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ON SPACES OF DISTRIBUTIONS STRONGLY REGULAR WITH RESPECT TO PARTIAL DIFFERENTIAL OPERATORS

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A distribution T in Ω is said to be strongly regular with respect to the differential operator $P(D)$, if $P^k(D)T$, $k = 0, 1, \cdots$, are of bounded order in any open set $\Omega' \subset\subset \Omega$. Necessary and sufficient conditions on the polynomials P and Q are established in order that a distribution T strongly regular with respect to $P(D)$ be strongly regular with respect to $Q(D)$.

Let $P(D)$ be a partial differential operator in R^n with constant coefficients and $P^k(D)$, $k = 1, 2, \cdots$, its successive iterations. The following result is due to L. Hörmander ([3], Theorem 3.6 and Remark on p. 233):

If $P(D)$ is hypoelliptic and T is a distribution such that $P^k(D)T$, $k = 1, 2, \cdots$, have a bounded order in any relatively compact open subset of R^n, then T is a C^∞-function.

In other words, the space $\mathscr{E}_P$ of distributions in R^n "strongly regular with respect to $P(D)$" is contained in the space $\mathscr{E}$ of C^∞-functions; in this case $\mathscr{E}_P = \mathscr{E}$. The concept of strong regularity with respect to $P(D)$ coincides with that of strong regularity in some variables (see [6], p. 453), when $P(D)$ is the Laplace operator in those variables.

Suppose now that given are two arbitrary partial differential operators $P(D)$ and $Q(D)$. Then the question arises: Under what conditions on P and Q is $\mathscr{E}_P \subset \mathscr{E}_Q$? In particular, if $P(D)$ is "Q-hypoelliptic," i.e. all solutions $U \in \mathscr{D}'$ of the equation

$$P(D)U = 0$$

are in $\mathscr{E}_Q$, must then be $\mathscr{E}_P \subset \mathscr{E}_Q$? The Q-hypoelliptic operators were studied (in a slightly different but equivalent version) and characterized by E. A. Gorin and V. V. Grušin [2].

In this paper we give necessary and sufficient conditions for the inclusion $\mathscr{E}_P(\Omega) \subset \mathscr{E}_Q(\Omega)$, where $\mathscr{E}_P(\Omega)$ and $\mathscr{E}_Q(\Omega)$ are the spaces of "strongly regular" distributions on an arbitrary open set $\Omega \subset R^n$. These conditions are, in general, stronger than the Q-hypeollipticity of $P(D)$. If the inclusion in question holds for every Q-hypoelliptic operator $P(D)$, then $Q(D)$ must be hypoelliptic and the problem reduces to that in Hörmander's theorem stated above.

1. The spaces $\mathscr{E}_P(\Omega)$ and $C_P^{m,\infty}(\Omega)$.

Let Ω be a nonempty open subset of R^n. A distribution $T \in \mathscr{D}'(\Omega)$ will be called strongly regular with respect to the differential operator $P(D)$, if to every open set Ω' having compact closure contained in Ω (we express this by writing $\Omega' \subset\subset \Omega$) there exists an integer $m \geqq 0$ such that $P^k(D)T$, $k = 0, 1, \cdots$, are all of order $\leqq m$ in Ω', i.e. the restrictions of $P^k(D)T$ to Ω' are all in $\mathscr{D}'^m(\Omega')$[1]. We denote by $\mathscr{E}_P(\Omega)$ the space of all distributions in Ω, which are strongly regular with respect to $P(D)$. We also denote by $C_P^{\mu,\infty}(\Omega)$, where μ is an integer $\geqq 0$, the space of all C^μ-functions in Ω such that $P^k(D)D^\alpha f$, $|\alpha| \leqq \mu$, $k = 0, 1, \cdots$, are continuous functions; here $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Consider now the spaces $\mathscr{E}_P(\Omega)$ and $\mathscr{E}_Q(\Omega)$ corresponding to the differential operators $P(D)$ and $Q(D)$ respectively.

