PACIFIC JOURNAL OF MATHEMATICS
Vol. 43, No. 2, 1972

HOMOTOPY AND ALGEBRAIC K-THEORY

BARRY DAYTON

A notion of homotopy is described on a category of rings. This is used to induce a notion of equivalence on the categories of projective modules and to construct a K-theory exact sequence. The topological K-theory exact sequence is then obtained from the algebraic K_0, K_1 sequence.

1. Homotopy.

In this section we describe the homotopy notion and the notion of equivalence it induces on the categories of projective modules.

A cartesian square of rings is a commutative diagram of rings

$$(*) \qquad \begin{array}{ccc} A & \xrightarrow{h_2} & A_2 \\ \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle f_2} \\ A_1 & \xrightarrow{f_1} & A_0 \end{array}$$

where $A = \{(a_1, a_2) \in A_1 \times A_2 | f_1(a_1) = f_2(a_2)\}$ and h_1, h_2 are restrictions of the coordinate projections. We will further assume that f_1 is surjective. If $\mathscr{K}$ is a category of rings and $F: \mathscr{K} \to \mathscr{K}$ is a functor we call F cartesian square preserving if the functor applied to a cartesian square gives a cartesian square.

DEFINITION 1.1. Let $\mathscr{K}$ be a category of rings. A homotopy theory $\mathscr{H}$ for $\mathscr{K}$ is an ordered quadruple $(I, \iota_0, \iota_1, \pi)$ where I is a cartesian square preserving functor and $\iota_0, \iota_1: I \to 1_{\mathscr{K}}$, $\pi: 1_{\mathscr{K}} \to I$ are natural transformations such that $\iota_0(A)\pi(A) = 1_A = \iota_1(A)\pi(A)$ for $A \in \mathscr{K}$.

For a homotopy theory $\mathscr{H} = (I, \iota_0, \iota_1, \pi)$ on $\mathscr{K}$ and $f, g: B \to A$ morphisms in $\mathscr{K}$ define $f \sim g$ if there exists a morphism $h: B \to IA$ in $\mathscr{K}$ such that $f = \iota_0 h$, $g = \iota_1 h$; h is called a homotopy of f to g. Let $\cong$ be the smallest equivalence relation on $\mathscr{K}(B, A)$ containing $\sim$; if $f \cong g$ we say f is homotopic to g.

Note that a homotopy theory gives rise to a homotopy category, i.e. a category whose objects are those of $\mathscr{K}$ and whose morphisms are homotopy classes of morphisms.

Let $\mathscr{L}$ be an arbitrary category and $G: \mathscr{K} \to \mathscr{L}$ be a covariant functor A homotopy theory $\mathscr{H} = (I, \iota_0, \iota_1, \pi)$ on $\mathscr{K}$ is called compatible with G if $G(\pi(A))$ is an isomophism for each $A \in \mathscr{K}$. Note that if $\mathscr{H}$ is compatible with G then $G(\iota_0) = G(\iota_1) = G(\pi)^{-1}$ consequently if $f \cong g$, then $G(f) = G(g)$.

For any ring A let $\underline{\underline{P}}(A)$ denote the category of finitely generated projective right A-modules. Given a ring homomorphism $f: A \to B$ denote by $\hat{f}: \underline{\underline{P}}(A) \to \underline{\underline{P}}(B)$ the covariant additive functor defined by $\hat{f}(M) = M \otimes_A B$ on objects M of $\underline{\underline{P}}(A)$ and $\hat{f}(\alpha) = \alpha \otimes 1$ on morphisms of $\underline{\underline{P}}(A)$. It is well known that if M is A-projective then $M \otimes_A B$ is B-projective.

If $A_0, A_1, \cdots, A_n, B_0, \cdots, B_e$ are rings, if $f_i: A_{i-1} \to A_i$ and $g_i: B_{i-1} \to B_i$ are ring homomorphisms, if $A_0 = B_0 = A$, $A_n = B_e = B$ and if $f_n f_{n-1} \cdots f_1 = g_e g_{e-1} \cdots g_1$, we denote by $\langle f_1, \cdots, f_n / g_1, \cdots, g_e \rangle$ the canonical natural equivalence $\hat{f}_n \cdots \hat{f}_1 \to \hat{g}_e \cdots \hat{g}_1$; it is straightforward to verify that

