(x_0)$. By the upper semicontinuity of f, there exists a neighborhood V of x_0 such that

$$f(y) \subset V_{r/3}(f(x_0))$$

for any $y \in V$. We may assume $V \subset S(x_0, r/3)$. Let $U = V_{r/3}(f(x_0))$. Then $z \in U$ implies

$$\begin{aligned} d(x_0, z) &\geqq d(x_0, f(x_0)) - d(z, f(x_0)) \quad \text{(by Lemma 1)} \\ &> r + a - r/3 = a + 2r/3\,. \end{aligned}$$

Therefore

$$d(x_0, U) = \inf\{d(x_0, z);\ z \in U\} \geqq a + 2r/3\,.$$

Thus $y \in V$ implies

$$\begin{aligned} d(y, f(y)) &\geqq d(y, U) \geqq d(x_0, U) - d(x_0, y) \quad \text{(by Lemma 1)} \\ &\geqq a + 2r/3 - r/3 = a + r/3 > a\,. \end{aligned}$$

Hence $y \in V$ implies $y \in A$. Thus A is open. Therefore g is lower semicontinuous.

2. Fixed point theorems.

First we state a well known fixed point theorem for a point-to-set contraction mapping (cf. [5] p. 479 for the proof): Let K be a nonempty bounded closed subset of a complete metric space (X, d). If $f\colon K \to b(K)$ is a contraction mapping, then f has a fixed point in K.

The space X in the sequel is assumed to be a Banach space unless otherwise stated.

DEFINITION 6. A mapping f from X into 2^X is said to be convex if for any $x, y \in X$ and $m = \lambda x + (1 - \lambda)y$ with $0 \leqq \lambda \leqq 1$, and any $x_1 \in f(x)$, $y_1 \in f(y)$, there exists $m_1 \in f(m)$ such that

$$\|m_1\| \leqq \lambda\|x_1\| + (1 - \lambda)\|y_1\|\,.$$

DEFINITION 7. A mapping $f: X \to 2^X$ is called semiconvex on X if for any $x, y \in X$, $m = \lambda x + (1-\lambda)y$, where $0 \leqq \lambda \leqq 1$, and any $x_1 \in f(x)$, $y_1 \in f(y)$, there exists $m_1 \in f(m)$ such that

$$\| m_1 \| \leqq \max \{\| x_1 \|, \| y_1 \|\} .$$

REMARK 2. A convex mapping is semiconvex, but the converse is not true. Take the mapping $f(x) = \sqrt{x}$, $x \in [0, 1]$, for instance. The map f is semiconvex because it is strictly increasing. But f is not convex, for example take $x = 1$ and $y = 0$,

$$m = 1/4 = 1/4 \cdot 1 + 3/4 \cdot 0 ,$$

then $f(1) = 1$, $f(0) = 0$, but

$$f(m) = \sqrt{1/4} = 1/2 \not\leqq 1/4 f(1) + 3/4 f(0) = 1/4 .$$

LEMMA 2. *Let $f: X \to 2^X$, and let $I: X \to X$ be the identity mapping. If $I - f$, where $(I - f)(x) = \{x - y;\ y \in f(x)\}$, is convex (semiconvex), then for any $x, y \in X$ and $m = \lambda x + (1-\lambda)y$, $0 \leqq \lambda \leqq 1$, we have*

$$d(m, f(m)) \leqq \lambda d(x, f(x)) + (1-\lambda) d(y, f(y)) .$$
$$(d(m, f(m)) \leqq \max \{d(x, f(x)), d(y, f(y))\}) .$$

Proof. Let $x_n \in f(x)$ be such that $\| x_n - x \| \to d(x, f(x))$ and $y_n \in f(y)$ be such that $\| y_n - y \| \to d(y, f(y))$. Let $I - f$ be a convex mapping, then there exists $m_n \in f(m)$ such that

$$\| m - m_n \| \leqq \lambda \| x - x_n \| + (1-\lambda) \| y - y_n \| .$$

Now

$$d(m, f(m)) \leqq \inf_{n \geqq 1} \| m - m_n \| \leqq \lambda \| x - x_n \| + (1-\lambda) \| y - y_n \|$$

for any $n \geqq 1$. Thus

$$d(m, f(m)) \leqq \lambda \| x - x_n \| + (1-\lambda) \| y - y_n \| \longrightarrow \lambda d(x, f(x)) + (1-\lambda) d(y, f(y)) .$$

Similarly one can prove that

$$d(m, f(m)) \leqq \max \{d(x, f(x)), d(y, f(y))\} ,$$

if $I - f$ is semiconvex.