THEOREM 1. *If $\mathscr{E}_P(\Omega) \subset \mathscr{E}_Q(\Omega)$, then to any open set $\Omega' \subset\subset \Omega$ there exists an integer $\mu \geqq 0$ such that the restriction mapping $f \to f \mid \Omega'$ maps $C_P^{\mu,\infty}(\Omega)$ into $C_Q^{0,\infty}(\Omega')$.*

Proof. Let Ω' be an open set satisfying the assumption $\Omega' \subset\subset \Omega$. We first prove the existence of nonnegative integers ν and m such that

$$\{Q^k(D)f \mid \Omega' : f \in C_P^{\nu,\infty}(\Omega),\ k = 0, 1, \cdots\} \subset \mathscr{D}'^m(\Omega') . \tag{1}$$

Suppose that inclusion (1) does not hold for any ν and m. Then to every ν and m there exist a function $f \in C_P^{\nu,\infty}(\Omega)$ and a k such that $Q^k(D)f \mid \Omega' \notin \mathscr{D}'^m(\Omega')$. Thus we can find strictly increasing sequences of positive integers ν_i, m_i and k_i, and a sequence of functions f_i with the following properties:

$$f_i \in C_P^{\nu_i,\infty}(\Omega) , \tag{2}$$

$$Q^k(D)f_i \mid \Omega' \in \mathscr{D}'^{m_i}(\Omega'),\ k = 0, 1, \cdots, \tag{3}$$

$$Q^{k_i}(D)f_i \mid \Omega' \text{ is of order } m_i, \tag{4}$$

$$qk_i < \nu_{i+1} , \tag{5}$$

where $i = 1, 2, \cdots$, and q is the order of the operator $Q(D)$.

We denote by Ω_i, $i = 1, 2, \cdots$, open subsets of Ω such that

$$\Omega_i \subset\subset \Omega_{i+1} \text{ and } \bigcup_{i=1}^{\infty} \Omega_i = \Omega . \tag{6}$$

Next we set

$$a_1 = 1 \text{ and } a_i = 2^{-i}M_i^{-1} , \qquad i = 2, 3, \cdots,$$

[1] $P^0(D)$ is the identity operator, i.e. $P^0(D)T = P$.

where

$$M_i = \sup \{ | P^k(D) f_i(x) | + | Q^l(D) f_i(x) | + 1 \}$$

and the supremum is taken over all $x \in \Omega_i$ and $k, l = 0, 1, \cdots, k_{i-1}$. Note that $Q^l(D) f_i$, $l = 0, 1, \cdots, k_{i-1}$, are continuous functions in Ω, because of (5).

The function

$$f = \sum_{i=1}^{\infty} a_i f_i$$

is defined and continuous in Ω, since the f_i's are continuous in Ω and the series converges there almost uniformly. Moreover, for any k we have (distributionally)

$$P^k(D) f = \sum_{i=1}^{\infty} a_i P^k(D) f_i \; . \tag{7}$$

But each term of the last series is a continuous function in Ω, by (1). Also

$$a_i \sup_{x \in \Omega_j} | P^k(D) f_i(x) | \leqq 2^{-i}$$

whenever $k < i$ and $j \leqq i$, by the definition of a_i. Hence it follows that the series (7) converges almost uniformly in Ω, for any k. Consequently $f \in C_P^{0,\infty}(\Omega) \subset \mathscr{E}_P(\Omega)$.

We now show that f is not in $\mathscr{E}_Q(\Omega)$, which is a contradiction to our hypothesis. We write

$$g_j = \sum_{i=1}^{j} a_i f_i \text{ and } h_j = \sum_{i=j+1}^{\infty} a_i f_i \; .$$

In view of (3) and (4), the restriction of $Q^{k_j}(D) g_j$ to Ω' is a distribution of order m_j. On the other hand, $Q^{k_j}(D) f_i$, $i = j+1, j+2, \cdots$, are continuous functions in Ω, because of (2) and (5). Furthermore, by the definition of the a_i's, the series

$$\sum_{i=j+1}^{\infty} a_i Q^{k_j}(D) f_i$$

converges almost uniformly in Ω, and so $Q^{k_j}(D) h_j$ is in Ω a continuous function. Thus

$$Q^{k_j}(D) f = Q^{k_j}(D) g_j + Q^{k_j}(D) h_j$$

is in Ω' a distribution of order m_j. Since $m_j \to \infty$, f is not in $\mathscr{E}_Q(\Omega)$. This contradiction proves (1).