$$\left\langle \frac{g_1, \cdots, g_e}{h_1, \cdots, h_k} \right\rangle \left\langle \frac{f_1, \cdots, f_n}{g_1, \cdots, g_e} \right\rangle = \left\langle \frac{f_1, \cdots, f_n}{h_1, \cdots, h_k} \right\rangle ,$$

that

$$\left\langle \frac{f_1, \cdots, f_n, h}{g_1, \cdots, g_e, h} \right\rangle = \hat{h} \left\langle \frac{f_1, \cdots, f_n}{g_1, \cdots, g_e} \right\rangle$$

whenever $h: B \to C$ and that

$$\left\langle \frac{h, f_1, \cdots, f_n}{h, g_1, \cdots, g_e} \right\rangle_M = \left\langle \frac{f_1, \cdots, f_n}{g_1, \cdots, g_e} \right\rangle_{\hat{h}M}$$

for $h: C \to A$ where the subscript M means that the natural equivalence is evaluated at the module $M \in \underline{\underline{P}}(C)$.

DEFINITION 1.2. A homotopy theory $\mathscr{H} = (I, \iota_0, \iota_1, \pi)$ in $\mathscr{K}$ induces an $\mathscr{H}$-equivalence $\cong_{\mathscr{H}}$ in each category $\underline{\underline{P}}(A)$, $A \in \mathscr{K}$ as follows: given $M, N \in \underline{\underline{P}}(A)$ write $M \sim_{\mathscr{H}} N$ if there is a $Q \in \underline{\underline{P}}(IA)$ such that $M \approx \iota_0 Q$, $N \approx \iota_1 Q$ and let $\cong$ be the smallest equivalence relation on the set of isomorphism classes of objects in $\underline{\underline{P}}(A)$ containing $\sim_{\mathscr{H}}$. If $M \cong_{\mathscr{H}} N$ we say that the modules are equivalent mod-$\mathscr{H}$.

The homotopy theory $\mathscr{H}$ in $\mathscr{K}$ also induces an equivalence relation $\cong_{\mathscr{H}}$ in the set $\mathrm{Iso}(M, N)$ of isomorphisms $M \to N$ of A-projectives by letting $\phi_0 \sim_{\mathscr{H}} \phi_1$ denote that there is an isomorphism $\theta: \hat{\pi}M \to \hat{\pi}N$ such that

$$\phi_j = \left\langle \frac{\pi, \iota_j}{1} \right\rangle_N (\hat{\iota}_j \theta) \left\langle \frac{1}{\pi, \iota_j} \right\rangle_M$$

for $j = 0, 1$ and letting $\cong_{\mathscr{H}}$ be the smallest equivalence relation containing $\sim_{\mathscr{H}}$ on the set $\mathrm{Iso}(M, N)$. If $\phi_0 \cong_{\mathscr{H}} \phi_1$ we say the isomorphisms are equivalent mod $\mathscr{H}$.

Note that if $M' \xrightarrow{\omega} M \underset{\phi_1}{\overset{\phi_0}{\rightrightarrows}} N \xrightarrow{\mu} N'$ are isomorphisms and if $\phi_0 \cong \phi_1$ mod $\mathscr{H}$ then also $\mu\phi_0\omega \cong \mu\phi_1\omega$ mod $\mathscr{H}$. It is not difficult to show that if $f\colon A \to B$ is a morphism in $\mathscr{K}$ then $M \cong N \bmod \mathscr{H}$ in $\underline{\underline{P}}(A)$ implies $\hat{f}M \cong \hat{f}N \bmod \mathscr{H}$ in $\underline{\underline{P}}(B)$ and $\phi_0 \cong \phi_1 \bmod \mathscr{H}$ implies $\hat{f}\phi_0 \cong \hat{f}\phi_1 \bmod \mathscr{H}$ in $\underline{\underline{P}}(B)$. It is also easily seen that if $f \cong g\colon A \to B$ and $M \in \underline{\underline{P}}(A)$ then $\hat{f}M \cong \hat{g}M \bmod \mathscr{H}$ in $\underline{\underline{P}}(B)$.