LEMMA 3. *Let $f: X \to cpt(X)$ be a mapping such that for any $x, y \in X$ and any $m = \lambda x + (1-\lambda)y$, $0 \leqq \lambda \leqq 1$, we have*

$$d(m, f(m)) \leqq \lambda d(x, f(x)) + (1 - \lambda)d(y, f(y))$$
$$(d(m, f(m)) \leqq \max \{d(x, f(x)), d(y, f(y))\} \textit{ respectively}) .$$

Then $I - f$ *is a convex mapping (semiconvex mapping respectively).*

Proof. Let $x_1 \in f(x)$, $y_1 \in f(y)$; we have

$$d(x, f(x)) \leqq \| x - x_1 \| \quad \text{and} \quad d(y, f(y)) \leqq \| y - y_1 \| .$$

Since $f(m)$ is compact, there is an $m_1 \in f(m)$ such that

$$\| m - m_1 \| = d(m, f(m)) \leqq \lambda d(x, f(x)) + (1 - \lambda)d(y, f(y)) .$$

Therefore $\| m - m_1 \| \leqq \lambda \| x - x_1 \| + (1 - \lambda) \| y - y_1 \|$. Hence $I - f$ is a convex mapping. Similarly one can prove, under the condition that $d(m, f(m)) \leqq \max \{d(x, f(x)), d(y, f(y))\}$, that $I - f$ is a semiconvex mapping.

Lemmas 2 and 3 characterize the convexity (semiconvexity) of $I - f$ in terms of the distance between a point and its image under f, where f is a mapping from X into $cpt(X)$. The following lemma is a simple consequence of Lemma 2.

LEMMA 4. *Let* $f: X \rightarrow 2^X$, *define*

$$H_r = \{x \in X: d(x, f(x)) \leqq r\} ,$$

where $r \geqq 0$. *If* $I - f$ *is a semiconvex mapping on* X, *then* H_r *is convex.*

THEOREM 1. *Let* K *be a nonempty weakly compact closed convex subset of* X. *If* $f: K \rightarrow 2^K$ *is upper semicontinuous and*

$$\inf \{d(x, f(x)); x \in K\} = 0 ,$$

and $I - f$ *is a semiconvex mapping on* K, *then* f *has a fixed point in* K.

Proof. Let $r > 0$, define H_r as in Lemma 4. We see that $H_r \neq \emptyset$ for any $r > 0$, since $\inf \{d(x, f(x)); x \in K\} = 0$. As f is upper semicontinuous, H_r is closed (by Proposition 2). The map $I - f$ is semiconvex, hence H_r is convex (by Lemma 4). The set H_r, being closed and convex, is weakly closed for each $r > 0$. The family $\{H_r; r>0\}$ has the finite intersection property. Therefore, by the weak compactness of K, we have $\bigcap_{r>0} H_r \neq \emptyset$. It is clear that any point in $\bigcap_{r>0} H_r$ is a fixed point of f.

REMARK 3. A convex mapping is semiconvex, therefore Theorem 1 extends Theorem 4.1 of Belluce and Kirk [1]. Example 4.1 and 4.2 in [1], though they are point-to-point mappings, serve the purposes of demonstrating that "$\inf\{d(x, f(x)); x \in K\} = 0$" or "$K$ is weakly compact" in Theorem 1 is indispensable. The following example, which is a special case of the example given by Kirk [4], shows that the semiconvexity of $I - f$ in Theorem 1 can not be removed.