Consider now the fundamental solution E of the iterated Laplace equation, i.e.

$$\Delta^{\gamma} E = \delta .$$

For sufficiently large γ, E is m times continuously differentiable. Therefore every distribution T on Ω' such that $\Delta^{\gamma} T \in \mathscr{D}'^{m}(\Omega')$ is, in fact, a continuous function (see [5], vol. 2, p. 47). We choose $\mu = 2\gamma + \nu$, where ν is the integer occurring in (1). Then, if $f \in C_P^{\mu,\infty}(\Omega)$, it follows that $\Delta^{\gamma} f \in C_P^{\nu,\infty}(\Omega)$ whence, in view of (1), $Q^k(D)\Delta^{\gamma} f \mid \Omega' = \Delta^{\gamma} Q^k(D) f \mid \Omega' \in \mathscr{D}'^{m}(\Omega')$. Thus, by what we said before, $Q^k(D) f \mid \Omega'$ is a continuous function, for every $k = 0, 1, \cdots$, i.e. $f \mid Q' \in C_Q^{0,\infty}(\Omega')$. The proof is complete.

2. Necessary conditions. We proceed to derive necessary conditions for the inclusion $\mathscr{E}_P(\Omega) \subset \mathscr{E}_Q(\Omega)$. In view of Theorem 1 it suffices to find necessary conditions for the inclusion

$$\{f \mid \Omega' : f \in C_P^{\mu,\infty}(\Omega)\} \subset C_Q^{0,\infty}(\Omega') . \tag{8}$$

We accomplish this by means of the standard argument based on the closed graph theorem and the Seidenberg-Tarski theorem (see [1]).

Let Ω_j, $j = 1, 2, \cdots$, be open sets satisfying conditions (6). We define the topology in $C_P^{\mu,\infty}(\Omega)$ by means of the semi-norms

$$v_j(f) = \sup |P^k(D) D^{\alpha} f(x)| ,$$

where the supremum is taken over all $x \in \Omega_j$, $|\alpha| \leqq \mu$ and $k \leqq j$. Similarly, if Ω'_j, $j = 1, 2, \cdots$, are open sets satisfying conditions analogous to (6) with Ω replaced by Ω', we define the topology in $C_Q^{0,\infty}(\Omega')$ by means of the semi-norms

$$w_j(f) = \sup_{x \in \Omega'_j, k \leqq j} |Q^k(D) f(x)| .$$

Then $C_P^{\mu,\infty}(\Omega)$ and $C_Q^{0,\infty}(\Omega')$ become Fréchet spaces. Moreover, the restriction mapping $C_P^{\mu,\infty}(\Omega) \to C_Q^{0,\infty}(\Omega')$ is closed and therefore continuous, by the closed graph theorem. Hence, to every integer $l > 0$, there exists an integer $k > 0$ and a constant $C > 0$ such that

$$w_l(f) \leqq C \max_{1 \leqq j \leqq k} v_j(f) , \tag{9}$$

for every $f \in C_P^{\mu,\infty}(\Omega)$. Applying condition (9) to the function

$$f(x) = e^{i\langle x, \zeta \rangle} ,$$

where $\zeta = \xi + i\eta$ and $\xi, \eta \in R^n$, we obtain the following lemma[2].

LEMMA 1. *If the inclusion* (8) *holds then, for every integer* $l > 0$, *we can find an integer* $k > 0$ *and constants* $C, c > 0$ *such that*

[2] We assume that $D^{\alpha} = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n}$, where $D_j = -i(\partial/\partial x_j)$.

$$|Q^l(\zeta)| \leqq C(1 + |\xi|^\mu)(1 + |P^k(\zeta)|)e^{c|\eta|} . \tag{10}$$

We denote by $N(P, a)$, V_a and W_a the sets of all $\zeta = \xi + i\eta \in C^n$ such that $|P(\zeta) \leqq a$, $|\eta| \leqq a$ and $|\xi| \leqq a$, respectively.

Lemma 2. *If condition* (10) *is satisfied, then* $Q(\zeta)$ *is bounded on every set* $N(P, a) \cap V_b$, $a, b \geqq 0$.