Given a ring with unit R, an R-algebra will mean a unitary R-algebra. If A is an R-algebra, then $a\colon R \to A$ will denote the unique R-algebra homomorphism such that $a(1) = 1$. In addition to the above results we then have:

LEMMA 1.3. *Let $\mathscr{K}$ be a category of R-algebras and R-algebra homomorphisms and let $\mathscr{H} = (I, \iota_0, \iota_1\pi)$ be a homotopy theory on $\mathscr{K}$. Let $f \cong g\colon A \to B$ in $\mathscr{K}$, let $M, N \in \underline{\underline{P}}(R)$ and let $\phi \in \operatorname{Iso}(\hat{a}M, \hat{a}N)$. Then*

$$\left\langle \frac{a,\, f}{b} \right\rangle_N (\hat{f}(\phi)) \left\langle \frac{b}{a,\, f} \right\rangle_M \cong \left\langle \frac{a,\, g}{b} \right\rangle_N (\hat{g}(\phi)) \left\langle \frac{b}{a,\, g} \right\rangle_M \bmod \mathscr{H}$$

in $\operatorname{Iso}(\hat{b}M, \hat{b}N)$.

Proof. We may assume $f \sim g$. Letting $h\colon A \to IB$ be a homotopy from f to g, define $\omega\colon \hat{\pi}\hat{b}M \to \hat{\pi}\hat{b}N$ by

$$\omega = \left\langle \frac{a,\, h}{b,\, \pi} \right\rangle_N (h(\phi)) \left\langle \frac{b,\, \pi}{a,\, h} \right\rangle_M .$$

It is easily verified that ω shows that the two isomorphisms are equivalent mod $\mathscr{H}$.

Equivalence mod $\mathscr{H}$ works well with cartesian squares. If (*) is a cartesian square we can construct the fiber product category $\underline{\underline{P}}(A) \times_{\underline{\underline{P}}(A_0)} \underline{\underline{P}}(A_2)$, [2, p. 358] in which objects are triples (M, ϕ, N) where $M \in \underline{\underline{P}}(A_1)$, $N \in \underline{\underline{P}}(A_2)$ and $\phi\colon \hat{f}_1 M \to \hat{f}_2 N$ is an isomorphism in $\underline{\underline{P}}(A_0)$; and the morphisms $(M, \phi, N) \to (M', \phi', N')$ are pairs (α, β) where $\alpha\colon M \to M' \in \underline{\underline{P}}(A_1)$, $\beta\colon N \to N' \in \underline{\underline{P}}(A_2)$ and $\phi'(\hat{f}\alpha) = (\hat{f}_2\beta)\phi$. By Milnor's theorem [2, p. 479] the functor $F\colon \underline{\underline{P}}(A) \to \underline{\underline{P}}(A_1) \times_{\underline{\underline{P}}(A_0)} \underline{\underline{P}}(A_2)$ given by $F(M) = (\hat{h}_1 M, \langle h_1 f_1 / h_2 f_2 \rangle_M, \hat{h}_2 M)$ and $F(\alpha) = (\hat{h}_1\alpha, \hat{h}_2\alpha)$ is an equivalence of categories. Making this identification, the following is a projective module analogue of a theorem on vector bundles. [1, Lemma 1.4.6].

PROPOSITION 1.4. *Let $\mathscr{H} = (I, \iota_0, \iota_1\pi)$ be a homotopy theory on $\mathscr{K}$ and (*) a cartesian square in $\mathscr{K}$. Let $M \in \underline{\underline{P}}(A)$, $N \in \underline{\underline{P}}(A)$ and $\phi_0 \cong \phi_1\colon \hat{f}_1 M \to \hat{f}_2 N \bmod \mathscr{H}$. Then $(M, \phi_0, N) \cong (M, \phi_1, N) \bmod \mathscr{H}$ in $\underline{\underline{P}}(A)$.*

Proof. Assume $\phi_0 \sim_{\mathscr{H}} \phi_1$ and let $\omega\colon \hat{\pi}\hat{f}_1 M \to \hat{\pi}\hat{f}_2 N$ show $\phi_0 \sim_{\mathscr{H}} \phi_1$.

Define $\omega': \widehat{If_1}\tilde{\pi}M \to \widehat{If_2}\tilde{\pi}N$ by

$$\omega' = \left\langle \frac{f_2, \pi}{\pi, If_2} \right\rangle_N (\omega) \left\langle \frac{\pi, If_1}{f_1, \pi} \right\rangle_M .$$

Since

$$\begin{array}{ccc} IA & \xrightarrow{Ih_2} & IA_2 \\ {\scriptstyle Ih_1}\downarrow & & \downarrow{\scriptstyle If_2} \\ IA_1 & \xrightarrow{If_1} & IA_0 \end{array}$$

is by hypothesis also a cartesian square we have $(\tilde{\pi}M, \omega', \tilde{\pi}N) \in \underline{\underline{P}}(IA)$ and direct calculation shows that $\hat{\iota}_j(\tilde{\pi}M, \omega', \tilde{\pi}N) \approx (M, \phi_j, N)$ for $j = 0, 1$.