EXAMPLE 2. Let $K = \{x \in l_2; \|x\| \leqq 1\}$ be the closed unit sphere of the Hilbert space l_2. Then K is closed, convex and weakly compact. Define f on K as follows: Let $x = (x_1, x_2, \cdots) \in K$, and let

$$f(x) = (1 - \|x\|, x_1, x_2, \cdots) .$$

Then $\|f(x)\| \leqq 1$ and $\|f(x) - f(y)\| \leqq \sqrt{2}\,\|x - y\|$. i.e., f is a continuous mapping on K into K. We claim that

$$\inf\{\|x - f(x)\|; x \in K\} = 0 .$$

Let $x^{(n)} = (x_1, x_2, \cdots) \in l_2$ be such that $x_1 = x_2 = \cdots = x_{n^2} = 1/n$ and $x_i = 0$ for $i > n^2$. Then $\|x^{(n)}\| = 1$ and

$$f(x^{(n)}) = (0, x_1, x_2, \cdots, x_{n^2}, 0, \cdots) .$$

We see that

$$\|x^{(n)} - f(x^{(n)})\| = \sqrt{2}/n \to 0 , \quad \text{as} \quad n \longrightarrow \infty .$$

Hence $\inf\{\|x - f(x)\|; x \in K\} = 0$. But $I - f$ is neither convex nor semiconvex. For instance, let $x = (1/2, 1/2, 0, \cdots)$, $y = (-1/2, -1/2, 0, \cdots)$. Then $f(x) = (1 - \sqrt{2}/2, 1/2, 1/2, 0, \cdots)$, $f(y) = (1 - \sqrt{2}/2, -1/2, -1/2, 0, \cdots)$, $\|x - f(x)\| = (\sqrt{4 - 2\sqrt{2}})/2 < 1$, $\|y - f(y)\| = (\sqrt{12 - 6\sqrt{2}}/2 < 1$. Take $m = 1/2(x + y)$, then $m = (0, 0, \cdots)$ and $f(m) = (1, 0, \cdots)$. Thus

$$\|m - f(m)\| = 1 > \max\{\|x - f\{x)\|, \|y - f(y)\|\} .$$

Therefore $I - f$ is not semiconvex and hence it is not convex. The map f has no fixed point, for if $f(x) = x$, where $x = (x_1, x_2, \cdots) \in K$, then $x_1 = x_2 = \cdots$, and $\sum_{i=1}^{\infty} x_i^2 < \infty$. Thus $x_i = 0$ for $i \geqq 1$. But then $f(x) = (1, 0, \cdots) \neq (0, 0, \cdots)$.

DEFINITION 8. A map $f: X \to 2^X$ is said to be asymptotically regular at x_0 if there exists a sequence of points such that $x_n \in f(x_{n-1})$ and $\|x_n - x_{n-1}\| \to 0$ as $n \to \infty$.

Definition 8 is an extension of the definition of asymtotically regular point-to-point mapping given by Browder and Petryshyn [2]. One immediate result of Theorem 1 is the following corollary which extends the first part of Theorem 4.3 by Belluce and Kirk [1].

COROLLARY 1. *If $f: K \to 2^K$ is asymptotically rgular at some point in K, where K is a nonempty closed convex weakly compact subset of X, and if f is upper semicontinuous in K such that $I - f$ is semiconvex, then f has a fixed point in K.*

Proof. Assume f is asymptotically regular at $x_0 \in K$; then there exists $x_n \in K$ such that $x_n \in f(x_{n-1})$, $n \geqq 1$, and $\| x_n - x_{n-1} \| \to 0$. Since $d(x_n, f(x_n)) \leqq \| x_{n+1} - x_n \| \to 0$, we have $\inf\{d(x, f(x)); x \in K\} = 0$; hence Corollary 1 follows Theorem 1.

THEOREM 2. *Let K be a nonempty weakly compact convex subset of X. If $f: K \to cc(K)$ is nonexpansive and if $I - f$ is semiconvex on K, then f has a fixed point in K.*

Proof. The map f is nonexpansive, so it is upper semi-continuous (by Proposition 1). Theorem 2 follows Theorem 1 provided that the condition "$\inf\{d(x, f(x)); x \in K\} = 0$" is satisfied. To prove this condition we have the following lemma.

LEMMA 5. *Let K be a nonempty bounded closed convex subset of X. If $f: K \to b(K)$ is nonexpansive, then* $\inf\{d(x, f(x)); x \in K\} = 0$.