Proof. Suppose there are $a, b \geqq 0$ such that $Q(\zeta)$ is not bounded on $N(P, a) \cap V_b$. Then the function

$$s(t) = \sup_{\zeta \in N(P,a) \cap V_b \cap W_t} |Q(\zeta)|$$

is defined and continuous for sufficiently large t, and

$$s(t) \longrightarrow \infty \text{ as } t \longrightarrow \infty . \tag{11}$$

But, for a given t, $s(t)$ is the largest of all s such that the equations and inequalities

$$\begin{gathered} |P(\xi + i\eta)|^2 \leqq a^2, \ |\eta|^2 \leqq b^2 , \\ |Q(\xi + i\eta)|^2 = s^2, \ |\xi|^2 \leqq t^2, \ s \geqq 0, \ t \geqq 0 , \end{gathered} \tag{12}$$

have a solution $\xi, \eta \in R^n$. Applying to (12) the Seidenberg-Tarski theorem and next a well-known argument (see [4], p. 276, or [6], p. 317) one shows easily that, for sufficiently large t, $s(t)$ is an algebraic function. We now expand $s(t)$ in a Puiseux series in a neighborhood of infinity and make use of (11). It follows that

$$s(t) > t^h$$

for some $h > 0$ and all t sufficiently large. On the other hand, $s(t)$ is assumed for some $\xi = \xi(t)$, $\eta = \eta(t)$, and

$$|\xi(t)| \leqq t .$$

Choosing in (10) $l > \mu h^{-1}$ we obtain a contradiction, which proves the lemma.

Theorem 2. *If* $\mathscr{E}_P(\Omega) \subset \mathscr{E}_Q(\Omega)$, *then the following equivalent conditions are satisfied*:

(I_1) $Q(\zeta)$ *is bounded on every set* $N(P, a) \cap V_b$.

(I_2) *For any* $a \geqq 0$ *there are constants* $C, h > 0$ *such that*

$$|Q(\zeta)|^h \leqq C(1 + |\eta|), \text{ for all } \zeta \in N(P, a) .$$

(I_3) *For any* $b \geqq 0$ *there are constants* $C', h' > 0$ *such that*

$$|Q(\zeta)|^{h'} \leqq C'(1 + |P(\zeta)|), \text{ for all } \zeta \in V_b .$$

Proof. In view of Theorem 1, Lemma 1 and Lemma 2, we need only to show that conditions (I_1)-(I_3) are equivalent. Also the implications $(I_2) \Rightarrow (I_1)$ and $(I_3) \Rightarrow (I_1)$ are obvious. We prove that $(I_1) \Rightarrow (I_2)$.

Consider the real polynomial

$$W(\xi, \eta, r, s, t) \\ = (a^2 - |P(\xi + i\eta)|^2 - r^2)^2 + (s^2 - |\eta|^2)^2 + (t^2 - |Q(\xi + i\eta)|^2)^2$$

of $2n + 3$ real variables. If $\xi, \eta \in R^n$ lie on the surface

$$W(\xi, \eta, r, s, t) = 0 \,, \tag{13}$$

then $\zeta = \xi + i\eta \in N(P, a)$. Moreover, by condition (I_1), the surface (13) is contained in a domain defined by an inequality

$$|s| > \varphi(|t|),$$

where $\varphi(\tau) \rightarrow \infty$ as $\tau \rightarrow \infty$. Applying now a theorem of Gorin ([1], Theorem 4.1) we conclude that there exist constants $C, h > 0$ satisfying condition (I_2). Thus $(I_1) \Rightarrow (I_2)$. The proof of the implication $(I_1) \Rightarrow (I_3)$ is similar.

3. Sufficient conditions. We now prove that conditions (I_1)-(I_3) are sufficient for the inclusion under consideration. Our first goal is to construct a sequence of suitable fundamental solutions for the operators $P^k(D)$, $k = 1, 2, \cdots$. We achieve this by modifying the construction of a fundamental solution for $P(D)$ given in [2].

In what follows p and q denote the orders of the differential operators $P(D)$ and $Q(D)$, respectively.

LEMMA 3. *Suppose that conditions* (I_1)-(I_3) *are satisfied. Then there exist continuous functions* F_k, $k = 1, 2, \cdots$, *in* R^n *with the following properties*:

(a) *For* $\nu = p + q + n$ *and any* k,

$$E_k = (\lambda - \Delta)^\nu F_k$$

is a fundamental solution for $P^k(D)$, *i.e.*

$$P^k(D)E_k = \delta$$

(b) $P^j(D)F_k = F_{k-j}$, *for* $j = 1, 2, \cdots, k - 1$.

(c) $Q^l(D)F_k$, $k, l = 1, 2, \cdots$, *are continuous functions in* $R^n \backslash \{0\}$.