2. A connecting homomorphism.

In this section we obtain an explicit formula for a connecting homomorphism useful in constructing algebraic K-theory exact sequences.

Let K_0, K_1 be the algebraic K_i functors [2, p. 445]. If $\mathscr{K}$ is a category of R-algebras and R-algebra homomorphisms define $\tilde{K}_i(A) = K_i(A)/\text{Im}\, K_i(a)$. If $f: A \to B$ is a morphism in $\mathscr{K}$ then $f \circ a = b$ and we let $\tilde{K}_i(f): \tilde{K}_i(A) \to \tilde{K}_i(B)$ be the induced map. It is simple to verify that $\tilde{K}_0, \tilde{K}_1$ are functors on $\mathscr{K}$ and moreover that $\tilde{K}_i(A)$ is isomorphic to the usual reduced group whenever A is an augmented R-algebra.

THEOREM 2.1. *Let $\mathscr{H}$ be a homotopy theory on a category $\mathscr{K}$ of R-algebras compatible with $\tilde{K}_0$. Let*

$$\begin{array}{ccc} B & \longrightarrow & R \\ \downarrow & & \downarrow{\scriptstyle a_0} \\ B_1 & \xrightarrow{g} & A_0 \end{array} \qquad\qquad \begin{array}{ccc} A & \longrightarrow & R \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle a_0} \\ A_1 & \xrightarrow{f} & A_0 \end{array}$$

be cartesian squares in $\mathscr{K}$, $h: B_1 \to A_1$ such that $fh \cong g$ and $\hat{K}_0(B_1) = 0$. Then there is a unique group homomorphism $\delta: \hat{K}_0(B) \to \hat{K}_0(A)$ such that

$$\delta[(\hat{b}_1M, \phi, N)] = \left[\left(\hat{a}_1M, \phi\left\langle \frac{a_1, f}{b_1, g} \right\rangle_M, N\right)\right]$$

for $M, N \in \underline{\underline{P}}(R)$.

Proof. For $Q = (\hat{b}_1M, \phi, N) \in \underline{\underline{P}}(B)$ define

$$DQ = \left(\hat{a}_1M, \phi\left\langle \frac{a_1, f}{b_1, g} \right\rangle_M, N\right) \in \underline{\underline{P}}(A) .$$

Once one has established

(i) If $Q_1 \approx Q_2$ then $DQ_1 \cong DQ_2 \bmod \mathscr{H}$.

(ii) $D(Q_1 \oplus Q_2) \approx DQ_1 \oplus DQ_2$

(iii) $D(\hat{b}M) = \hat{a}M$

(iv) every element of $\hat{K}_0(B)$ is of the form $[Q]$

it follows easily that δ is well defined, unique and a group homomorphism. Because proofs of assertions (ii)—(iv) are themselves straightforward and do not depend on homotopy, we will prove only (i). Suppose then that (α, β): $(\hat{b}_1M, \phi, N) \to (\hat{b}M', \phi', N')$ is an isomorphism. Then we have $\phi' = \hat{a}_0(\beta)(\phi)g(\alpha^{-1})$. By Lemma 1.3

$$\left\langle \frac{b_1, g}{a_0} \right\rangle_M \hat{g}(\alpha^{-1}) \left\langle \frac{a_0}{b_1, g} \right\rangle_{M'} \cong \left\langle \frac{b_1, fh}{b_1, g} \right\rangle_M \widehat{fh}(\alpha^{-1}) \left\langle \frac{a_0}{b_1, h} \right\rangle_{M'} \bmod \mathscr{H}.$$