Proof. Let $x_0 \in K$. Denote $K_0 = \{x - x_0; x \in K\}$, then K_0 is a bounded closed convex subset of X and K_0 contains 0. Let $0 \leqq k < 1$, define f_k on K_0 as follows:

$$f_k(x - x_0) = k(f(x) - x_0) .$$

Then $f_k(x - x_0) \subset K_0$ for any $x - x_0 \in K_0$, since K_0 is convex and contains zero element. As f is nonexpansive, f_k is contraction. By the fixed point theorem for point-to-set contraction mapping, there exists $x_k \in K$ such that

$$x_k - x_0 \in f_k(x_k - x_0) = k(f(x_k) - x_0) .$$

Thus there is $y_k \in f(x_k)$ such that $x_k - x_0 = k(y_k - x_0)$. Now

$$\begin{aligned} d(x_k, f(x_k)) &= \inf\{\| x_k - y \|; y \in f(x_k)\} \leqq \| x_k - y_k \|| \\ &= \| x_0 + k(y_k - x) - y_k \| = (1 - k) \| y_k - x_0 \| . \end{aligned}$$

Therefore

$$0 \leqq \inf_{x \in K} d(x, f(x)) \leqq \inf_{0 \leqq k < 1} d(x_k, f(x_k))$$
$$\leqq \inf_{0 \leqq k < 1} (1 - k) \| x_0 - y_k \| = 0 ,$$

since the set $\{\| x_0 - y_k \|; 0 \leqq k < 1\}$ is bounded. Hence

$$\inf \{d(x, f(x)); x \in K\} = 0 .$$

3. The set of fixed points of a point-to-set mapping. Let K be a closed convex subset of a Banach space X. Denote by W the set of fixed points of a mapping $f: K \rightarrow 2^K$. Throught this section we assume W to be nonempty.

DEFINITION 9. A mapping $f: X \rightarrow b(X)$ is strictly nonexpansive if $D(f(x), f(y)) < \| x - y \|$ for any $x, y \in X$ and $x \neq y$.

If f is a point-to-point mapping, then the following properties are true.

(A) If f is strictly nonexpansive, then W is a singleton.

(B) If f is nonexpansive and the norm of the Banach space is strictly convex, then W is convex.

Statement (A) is no longer true for point-to-set mapping. For example, let K be a set containing more than two points, then the set of fixed points of the mapping $f: K \rightarrow 2^K$, such that $f(x) = K$ for any $x \in K$, is K itself which is not a singleton.

Statement (B) is obviously not true for a point-to-set mapping. However, as the next example shows, statement (B) is also not true for point-to-set mappings such that the image of each point is a nonempty compact convex set; note that the domain K in our example is also convex.

EXAMPLE 3. Let $K = [0, 1] \times [0, 1]$ with the usual norm. Define $f: K \rightarrow cc(K)$ by

$$f((x_1, x_2)) = \text{the triangle with vertices } (0, 0),\ (x_1, 0) \text{ and } (0, x_2) .$$

Note that $f((x_1, x_2))$ is a degenerate triangle if $x_1 x_2 = 0$. We see that f is nonexpansive and the norm in R^2 is strictly convex. But the set W of fixed points of f is

$$W = \{(x_1, x_2);\ (x_1, x_2) \in K \text{ and } x_1 x_2 = 0\}$$

which is not convex.

For a point-to-set mapping f, we have several choices for values of f, e.g., $f(x) \in k(X)$, $f(x) \in cpt(X)$ or $f(x) \in cc(X)$; among them, $f(x) \in cc(X)$ is the strongest assumption. For example, let K be a compact convex subset of X, and let $g: X \rightarrow cpt(X)$ be an upper semi-continuous mapping such that $g(x) \subset K$ for any $x \in K$, then g does not always have a fixed point (e.g., the map G of Strother [6], p. 990). But if we simply change g as a mapping into $cc(X)$ instead of into $cpt(X)$, then g has a fixed point (see K. Fan [3]). In Example 3, although we have imposed the strongest condition on the values of f, i.e., $f(x) \in cc(K)$, that condition does not force f to satisfy statement (B). However the following proposition shows us a sufficient condition for W to be convex.