(d) *For any* l *there is a* k *such that* $Q^l(D)F_k$ *is a continuous function in* R^n.

Proof. For any $\xi' = (\xi_1, \cdots, \xi_{n-1}) \in R^{n-1}$, consider the subset of the complex ζ_n-plane

$$U(\xi') = \{\zeta_n \in C: |P(\xi', \zeta_n)| \leqq 1 \text{ or } |\lambda + |\xi'|^2 + \zeta_n^2| \leqq 1\},$$

where $\lambda > 2p$. There exist constants $C, h > 0$ such that

$$|Q(\xi', \zeta_n)| \leqq C(1 + |\eta_n|^h), \tag{14}$$

for all $\xi' \in R^{n-1}$ and $\zeta_n = \xi_n + i\eta \in U(\xi')$. This follows from (I_2), when $|P(\xi', \zeta_n)| \leqq 1$ and can be easily verified in the other case.

Let $U^-(\xi')$ be the union of all connected components of $U(\xi')$ having nonempty intersections with $C^- = \{\zeta_n \in C: \eta_n < 0\}$. We denote by $L(\xi')$ the boundary of $C^- \cup U^-(\xi')$.

If $\zeta_n \in L(\xi')$, we have

$$|P(\xi', \zeta_n)| \geqq 1; \tag{15}$$

also there are constants $C', h' > 0$ (independent of ξ') such that

$$|Q(\xi', \zeta_n)| \leqq C' |P(\xi', \zeta_n)|^{h'}. \tag{16}$$

Inequality (16) is implied by (I_3) and (15), since $(\xi', \zeta_n) \in V_{2p}$, when $\zeta_n \in L(\xi')$.

For $k = 1, 2, \cdots$, we set

$$F_k(x) = \frac{1}{(2\pi)^n} \int_{R^{n-1}} \left\{ \int_{L(\xi')} \frac{e^{i\langle x, \zeta \rangle}}{(\lambda + |\xi'|^2 + \zeta_n^2)^\nu P^k(\zeta)} d\zeta_n \right\} d\xi'.$$

The functions F_k are obviously continuous, because of (15). We claim that they satisfy the conditions (a)-(d).

Conditions (a) and (b) follow from general properties of the Fourier transforms of distributions.

The verification of condition (c) can be carried out in the same way as in [2] (see the proof of Lemma 4). We give a brief sketch of the argument.

Suppose first that, for a given k, $F_k^{(j)}$ is a function obtained by a construction as above, where the contour of integration (corresponding to $L(\xi')$) lies in the complex ζ_j-plane; in particular $F_k^{(n)} = F_k$. Then

$$Q^l(D)[F_k - F_k^{(j)}],\ j = 1, \cdots, n-1;\ l = 1, 2, \cdots,$$

are continuous functions in R^n; we omit the easy proof of this fact. Thus condition (c) will be verified, if we show that $Q^l(D)F_k^{(j)}$, $l = 1, 2, \cdots$, are continuous for $x_j \neq 0$ $(j = 1, \cdots, n)$.

Consider, for example, the function F_k and let $x_n < 0$. In this case the contour $L(\xi')$ can be replaced by the boundary $V^-(\xi')$ of $U^-(\xi')$. By (14), there are positive constants C_1 and C_2 such that

$$\eta_n \leqq -C_1 |Q(\xi', \zeta_n)|^{1/h} + C_2$$

for all $\xi' \in R^{n-1}$ and $\zeta_n \in V^-(\xi')$. Hence, if $\zeta = (\xi', \zeta_n)$, we have

$$| Q^l(\zeta)e^{i\langle x,\zeta\rangle} | \leqq | Q(\zeta) |^l \exp \{x_n(C_1 | Q(\zeta) |^{1/h} - C_2)\} .$$

It follows that the integral

$$\int_{R^{n-1}} \left\{\int_{V^-(\xi')} \frac{Q^l(\zeta)e^{i\langle x,\zeta\rangle}}{(\lambda + | \xi' |^2 + \zeta_n^2)^\nu P^k(\zeta)} d\zeta_n\right\} d\xi'$$

converges absolutely and coincides with $Q^l(D)F_k(x)$, for every l.

In case $x_n > 0$ we can reason similarly, replacing $L(\xi')$ by a contour $V^+(\xi')$ lying entirely in the half-plane $\eta_n \geqq 0$.