A direct computation gives

$$\hat{g}(\alpha^{-1}) \left\langle \frac{a_1, f}{b_1, g} \right\rangle_M \cong \left\langle \frac{a_1, f}{b_1, g} \right\rangle_M \hat{f}\left(\left\langle \frac{b_1, h}{a_1} \right\rangle_M \hat{h}(\alpha^{-1}) \left\langle \frac{a_1}{b_1, h} \right\rangle_{M'} \right) \bmod \mathscr{H},$$

so

$$\phi' \left\langle \frac{a_1, f}{b_1, g} \right\rangle_{M'} \cong \hat{a}_0(\beta)(\phi) \left\langle \frac{a_1, f}{b_1, g} \right\rangle_M \hat{f}(\gamma)$$

where

$$\gamma = \left\langle \frac{b_1, h}{a_1} \right\rangle_M (\hat{h}(\alpha^{-1})) \left\langle \frac{a_1}{b_1, h} \right\rangle_{M'}.$$

Therefore (using Proposition 1.4)

$$\left(\hat{a}_1 M', \phi' \left\langle \frac{a_1, f}{b_1, g} \right\rangle_{M'}, N' \right) \cong \left(\hat{a}_1 M', a_0(\beta)(\phi) \left\langle \frac{a_1, f}{b_1, g} \right\rangle_M \hat{f}(\gamma), N' \right) \bmod \mathscr{H}.$$

Since (γ, β^{-1}) is an isomorphism from this latter module to

$$\left(\hat{a}_1 M, \phi \left\langle \frac{a_1, f}{b_1, g} \right\rangle_M, N \right)$$

the assertion (i) is proved.

3. An exact sequence. In this section we use the homomorphism of 2.1 and the standard K_0, K_1 exact sequence to construct a 5-term exact sequence.

An R-algebra A is called proper if the morphism $K_0(a)$: $K_0(R) \to K_0(A)$ is injective. We note that either of the following two conditions is sufficient to insure that an R-algebra A is proper:

(i) A has as an augmentation, i.e. there is a e: $A \to R$ such that $ea = 1_R$

(ii) R is a principal ideal domain and A is a commutative R algebra.

LEMMA 3.1. *Let* (*) *be a cartesian square of proper R-algebra. Then there is an exact sequence*

$$\begin{aligned}\tilde{K}_1(A) \longrightarrow \tilde{K}_1(A_1) \oplus \tilde{K}_1(A_2) \longrightarrow \tilde{K}_1(A_0) \xrightarrow{\tilde{\partial}} \tilde{K}_0(A) \\ \longrightarrow \tilde{K}_0(A_1) \oplus \tilde{K}_0(A_2) \longrightarrow \tilde{K}_0(A_0)\end{aligned}$$

which is functorial with respect to transformations of cartesian squares.

Proof. Since

$$\begin{array}{ccc} R & \longrightarrow & R \\ \downarrow & & \downarrow \\ R & \longrightarrow & R \end{array}$$

is a cartesian square, by [2, p. 481] we have the commutative diagram

$$\begin{array}{ccccccccccc}
 & & & & & & 0 & & 0 & & 0 \\
 & & & & & & \downarrow & & \downarrow & & \downarrow \\
K_1(R) & \longrightarrow & K_1(R) \oplus K_1(R) & \longrightarrow & K_1(R) & \longrightarrow & K_0(R) & \longrightarrow & K_0(R) \oplus K_0(R) & \longrightarrow & R_0(R) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
K_1(A) & \longrightarrow & K_1(A_1) \oplus K_1(A_2) & \longrightarrow & K_1(A_0) & \xrightarrow{\partial} & K_0(A) & \longrightarrow & K_0(A_1) \oplus K_0(A_2) & \longrightarrow & K_0(A_0) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\tilde{K}_1(A) & \longrightarrow & \tilde{K}_1(A_1) \oplus \tilde{K}_1(A_2) & \longrightarrow & \tilde{K}_1(A_0) & \xrightarrow{\tilde{\partial}} & \tilde{K}_0(A) & \longrightarrow & \tilde{K}_0(A_1) \oplus \tilde{K}_0(A_2) & \longrightarrow & \tilde{K}_0(A_0) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0 & & 0 & & 0
\end{array}$$

where the columns and the first two rows are exact. An easy chase shows that the third row is exact.