PROPOSITION 3. *Let $f: K \rightarrow 2^K$ be a mapping such that $I-f$ is a semiconvex mapping on K. Then W is convex.*

Proof. If $I - f$ is semiconvex on K, then Lemma 4 shows that the set $H_r = \{x \in K; d(x, f(x)) \leqq r\}$ is convex. Hence $W = H_0$ is convex.

Statement (A) can be rephrased as follows:

(A′) If f is strictly nonexpansive, then there is x_0 in W such that $W \subset f(x_0)$.

For a point-to-point mapping f, statement (A') implicitly shows W to be a singleton. As for a point-to-set mapping f, statement (A') does not require W to be a singleton, and on the other hand it does not rule out the possibility that W is a singleton. Therefore, it is reasonable to define W to be a singleton in a generalized sense if there exists $x_0 \in W$ such that $W \subset f(x_0)$. Unfortunately even for a strictly nonexpansive mapping f on K into $cc(K)$, the set W of fixed points of f is not necessarily a singleton in a generalized sense.

EXAMPLE 4. Let $K = [0, 1] \times [0, 1]$, a subset of R^2 with the usual metric. Define f; $K \rightarrow cc(K)$ as follows:

$$f((x_1, x_2)) = \text{the triangle with vertices } (x_1/2, 0),\ (x_1/2, 1) \text{ and } (1, 0)\ .$$

Let $x = (x_1, x_2)$, $y = (y_1, y_2) \in K$, with $x \neq y$, then

$$D(f(x), f(y)) = 1/2\,|\,x_1 - y_1\,| < d(x, y)\ .$$

Hence f is strictly nonexpansive. The set W of fixed points of f is the set bounded by positive x, y axes and a branch of hyperbola $2x + 2y - xy - 2 = 0$. i.e.,

$$W = \{(x, y) \in K;\ 2x + 2y - xy - 2 \leqq 0\} .$$

By an inspection of the shape of the set W, one sees that $W \not\subset f((x, y))$ for any $(x, y) \in K$. Hence W is not a singleton in a generalized sense.

The question arises: Is W a singleton in a generalized sense if f is nonexpansive and $I - f$ is convex? The answer is no. Let us consider the following example.

EXAMPLE 5. Let $K = [0, 1] \times [0, 1]$ with the usual metric. Let $(x, y) \in K$, define

$$f((x, y)) = \text{the segment } \{(t, y);\ 0 \leqq t \leqq x/2\} .$$

Then $f: K \to cc(K)$ is nonexpansive. $I - f$ is a convex mapping. To show it, let $P = (x_1, y_1)$, $Q = (x_2, y_2)$ both in K, and let

$$M = \lambda P + (1 - \lambda)Q ,$$

for some $0 \leqq \lambda \leqq 1$. Then

$$\begin{aligned} d(P, f(P)) &= x_1/2 , \\ d(Q, f(Q)) &= x_2/2 , \\ d(M, f(M)) &= 1/2(\lambda x_1 + (1 - \lambda)x_2) \\ &= \lambda d(P, f(P)) + (1 - \lambda)d(Q, f(Q)) . \end{aligned}$$

By Lemma 3, we see that $I - f$ is convex on K. Now the set of fixed points of f is $W = \{(0, y);\ 0 \leqq y \leqq 1\}$. But $W \not\subset f((x, y))$ for any $(x, y) \in K$. Hence W is not a singleton in the generalized sense.

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REFERENCES

1. L. P. Belluce and W. A. Kirk, *Some fixed point theorems in metric and Banach spaces*, Canad. Math. Bull., **12** (1969), 481–491.

2. F. E. Browder and W. V. Petryshyn, *The solution by iteration of nonlinear functional equations in Banach spaces*, Bull. Amer. Math. Soc., **72** (1966), 571–575.

3. K. Fan, *Fixed points and minimax theorem in locally convex topological spaces*, Nat. Acad. of Sci. Proc., **38** (1952), 121-126.
4. W. A. Kirk, *A fixed point theorem for mappings which do not increase distances*, Amer. Math. Monthly, **72** (1965), 1004-1006.
5. S. B. Nadler, Jr, *Multivalued contractive mappings*, Pacific J. Math. **30** (1969), 475-488.
6. W. L. Strother, *On an open question concerning fixed points* Proc. Amer. Math. Soc., **4** (1953), 988-993.

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