Condition (d) is a consequence of inequality (16). In fact,

$$\frac{Q^l(\xi', \zeta_n)}{P^k(\xi', \zeta_n)}$$

is bounded for $\xi' \in R^{n-1}$, $\zeta_n \in L(\xi')$, whenever $k \geqq h'l$.

Lemma 3 is now established.

THEOREM 3. *If conditions* $(\mathrm{I}_1) - (\mathrm{I}_3)$ *are satisfied, the* $\mathscr{E}_P(\Omega) \subset \mathscr{E}_Q(\Omega)$, *for any open set* $\Omega \subset R^n$.

Proof. Assume that $T \in \mathscr{E}_P(\Omega)$ and fix an arbitrary open set $\Omega' \subset\subset \Omega$. We have to show that the restrictions of $Q^l(D)T$, $l = 0, 1, \cdots$, to Ω' are all in a space $\mathscr{D}'^m(\Omega')$.

By Lemma 3, there are fundamental solutions E_k for the operators $P^k(D)$, $k = 1, 2, \cdots$, representable according to (a) with the functions F_k satisfying conditions (b) – (d). Let l be given and let k be the integer corresponding to l in condition (d).

There are open sets Ω_j, $j = 0, 1, \cdots, k+1$, such that

$$\Omega' \subset\subset \Omega_{k+1} \subset\subset \Omega_k \subset\subset \cdots \subset\subset \Omega_0 \subset\subset \Omega . \tag{17}$$

Since $T \in \mathscr{E}_P(\Omega)$, the restrictions of $P^j(D)T$, $j = 0, 1, \cdots$, to Ω_0 are all of order $\leqq m_0$, say. For every $j = 1, 2, \cdots, k+1$, we now choose a function $\varphi_j \in \mathscr{D}(\Omega_{j-1})$ such that $\varphi = 1$ on Ω_j. Then the distributions

$$S_1 = \varphi_1 T, S_j = \varphi_j P(D) S_{j-1}, j = 2, 3, \cdots, k+1 ,$$

are all of order $\leqq m_0$. Moreover

$$S_1 = T \text{ on } \Omega_1 \tag{18}$$

and

$$P(D)S_j - S_{j+1} = 0 \text{ on } \Omega_{j+1} , \qquad j = 1, \cdots, k. \tag{19}$$

Making use of (a) we may write

$$S_1 = \sum_{j=1}^{k} [P(D)S_j - S_{j+1}] * E_j + S_{k+1} * E_k ,$$

whence

$$Q^l(D)S_1 = \sum_{j=1}^{k} [P(D)S_j - S_{j+1}]*Q^l(D)E_j + S_{k+1}*Q^l(D)E_k \; ; \tag{20}$$

here $*$ denotes the convolution. By (19), the "values" on Ω' of each convolution

$$[P(D)S_j - S_{j+1}]*Q^l(D)E_j$$

depend on the values of $Q^l(D)E_j$ outside a neighborhood of the origin (see [5], Chapter VI, Theorem III). Therefore the restriction to Ω' of the sum in (20) is a distribution of order $\leqq m_0 + p + 2\nu$. On the other hand, the last term in (20) is of order $\leqq m_0 + p + 2\nu$, because of (a) and (d). Hence the restriction of $Q^l(D)S_1$ to Ω' is of order $\leqq m = m_0 + p + 2\nu$ and m_0 can be chosen the same for all l. Since, by (18), the restrictions of $Q^l(D)S_1$ and $Q^l(D)T$ to Ω' coincide, the theorem is proved.

Combining Theorem 2 with Theorem 3 we obtain the following corollary.

COROLLARY. *Each of the conditions* $(I_1) - (I_3)$ *is necessary and sufficient for the inclusion* $\mathscr{E}_P(\Omega) \subset \mathscr{E}_Q(\Omega)$, *where* Ω *is any nonempty open set.*

REMARK. Suppose that

$$Q(\zeta) = P(\zeta) \sum_{j=1}^{n} \zeta_j^2$$

where $P(\zeta)$ is an arbitrary polynomial. Then the operator $P(D)$ is Q-hypoelliptic (see [2], Theorem 1), but condition (I_3) is not satisfied, unless $P(D)$ (and consequently $Q(D)$) is hypoelliptic.

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