We wish to give an explicit formula for the morphism $\tilde{\partial}$. For this we have:

LEMMA 3.2. *Let A, A_0 and A_1 be proper R-algebras and*

$$\begin{array}{ccc} A & \xrightarrow{e} & R \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle a_0} \\ A_1 & \xrightarrow{f} & A_0 \end{array}$$

be a cartesian square. Then the connecting homomorphism of 3.1 *is* *given by*

$$\tilde{\partial}[\hat{a}_0 M, \alpha] = \left[\left(\hat{a}M, \alpha\left\langle\frac{a_1, f}{a_0}\right\rangle_M, M\right)\right] \quad for \quad M \in \underline{\underline{P}}(R) .$$

Proof. Since the full subcategory of $\underline{\underline{P}}(A_0)$ with objects $\hat{a}_0 M$, $M \in \underline{\underline{P}}(R)$ is cofinal, $K_1(A_0)$ and hence $\tilde{K}_1(A_0)$ is generated by elements of the form $[\hat{a}_0 M, \alpha]$ [2, p. 355]. But

$$\begin{aligned}\partial[\hat{a}_0 M, \alpha] &= \partial\left[\hat{f}\hat{f}'\hat{a}M, \left\langle\frac{a_0}{a, f, f'}\right\rangle_M \alpha\left\langle\frac{a, f', f}{a_0}\right\rangle_M\right] \\ &= \left[\left(\hat{f}'\hat{a}M, \left\langle\frac{a_0}{a, \varepsilon, a_0}\right\rangle_M \alpha\left\langle\frac{a, f', f}{a_0}\right\rangle_M, \hat{\varepsilon}\hat{a}M\right)\right] - [\hat{a}M] \\ &= \left[\left(\hat{a}, M, \alpha\left\langle\frac{a_1, f}{a_1}\right\rangle_M, M\right)\right] + 0\end{aligned}$$

from [2, 4.3 p. 365] since $[\hat{a}M] \in \operatorname{Im} K_0(a)$.

In order to apply 2.1 we need

LEMMA 3.3. *Under the hypotheses of Theorem* 2.1 *the diagram*

$$\begin{array}{ccccc}\tilde{K}_1(A_0) & \xrightarrow{\tilde{\partial}'} & \tilde{K}_0(B) & \longrightarrow & \tilde{K}_0(B_1) = 0 \\ \downarrow 1 & & \downarrow \delta & & \downarrow \\ \tilde{K}_1(A_0) & \xrightarrow{\tilde{\partial}} & \tilde{K}_0(A) & \xrightarrow{\tilde{K}_0(f')} & \tilde{K}_0(A_1)\end{array}$$

commutes.

Proof.

$$\begin{aligned}\delta\tilde{\partial}'[\hat{a}'M, \alpha] &= \delta\left[\left(\hat{b}, M, \alpha\left\langle\frac{b_1, g}{a'}\right\rangle_M, M\right)\right] = \left[\left(\hat{a}_1 M, \alpha\left\langle\frac{b_1, g}{a_0}\right\rangle_M\left\langle\frac{a_1, f}{b_1, g}\right\rangle_M, M\right)\right] \\ &= \left[\left(\hat{a}_1 M, \alpha\left\langle\frac{a_1, f}{a'}\right\rangle, M\right)\right] = \tilde{\partial}[\hat{a}_0 M, \alpha] .\end{aligned}$$

Also since $\tilde{K}_0(B_1) = 0$ it can be seen that if

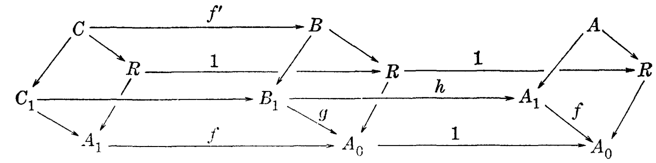
$$[N] \in \tilde{K}_0(B),\ [N] = [(\hat{b}_1 M, \phi, N)],\ M, N \in P(R) .$$

Thus

$$\tilde{K}_0(f')\delta[(\hat{b}_1 M, \phi, N)] = \tilde{K}_0(f')\left[\left(\hat{a}_1 M, \phi\left\langle\frac{a_1, f}{h_1, g}\right\rangle_M, N\right)\right] = [\hat{a}_1 M] = 0 .$$

THEOREM 3.4. *Let $\mathscr{K}$ be a category of proper R-algebras and $\mathscr{H}$ be a homotopy theory on $\mathscr{K}$ compatible with $\tilde{K}_0$. Let*

(**)

