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THE NON-CONJUGACY OF CERTAIN ALGEBRAS OF OPERATORS

JULIEN HENNEFELD

Let E be a Banach space and $B(E)$ be the space of all bounded linear operators on E. It was shown by Schatten, that if E is a conjugate space then $B(E)$ is isometrically isomorphic to a conjugate space. The fact that for an arbitrary Banach space, the unit ball of $B(E)$ has extreme points suggests that $B(E)$ might always be a conjugate space. In this paper it is proved that if E has an unconditional basis and is not isomorphic to a conjugate space, then $B(E)$ is not isomorphic to a conjugate space. An even stronger result is proved.

Furthermore, it is shown that if E has an unconditional basis or a complemented subspace with an unconditional basis, then the space of all compact linear operators on E is not isomorphic to a conjugate space.

The result of Schatten is proved in [3; p. 4]. It is a theorem of Kakutani, that the identity of a Banach algebra is an extreme point of the unit ball. It follows that the invertible elements of norm one, whose inverses also have norm one, are extreme points of the unit ball. Hence, one cannot readily invoke the Krein Millman Theorem to prove non-conjugacy of $B(E)$. For X and E Banach spaces let $B(X, E)$ denote the space of all bounded linear operators from X into E.

THEOREM 2.1. (Bessaga-Pełczynski). *A conjugate space contains no complemented subspace isomorphic to* c_0.

Proof. See [1; p. 250].

THEOREM 2.2. *Let X, E be Banach spaces.*

(1) *If E has an unconditional basis $\{e_i\}$ and E is not isomorphic to a conjugate space, then $B(X, E)$ is not isomorphic to a conjugate space.*

(2) *If E has a complemented subspace which is not isomorphic to a conjugate space and which has an unconditional basis, then $B(X, E)$ is not isomorphic to a conjugate space.*

Proof. (1) Since E is not isomorphic to a conjugate space, the basis $\{e_i\}$ is not boundedly complete [2; Cor. 12, p. 37]. Since $\{e_i\}$ is also unconditional, E cannot be weakly sequentially complete and hence has a subspace isomorphic to c_0 by [2; Thm. 5, p. 39 and Thm.

6, p. 71]. Then since E is separable this subspace isomorphic to c_0 must be complemented [2; p. 92].

Let Q be a projection from E onto M_0, the subspace of E isomorphic to c_0. Fix $x_0 \in X$. Let R be a projection from X to $[x_0]$. Define $\mathscr{P}: B(X, E) \to B(X, E)$ by $\mathscr{P}T = QTR$ for each $T \in B(X, E)$. Then $\mathscr{P}(\mathscr{P}T) = QQTRR = QTR$ and hence $\mathscr{P}$ is a bounded projection. The map which sends $\mathscr{P}T$ onto $\mathscr{P}Tx_0$ for each $T \in B(X, E)$ is a one-to-one, bounded map from the image of $\mathscr{P}$ onto M_0. Hence $B(X, E)$ has a complemented subspace isomorphic to c_0, and by Theorem 2.1 $B(X, E)$ cannot be isomorphic to a conjugate space.

(2) E still has a complemented subspace isomorphic to c_0.

THEOREM 2.3. *Let E have an unconditional basis $\{e_i\}$. Then $\mathscr{C}(E)$, the space of compact linear operators from E to E, is not isomorphic to a conjugate space.*

Proof. The map which sends a compact operator A onto the operator whose matrix with respect to $\{e_i\}$ consists of the diagonal of the matrix of A, is a bounded projection from $\mathscr{C}(E)$ onto a subspace isomorphic to c_0 [4; p. 493]. Then apply Theorem 2.1.

COROLLARY 2.3. *Let E have a complemented subspace M with an unconditional basis. Then $\mathscr{C}(E)$ is not isomorphic to a conjugate space.*

Proof. Let $Q: E \to M$ be a bounded projection. Define $\mathscr{P}: \mathscr{C}(E) \to \mathscr{C}(E)$ by $\mathscr{P}A = QAQ$ for each $A \in \mathscr{C}(E)$. Then $\mathscr{P}$ is a projection onto a subspace isomorphic to $\mathscr{C}(M)$. Since $\mathscr{C}(M)$ has a complemented subspace isomorphic to c_0 so does $\mathscr{C}(E)$.

REMARK. It is an open question whether a separable Banach space has a complemented subspace with an unconditional basis. It is a reasonable conjecture that for any separable Banach space E, $\mathscr{C}(E)$ is not isomorphic to a conjugate space.

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BOSTON COLLEGE