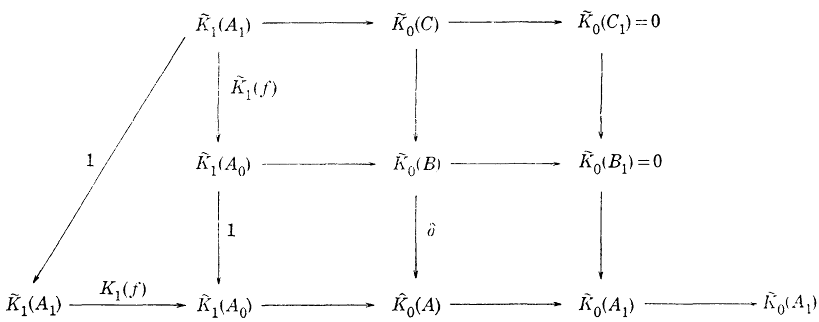


be a diagram in $\mathscr{K}$ where $fh \cong g$, all other squares commute and the vertical squares are cartesian. If $\tilde{K}_0(C_1) = \tilde{K}_0(B_1) = 0$ then

$$\tilde{K}_0(C) \xrightarrow{\tilde{K}_0(f')} \tilde{K}_0(B) \xrightarrow{\delta} \tilde{K}_0(A) \longrightarrow \tilde{K}_0(A_1) \longrightarrow \tilde{K}_0(A_0)$$

is exact

Proof. From 3.1 and 3.3 we get a commutative diagram



where the rows are exact. A diagram chase gives the result.

4. **The topological K-theory exact sequence.** In this section we use 3.4 to construct the topological K-Theory exact sequence.

Let R denote the real or complex numbers. For a compact Hausdorff space X let CX be the ring of continuous R-valued functions and for a continuous function $f: X \to Y$ let $f^*: CY \to CX$ be the induced ring homomorphism. Denote the one point space by $*$ and take $\mathscr{K}$ to be the category of rings CX and ring homomorphisms. We will consider $\mathscr{K}$ to be a category of $C^* = R$ algebras. Define $J: \mathscr{K} \to \mathscr{K}$ by $JCX = C(X \times I)$ where I denotes the unit interval and $J(f) = (f \times 1)^*$. Define ι_0, ι_1, π by i_0^*, i_1^*, π^* where $i_j: X \to I$ is given by $i_j(x) = (x, j)$ and $\pi(x, t) = x$, $\pi: X \times I \to X$. It follows easily that $\mathscr{H} = (J, \iota_0, \iota_1, \pi)$ is a homotopy theory on $\mathscr{K}$. We recall that $K_0^T(X) = K_0(CX)$ where K_0^T is topological K_0 functor. If X is a pointed space the reduced group as defined above coincides with the usual reduced group. It follows from standard results on vector bundles [1, Lemma 1.4.3] and on the correspondence between vector bundles over X and projective modules over CX that $\mathscr{H}$ is compatible with K_0^T. Alternatively it can be easily proved directly that if $M, N \in \underline{\underline{P}}(X)$ then $M \cong$

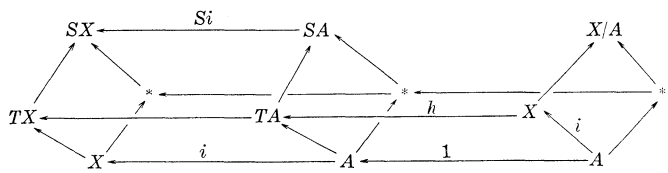
$N \bmod \mathscr{H}$ if and only if $M \approx M$.

We then have

THEOREM 4.1. *Let X be a compact Hausdorff space, $A \subset X$ a closed subspace. Let SA, SX denote the suspensions of A, X respectively. Then there is an exact sequence*

$$\tilde{K}_0^T(SX) \longrightarrow \tilde{K}_0^T(SA) \longrightarrow \tilde{K}_0^T(X/A) \longrightarrow \tilde{K}_0^T(X) \longrightarrow \tilde{K}_0^T(A)$$

Proof. Consider the diagram



where TX denotes the cone on X and h is any continuous function. Applying the functor C we get a diagram of the form (*) and it is not hard to show that the vertical squares are cartesian. Since TA is contractible $hi \cong j$ so $i^*h^* \cong j^*$. Thus theorem (3.4) applies to give the desired exact sequence.

The long exact K-theory sequence follows in the usual manner by splicing sequences of this form together.

REFERENCES

1. M. F. Atiyah, *K-Theory*, Benjamin, New York, (1966).
2. Hyman Bass, *K-Theory*, Benjamin, New York, (1968).

Received August 10, 1971

HARVEY MUDD COLLEGE
AND
NORTHEASTERN ILLINOIS UNIVERSITY